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REARICK'S ISOMORPHISM AND A CHARACTERIZATION OF ψ–ADDITIVE FUNCTIONS

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Abstract. Let $\mathbb{A}_1$ denote the set of arithmetic functions f with $f(1)$ real and $P' \subseteq \mathbb{A}_1$ denote the set of arithmetic functions f with $f(1) > 0$. If ψ denotes Lehmer's convolution, placing mild conditions on ψ it can be shown that the $\psi-$analogue of Rearick's ([7]) *logarithmic operator* $L : (P', \psi) \to \to (\mathbb{A}_1, +)$ defined by $Lf(1) = \log f(1)$ and $Lf(n) = (fh\psi f^{-1})(n)$, if $n > 1$, where h is any $\psi-$additive function with $h(n) \neq 0$ for all $n > 1$, is a *group isomorphism.* In this paper we prove the converse when ψ is a *Lehmer-Narkiewicz convolution.*

1. Introduction

An *arithmetic function* is a complex-valued function defined on the set of positive integers $\mathbb{Z}^+$. The set of arithmetic functions will be denoted by $\mathbb{A}$.

The classical Dirichlet convolution denoted by D is defined by

$$(1.1) \qquad (f\ D\ g)(n) = \sum_{d|n} f(d)g(n/d),$$

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for $f, g \in \mathbb{A}$ and $n \in \mathbb{Z}^+$; the sum on the right hand side of (1.1) is taken over all positive divisors d of n.

Let $\mathbb{A}_1$ denote the set of arithmetic functions f with $f(1)$ real and

$$P' = \{f \in \mathbb{A}_1 \;:\; f(1) > 0\}. \tag{1.2}$$

It is well-known that an arithmetic function f which is not identically zero is said to be *multiplicative* if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}^+$ with $(m,n) = 1$; here, as usual, the symbol (a,b) denotes the greatest common divisor of a and b. The set of multiplicative functions will be denoted by M.

In 1968, David Rearick (cf. [7], Theorem 9) among other things proved that the groups (P', D), $(\mathbb{A}_1, +)$ and (M, D) are all isomorphic. In fact, Rearick (cf. [7], Theorems 2 and 3) showed that the logarithmic operator $L : (P', D) \to (\mathbb{A}_1, +)$ defined by

$$Lf(1) = \log f(1), \tag{1.3}$$

and for $n > 1$

$$Lf(n) = \sum_{d|n} f(d) f^{-1}(n/d) \log d \tag{1.4}$$

is an *isomorphism,* where f^{-1} is the inverse of f with respect to the Dirichlet convolution D so that

$$\sum_{d|n} f(d) f^{-1}(n/d) = e(n)$$

for all $n \in \mathbb{Z}^+$, where

$$e(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{1.5}$$

A divisor d of n is called a *unitary* divisor (cf. [1]) and write $d \| n$ if $(d, n/d) = 1$. The unitary convolution (cf. [1]) denoted by U is defined by

$$(f\ U\ g)(n) = \sum_{d\|n} f(d) g(n/d), \tag{1.6}$$

for $f, g \in \mathbb{A}$ and $n \in \mathbb{Z}^+$.

The unitary convolution was originally introduced by R.Vaidyanathaswamy (cf. [13]) under the name of "compounding operation".

It is interesting to note that Rearick (cf. [7], Theorem 9) also observed that the results mentioned above in the case of Dirichlet convolution can be extended to the unitary convolution. That is, Rearick proved that the groups (P', U), $(\mathbb{A}_1, +)$ and (M, U) are isomorphic.

As in the case of Dirichlet convolution, Rearick (cf. [7]) proved that the *logarithmic operator* $L : (P', U) \to (\mathbb{A}_1, +)$ defined by

$$Lf(1) = \log f(1), \tag{1.7}$$

and for $n > 1$,

$$Lf(n) = \sum_{d \| n} f(d) f^{-1}(n/d) \log d \tag{1.8}$$

is an *isomorphism,* where f^{-1} is the inverse of f with respect to the unitary convolution U so that

$$\sum_{d \| n} f(d) f^{-1}(n/d) = e(n)$$

for all $n \in \mathbb{Z}^+$, where $e(n)$ is as given in (1.5).

Let $\emptyset \neq T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi \ : \ T \longrightarrow \mathbb{Z}^+$ be a mapping satisfying the following conditions :

(1.9) For each $n \in \mathbb{Z}^+$, $\psi(x, y) = n$ has a finite number of solutions.

(1.10) If $(x, y) \in T$, then $(y, x) \in T$ and $\psi(x, y) = \psi(y, x)$.

(1.11) The statements "$(x, y) \in T, (\psi(x, y), z) \in T$" and "$(y, z) \in T, (x, \psi(y, z)) \in T$" are equivalent; if one of these conditions holds, we have $\psi(\psi(x, y), z) = \psi(x, \psi(y, z))$.

If $f, g \in \mathbb{A}$, then the ψ-product of f and g denoted by $f\psi g \in \mathbb{A}$ is defined by

$$(f\psi g)(n) = \sum_{\psi(x,y)=n} f(x) g(y) \tag{1.12}$$

for all $n \in \mathbb{Z}^+$. The binary operation ψ in (1.12) is due to D.H. Lehmer [3]. It is easily seen that $(\mathbb{A}, +, \psi)$ is a commutative ring (cf. [3]).

Clearly, the Dirichlet and unitary convolutions arise as special cases of the ψ-convolution. Indeed, let $\psi(x, y) = xy$ for all $(x, y) \in T$. If $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ then ψ in (1.12) reduces to the Dirichlet convolution. If $T = \{(x, y) \in \mathbb{Z}^+ \times \times \mathbb{Z}^+ : \quad (x, y) = 1\}$, then ψ reduces to the unitary convolution [1]. More generally, if $T = \bigcup_{n=1}^{\infty} \{(d, n/d) : d \in A(n)\}$, where A is Narkiewicz's convolution [4], then ψ reduces to the A-convolution. Thus the binary operation in (1.12) is more general than Narkiewicz's A-convolution.

An arithmetic function h is said to be additive if $h(mn) = h(m) + h(n)$ whenever $(m, n) = 1$; h is said to be completely additive if $h(mn) = h(m) + h(n)$ for all $m, n \in \mathbb{Z}^+$. It is easily seen that $L : (P', D) \to (\mathbb{A}_1, +)$ remains isomorphism if $\log d$ in (1.4) is replaced by $h(d)$ where $h \in \mathbb{A}$ is completely additive and non-zero on $\mathbb{Z}^+ - \{1\}$. Similarly, $L : (P', U) \to (\mathbb{A}_1, +)$ remains isomorphism if $\log d$ in (1.8) is replaced by $h(d)$ where $h \in \mathbb{A}$ is additive and non-zero on $\mathbb{Z}^+ - \{1\}$.

$h \in \mathbb{A}$ is said to be ψ-additive if ψ satisfies conditions (1.10) and (1.11), and $h(\psi(m, n)) = h(m) + h(n)$, for all $(m, n) \in T$.

If h is a $\psi-$additive function, under mild additional conditions on ψ, it is easy to observe (see §3 for precise statements) that $L : (P', \psi) \to (\mathbb{A}_1, +)$ defined by

$$Lf(1) = \log f(1), \tag{1.13}$$

and if $n > 1$,

$$Lf(n) = \sum_{\psi(x,y)=n} f(x) f^{-1}(y) h(x) \tag{1.14}$$

is a homomorphism, where f^{-1} is the inverse of f with respect to ψ-convolution. Further if $h(n) \neq 0$ for all $n > 1$, then L is an isomorphism.

The main purpose of this paper is to prove that (see Theorem 4.1) if ψ is a Lehmer-Narkiewicz convolution (for undefined notions in this section, see §2), L in (1.13)-(1.14) is an isomorphism, $h \in \mathbb{A}$ and $h(1) = 0$, then h must be a ψ-additive function which is non-zero on $\mathbb{Z}^+ - \{1\}$.

In §2, we develop preliminaries.

2. Preliminaries

The following results (Lemmas 2.1 and 2.2) describe necessary and sufficient conditions concerning the existence of unity and inverses in $(\mathbb{A}, +, \psi)$.

Lemma 2.1. (cf. [9], Theorem 2.2). *Let ψ satisfy (1.9) - (1.12) implying that $(\mathbb{A}, +, \psi)$ is a commutative ring. Let $\psi(x, y) \geq \max\{x, y\}$ for all $x, y \in T$. Then $(\mathbb{A}, +, \psi)$ possesses the unity if and only if for each $k \in \mathbb{Z}^+$, $\psi(x, k) = k$ has a solution. In such a case if g stands for the unity, then for each $k \in \mathbb{Z}^+$,*

$$g(k) = \begin{cases} 1 - \sum\limits_{\substack{\psi(x,k)=k,\\ x<k}} g(x), & \text{if } \psi(k, k) = k, \\ 0\,, & \text{if } \psi(k, k) \neq k. \end{cases} \tag{2.1}$$

Remark 2.1. It has been established by J.L. Nicolas and V. Sitaramaiah (cf. [5], Theorem 3.1) that if $(\mathbb{A}, +, \psi)$ is a commutative ring then it possesses unity if and only if for each $k \in \mathbb{Z}^+$, $\psi(x, k) = k$ has a solution.

Lemma 2.2. (cf. [8], also see [10], Remark 1.1). *Let ψ satisfy (1.9) - (1.12) and $\psi(x, y) \geq \max\{x, y\}$ for all $x, y \in T$. For each $k \in \mathbb{Z}^+$, let the equation $\psi(x, k) = k$ have a solution so that the unity exists in $(\mathbb{A}, +, \psi)$. Let g denote the unity. Then $f \in \mathbb{A}$ is invertible with respect to ψ if and only if*

$$S_f(k) \overset{\text{def}}{=} \sum_{\psi(x,k)=k} f(x) \neq 0,$$

for all $k \in \mathbb{Z}^+$. In such a case, this inverse denoted by $f^{-1}(k)$ can be computed by

$$f^{-1}(1) = \frac{1}{f(1)},$$

and for $k > 1$,

$$f^{-1}(k) = (S_f(k))^{-1} \left[g(k) - \sum_{\substack{\psi(x,y)=k\\ y<k}} f(x) f^{-1}(y) \right].$$

Remark 2.2. If ψ satisfies (1.9) - (1.12) and $\psi(1, k) = k$ for all $k \in \mathbb{Z}^+$, then the function e defined in (1.5) is the unity in the ring $(\mathbb{A}, +, \psi)$. Further, if $\psi(x, y) \geq \max\{x, y\}$ for all $x, y \in T$ and for each $k \in \mathbb{Z}^+$, $\psi(x, k) = k$ if and only if $x = 1$, by Lemma 2.2, it follows that $f \in \mathbb{A}$ is invertible with respect to ψ if and only if $f(1) \neq 0$.

Definition 2.1. If ψ satisfies (1.9) - (1.12), then ψ is said to be multiplicativity preserving if $f\,\psi\,g$ is multiplicative whenever f and g are (see [10]).

The following results (Lemmas 2.3 and 2.4) give a characterization of multiplicativity preserving ψ-functions which are onto:

Lemma 2.3. (cf. [11], Theorem 3.1) *Suppose that the binary operation ψ in (1.12) is multiplicativity preserving and for each $k \in \mathbb{Z}^+$, the equation $\psi(x,k) = k$ has a solution. Let $x = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $y = \prod_{i=1}^{r} p_i^{\beta_i}$, where $p_1, p_2, \ldots, p_r$ are distinct primes, α_i and β_i are non-negative integers. Then we have*

(a) $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i = 1, 2, \ldots, r$.

(b) If $(x,y) \in T$ then

$$\psi(x,y) = \prod_{i=1}^{r} p_i^{\theta_{p_i}(\alpha_i,\beta_i)}, \tag{2.2}$$

where $\theta_p(\alpha,\beta)$ is a non-negative integer satisfying the following properties:

(i) $\theta_p(\alpha,\beta)$ is a non-negative integer defined for non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$.

(ii) For each integer $\gamma \geq 0$, $\theta_p(\alpha,\beta) = \gamma$ has a finite number of solutions.

(iii) $\theta_p(\alpha,\beta) = 0$ if and only if $\alpha = \beta = 0$.

(iv) $\theta_p(\alpha,\beta) = \theta_p(\beta,\alpha)$.

(v) For each $\gamma \geq 0$, $\theta_p(\alpha,\gamma) = \gamma$ has a solution.

(vi) For non-negative integers α, β, γ and for any prime p, the statements "$(p^\beta, p^\gamma) \in T$, $(p^\alpha, p^{\theta_p(\beta,\gamma)}) \in T$" and "$(p^\alpha, p^\beta) \in T$ and $(p^{\theta_p(\alpha,\beta)}, p^\gamma) \in T$" are equivalent; if one of these conditions holds, we have

$$\theta_p(\alpha, \theta_p(\beta,\gamma)) = \theta_p(\theta_p(\alpha,\beta),\gamma).$$

Lemma 2.4. (cf. [11], Theorem 3.2) *Let $T \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ be such that*

(a) $(x,y) \in T$ if and only if $(y,x) \in T$.

(b) If x and y are given as in Lemma 2.3 then $(x,y) \in T$ if and only if $(p_i^{\alpha_i}, p_i^{\beta_i}) \in T$ for $i = 1, 2, \ldots, r$.

Further, for each prime p and non-negative integers α, β such that $(p^\alpha, p^\beta) \in T$, let $\theta_p(\alpha,\beta)$ be a non-negative integer satisfying (i) - (vi) of Lemma

2.3. If for $(x,y) \in T$, $\psi(x,y)$ is defined by (2.2), then ψ is mulitiplicativity preserving and for each $k \in \mathbb{Z}^+$, $\psi(x,k) = k$ has a solution.

Lemma 2.5. (cf. [11], Theorem 3.3) *Let ψ be given as in Lemma 2.4 and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. If M denotes the set of all multiplicative functions which are invertible with respect to ψ then (M,ψ) is a commutative group in which the function g defined in (2.1) is the identity.*

Remark 2.3. *For ψ and θ_p given in Lemma 2.4, we have that $\psi(x,y) \geq$ $\geq \max\{x,y\}$ for all $(x,y) \in T$ is equivalent to saying that $\theta_p(\alpha,\beta) \geq \max\{\alpha,\beta\}$ for all non-negative integers α and β such that $(p^\alpha, p^\beta) \in T$. In such a case, it is clear that $\psi(x,y) = n$ implies that $x|n$ and $y|n$; it may also be noted that if $\psi(1,n) = n$ for all $n \in \mathbb{Z}^+$, then $\psi(x,y) = xy$ whenever $(x,y) = 1$ (see also [10], Lemmas 2.1 and 2.2) and $\theta_p(0,\alpha) = \alpha$ for all non-negative integers α.*

Definition 2.2. (see [12]) Let ψ be multiplicativity preserving with $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ and $\psi(1,k) = k$ for all $k \in \mathbb{Z}^+$. Let T and θ_p be as in Lemma 2.4. Then ψ is called a Lehmer-Narkiewicz convolution or simply an L-N convolution if θ_p satisfies the following conditions for all primes p:

$$(i) \quad (\theta_p(\alpha,\beta) = \theta_p(\alpha,\gamma)) \text{ implies that } (\beta = \gamma),$$

and

$$(ii) \quad (\theta_p(\alpha,\beta) = \theta_p(\gamma,\delta)) \text{ implies that } \begin{cases} \alpha = \theta_p(\gamma, c) & \text{for some } c \geq 0, \\ \text{or } \beta = \theta_p(\delta, d) & \text{for some } d \geq 0. \end{cases}$$

Definition 2.3. (see Narkiewicz [4]) A binary operation B in $\mathbb{A}$ is called a regular convolution if the following conditions hold:

(i) The triple $(F\,,+\,,B)$ is a commutative ring with unity (here $'+'$ denotes the usual point-wise addition).

(ii) B is multiplicativity preserving; that is $f\,B\,g$ are multiplicative whenever $f\,,g \in \mathbb{A}$ are multiplicative.

(iii) The function $1 \in \mathbb{A}$ defined by $1(n) = 1$ for all $n \in \mathbb{Z}^+$ has an inverse μ_B with respect to B and μ_B is 0 or -1 at prime powers.

Definition 2.4. (cf. [12]) Let ψ satisfy (1.9) - (1.12). The binary operation ψ in (1.12) is called a regular $\psi-$convolution if it satisfies Definition 2.3.

Remark 2.4. Let ψ satisfy (1.9) - (1.12) and $\psi(x,y) \geq \max\{x,y\}$, for all $(x,y) \in T$. It has recently been established (cf. [6], Theorem 3.1) that ψ is regular convolution if and only if ψ is a Lehmer-Narkiewicz convolution.

In what follows ψ denotes a Lehmer-Narkiewicz convolution. For convenience, we now enlist the properties of these convolution (cf. [12] and [6]):

Theorem 2.1. *We have*

(I) The triple $(\mathbb{A},+,\psi)$ *is a commutative ring with unity e given by (1.5).*

(II) $\psi(x,y) \geq \max\{x,y\}$, *for all* $(x,y) \in T$.

(III) For each $k \in \mathbb{Z}^+, \psi(x,k) = k$ *if and only if* $x = 1$.

(IV) $\psi(x,k) = \psi(x,\ell)$ *implies* $k = \ell$.

(V) $f \in \mathbb{A}$ *is invertible with respect to* ψ *if and only if* $f(1) \neq 0$.

(VI) ψ *is multiplicativity preserving, that is, whenever* $f, g \in \mathbb{A}$ *are multiplicative, then so is* $f \, \psi \, g$.

(VII) The set of multiplicative functions forms a group with respect to ψ *and with e as identity. In particular, if* f *is multiplicative then the inverse of* f *with respect to* ψ, *namely,* f^{-1} *is also multiplicative.*

(VIII) The ψ*-analogue of the Möbius function denoted by* μ_ψ *is the inverse of the constant function 1 and* μ_ψ *is multiplicative. Clearly*

$$\sum_{\psi(x,y)=n} \mu_\psi(x) = e(n), \tag{2.4}$$

for all $n \in \mathbb{Z}^+$.

(IX) For each prime p and non-negative integers α *and* β *with* $\left(p^\alpha, p^\beta\right) \in$ $\in T$ *let* $\theta_p(\alpha,\beta) = \theta(\alpha,\beta)$ *be the non-negative integer given in Lemma 2.3. By taking* $n = p^\alpha > 1$ *in (2.4), we obtain*

$$\sum_{\theta(a,b)=\alpha} \mu_\psi\left(p^a\right) = 0. \tag{2.5}$$

(X) For each prime p and any non-negative integer α *let* $S_{p,\alpha} \subseteq$ $\subseteq (\mathbb{Z}^+ \bigcup\{0\}) \times (\mathbb{Z}^+ \bigcup\{0\})$ *be defined by*

$$S_{p,\alpha} = S_\alpha = \{(a,b) \; : \theta(a,b) = \alpha\}. \tag{2.6}$$

(a) If

$$S_\alpha = \{0 = a_0 < a_1 < a_2 < \ldots < a_k = \alpha\}, \tag{2.7}$$

then for $i = 1, 2, \ldots, k$

$$S_{a_i} = \{a_0, a_1, \ldots, a_i\}.$$

(b) p^{α} *is called* $\psi-$*primitive if* $S_{\alpha} = \{0\,,\,\alpha\}$. *From (a) it is clear that* p^{a_1} *is* $\psi-$*primitive and* p^{a_i} *is not* $\psi-$*primitive for* $i = 2, 3, \ldots k$, *if* $k \geq 2$.

(c) The least positive integer in S_{α} *is denoted by* $\tau_{\psi}(p^{\alpha})$. *The rank of* p^{α} *denoted by* $r_p(\alpha)$ *or simply by* $r(\alpha)$ *is the number of elements in* $S_{\alpha} - \{0\}$. *Clearly,* $a_1 = \tau_{\psi}(p^{\alpha})$ *and* $r(a_i) = i$ *for* $i = 1, 2, \ldots k$. *Also,*

$$\mu_{\psi}\,(p^{\alpha}) = \begin{cases} -1, & \text{if } p^{\alpha} \text{ is } \psi\text{-primitive,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

In particular,

$$\mu_{\psi}(p^{a_1}) = -1 \text{ and } \mu_{\psi}(p^{a_i}) = 0, \tag{2.9}$$

for $i = 2, 3, \ldots, k$.

(d) $\theta(\alpha, \beta) = \theta(\alpha, \gamma)$ *implies* $\beta = \gamma$.

(e) $\theta(x, \alpha) = \alpha$ *if and only if* $x = 0$.

(f) If $a_i\,, a_j$ and $a_{i+j} \in S_{\alpha}$, *then* $\theta(a_i, a_j) = a_{i+j}$ *(i and j need not be distinct).*

(g) If $0 \leq \ell \leq k$ *then the solutions of* $\theta(x, y) = a_{\ell}$ *are precisely* $\{(a_i, a_j)\ :\ i + j = \ell,\ i\,j \geq 0.\}$

The following theorem characterizes the Lehmer-Narkiewicz convolutions (or simply L-N convolutions) in a very effective way:

Theorem 2.2. (cf. [12], Corollary 4.1) *For each prime p, let* π_p *denote a class of subsets of non-negative integers such that*

(i) the union of all members of π_p *is the set of non-negative integers;*

(ii) each member of π_p *contains zero;*

(iii) no two members of π_p *contain a positive integer in common.*

If $S \in \pi_p$ *and* $S = \{a_0, a_1, a_2, \ldots\}$ *with* $0 = a_0 < a_1 < a_2 < \cdots$, *we define* $\theta_p(a_i, a_j) = a_{i+j}$, *if* a_i, a_j *and* $a_{i+j} \in S$ *(i and j need not be distinct). If* ψ *and* T *are as given in Lemma 2.4 then* ψ *is an L-N convolution and is also a regular convolution. Also, every L-N convolution can be obtained in this way.*

For each prime p, if $\pi_p\ :\ \{0, 2, 3\}\,; \{0, 4, 5\}\,; \{0, 6\}\,; \{0, 7\}\,; \{0, 8\};\ldots.$ then the corresponding ψ convolution is an L-N convolution, but not a regular Narkiewicz convolution [4].

3. ψ-analogues of some results of Rearick

We recall that if ψ satisfies the conditions (1.10) and (1.11) then $h \in \mathbb{A}$ is called $\psi-$additive if $h(\psi(x,y)) = h(x) + h(y)$, for all $(x,y) \in T$. It is clear that $h(1) = 0$ if $\psi(1,1) = 1$ and h is ψ-additive.

The following results (Theorems 3.1-3.4) can be established on lines similar to Theorems 1 to 4 in Rearick [7]:

Theorem 3.1. *Let ψ satisfy (1.9)-(1.12) and $\psi(x,y) \geq \max\{x\,, y\}$, for all $(x,y) \in T$. Further suppose that for each $k \in \mathbb{Z}^+\,, \psi(x,k) = k$ if and only if $x = 1$. Let $h \in \mathbb{A}$ be $\psi-$additive and $L\,:(P',\psi) \to (\mathbb{A}_1,+)$ be the logarithmic operator given in (1.13) - (1.14). Then L is a homomorphism.*

Theorem 3.2. (under the hypothesis of Theorem 3.1) *If $h(n) \neq 0$, for all $n > 1$, then L is an isomorphism.*

Theorem 3.3. (under the hypothesis of Theorem 3.2) *Let ψ be multiplicativity preserving. Then $f \in P'$ is multiplicative if and only if $Lf(n) = 0$, whenever n is not a prime power.*

Theorem 3.4. (under the hypothesis of Theorem 3.3) *The groups (M,ψ) and $(\mathbb{A}_1,+)$ are isomorphic.*

Remark 3.1. Suppose that ψ satisfies (1.9)-(1.12) and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$. It can be shown that a necessary and sufficient condition for every $f \in P'$ is invertible with respect to ψ is that for each $k \in \mathbb{Z}^+$, $\psi(x,k) = k$ if and only if $x = 1$. Since the *logarithmic operator* defined in (1.13) and (1.14) involves f^{-1} for $f \in P'$, this condition imposed in Theorem 3.1 is justified.

Remark 3.2. If ψ is multiplicativity preserving, $\psi(1,k) = k$ for all $k \in \in \mathbb{Z}^+$, and $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$, then by Remark 2.3 we have $\psi(x,y) = xy$ whenever $(x,y) = 1$. If $\theta = \theta_p$ is as given in Lemma 2.3, then an additive arithmetic function h is ψ-additive if and only if $h\left(p^{\theta(\alpha,\beta)}\right) = = h(p^\alpha) + h\left(p^\beta\right)$, for all non-negative integers α and β such that $(p^\alpha, p^\beta) \in T$.

Example 3.1. Let $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $r \geq 0$ be an integer. For each prime p, let $\theta_p(\alpha,\beta) = \alpha + \beta + r\alpha\beta$, for non-negative integers α and β. If $x = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $y = \prod_{i=1}^{k} p_i^{\beta_i}$, where $p_1\,, p_2\,, \ldots, p_k$ are distinct primes, α_i and β_i are non-negative integers for $i = 1\,, 2\,, \ldots\,, k$, we define $\psi_r(x,y) = \prod_{i=1}^{k} p_i^{\alpha_i+\beta_i+r\alpha_i\beta_i}$. Then ψ_r satisfies the hypothesis of Theorems 3.1 - 3.4. Let $\theta = \theta_p$. We note that $\theta(\alpha,\beta) = n$ if and only if $rn + 1 = (r\alpha + 1)(r\beta + 1)$. For a completely additive function g define the additive function h at prime powers $p^n > 1$ by $h(p^n) = g(rn+1)$. Then $h\left(p^{\theta(\alpha,\beta)}\right) = h(p^\alpha) + h\left(p^\beta\right)$, for all non-negative integers α and β. Hence by Remark 3.2, h is ψ_r additive. If $g(x) > 0$ for all $x > 1$, it follows that $h(n) > 0$ for $n > 1$. One can take $g(n) = \Omega(n)$ or $\log n$, where $\Omega(n)$ is the total number of prime factors of n if $n > 1$ and $\Omega(1) = 0$. It follows that the groups $(P', \psi_r), (M, \psi_r)$ and $(\mathbb{A}_1, +)$ are isomorphic. Clearly ψ_0 is the Dirichlet convolution; ψ_1 is due to D.H. Lehmer [3] and ψ_r for $r \geq 2$ is due to V. Sitaramaiah and M.V. Subbarao [10]. It is not difficult to see that ψ_r is not a regular convolution for $r \geq 2$.

Example 3.2. Let $T = \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi(x,y) = x+y-1$ for all $(x,y) \in T$. It is not difficult to see that $h \in \mathbb{A}$ is $\psi-$additive if and only if $h(n) = (n-1)h(2)$ for all $n \geq 1$. In particular, if $h(2) \neq 0$ then $h(n) \neq 0$ for all $n \geq 2$. Hence ψ satisfies Theorems 3.1 and 3.2 so the groups (P', ψ) and $(\mathbb{A}_1, +)$ are isomorphic. Here, ψ is not multiplicativity preserving.

Example 3.3. Let ψ be an L-N-convolution. On lines similar to that of Theorem 4.1 in [6], it can be shown that an additive arithmetic function h is ψ-additive if and only if $h(p^\alpha) = r(\alpha)h(p^{a_1})$ where $a_1 = \tau_\psi(p^\alpha)$ and $r(\alpha) = |S_{p,\alpha} - \{0\}|$. It is clear that one can find a ψ-additive function h not vanishing on $\mathbb{Z}^+ - \{1\}$. For example the additive function h defined at any prime power $p^\alpha > 1$ by $h(p^\alpha) = r(\alpha)$ serves the purpose. Hence Theorems 3.1-3.4 are applicable so the groups $(P', \psi), (\mathbb{A}_1, +)$ and (M, ψ) are isomorphic. We may note that Dirichlet and unitary convolutions are L-N-convolutions.

Example 3.4. Let $\beta \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. β is said to be a *basic sequence* if (i) $(a,b) \in \beta$ implies that $(b,a) \in \beta$; (ii) $(a,bc) \in \beta$ if and only if (a,b) and (a,c) are in β; (iii) $(1,n) \in \beta$ for every $n \in \mathbb{Z}^+$. If we take $T = \beta$ and $\psi(x,y) = xy$ for all $(x,y) \in T$, then it is easily seen that ψ satisfies Theorems 3.1 and 3.2. If $h(n) = \log n$ for all $n \in \mathbb{Z}^+$, then h is $\psi-$additive. The ψ-convolution in this example reduces to the *basic convolution* introduced by Smith [2]. It follows by Theorems 3.1 and 3.2 that the groups (P', ψ) and $(\mathbb{A}_1, +)$ are isomorphic; these results were originally due to Smith (cf. [2], Theorem 2).

4. A characterization

Throughout this section we assume that ψ is a *Lehmer-Narkiewicz convolution* and L is the *logarithmic operator* defined in (1.13) and (1.14). We make use of the multiplicativity properties of the functions $d_\psi = 1\,\psi\,1$, μ_ψ, and $d_\psi^{-1} = \mu_\psi\,\psi\,\mu_\psi$. We recall that 1 denotes the constant function 1 and μ_ψ

denotes the inverse of the function 1 with respect to ψ, the ψ-analogue of the Möbius function μ. We shall write $L(f)$ instead of Lf.

We begin with

Lemma 4.1. *Suppose h is an arithmetic function. If $p_1^{\alpha_1}, p_2^{\alpha_2} ..., p_r^{\alpha_r}$ are any r distinct ψ primitive elements, then*

$$L(d_\psi)\,(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = \tag{4.1}$$

$$= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) h\left(\prod_{j=1}^{r} p_j^{x_j}\right) =$$

$$= 2^r\left\{\sum_{k=0}^{r-1}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \dots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) + (-1)^r h(1)\right\}.$$

Proof. We shall prove (4.1) by induction on r. If $r = 1$,

$$\begin{aligned}
&\sum_{\theta(x_1,y_1)=\alpha_1} d_\psi(p_1^{x_1}) d_\psi^{-1}(p_1^{y_1}) h(p_1^{x_1}) = \\
&= d_\psi(1) d_\psi^{-1}(p_1^{\alpha_1}) h(1) + d_\psi(p_1^{\alpha_1}) d_\psi^{-1}(1) h(p_1^{\alpha_1}) = \\
&= 2\left\{h(p_1^{\alpha_1}) - h(1)\right\}
\end{aligned} \tag{4.1}$$

since $p_1^{\alpha_1}$ is ψ-primitive. Thus the identity in (4.1) holds good when $r = 1$. We assume (4.1) for some positive integer r. Suppose that $p_1^{\alpha_1}, p_2^{\alpha_2} ..., p_r^{\alpha_r}, p_{r+1}^{\alpha_{r+1}}$ are any $r+1$ distinct ψ primitive elements. If Σ denotes the sum on the left hand side of (4.1) for $r+1$, we obtain

$$\begin{aligned}
\Sigma = &\sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) \times \\
&\times \sum_{\theta(x_{r+1},y_{r+1})=\alpha_{r+1}} d_\psi(p_{r+1}^{x_{r+1}}) d_\psi^{-1}(p_{r+1}^{y_{r+1}}) h\left(\prod_{j=1}^{r+1} p_j^{x_j}\right) = \\
= &\sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) \times \\
&\times \left\{-2h\left(\prod_{j=1}^{r} p_j^{x_j}\right) + 2h\left(p_{r+1}^{\alpha_{r+1}} \prod_{j=1}^{r} p_j^{x_j}\right)\right\} = \\
= &2\left\{\Sigma_1 - \Sigma_2\right\},
\end{aligned} \tag{4.2}$$

where

$$\Sigma_1 = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) h\left(p_{r+1}^{\alpha_{r+1}} \prod_{j=1}^{r} p_j^{x_j}\right) \tag{4.3}$$

and

$$\Sigma_2 = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) h\left(\prod_{j=1}^{r} p_j^{x_j}\right). \tag{4.4}$$

For non-negative integers $x_1, x_2, \ldots, x_r$ if we define

$$g(p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r}) = h(p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r} p_{r+1}^{\alpha_{r+1}}),$$

noting that $g(1) = h(p_{r+1}^{\alpha_{r+1}})$, we obtain from (4.1),

$$\Sigma_1 = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} d_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) d_\psi^{-1}\left(\prod_{j=1}^{r} p_j^{y_j}\right) g\left(p_1^{x_1} p_2^{x_2} \ldots p_r^{x_r}\right) = \tag{4.5}$$

$$2^r \left\{ \sum_{k=0}^{r-1} (-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}} p_{r+1}^{\alpha_{r+1}}\right) + (-1)^r h\left(p_{r+1}^{\alpha_{r+1}}\right) \right\}.$$

Substituting (4.5) and (4.1) into (4.2), we obtain

$$\Sigma = 2^{r+1}\left(\Sigma_4 + (-1)^{r+1} h(1)\right), \tag{4.6}$$

where

$$\begin{aligned} \Sigma_4 = {} & \sum_{k=0}^{r-1} (-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}} p_{r+1}^{\alpha_{r+1}}\right) + \\ & + (-1)^r h\left(p_{r+1}^{\alpha_{r+1}}\right) - \sum_{k=0}^{r-1} \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right). \end{aligned} \tag{4.7}$$

In view of (4.6), (4.7) and considering the right hand side of (4.1) when r is replaced by $r+1$, it remains to prove that

$$\Sigma_4 = \sum_{k=0}^{r} (-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r+1-k}\le r+1} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right). \tag{4.8}$$

Let Σ_5 denote the sum on the right hand side of (4.8). Splitting the inner sum in Σ_5 according to $i_{r+1-k} = r+1$ or $i_{r+1-k} \le r$ we obtain

$$\begin{aligned}\Sigma_5 =& \sum_{k=0}^{r}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}} p_{r+1}^{\alpha_{r+1}}\right) + \\ &+ \sum_{k=0}^{r}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r+1-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right) = \\ =&\Sigma_6 + \Sigma_7,\end{aligned} \tag{4.9}$$

say. Consider the sum Σ_6. In this sum the term corresponding to $k = r$ is $(-1)^r h\left(p_{r+1}^{\alpha_{r+1}}\right)$. Hence

$$\Sigma_6 = \sum_{k=0}^{r-1}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}} p_{i_{r+1}}^{\alpha_{i_{r+1}}}\right) + (-1)^r h\left(p_{r+1}^{\alpha_{r+1}}\right). \tag{4.10}$$

In the sum Σ_7, the inner sum is empty when $k = 0$. Hence by using the substitution $k \leftarrow k-1$, we see that

$$\Sigma_7 = -\sum_{k=0}^{r-1}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right). \tag{4.11}$$

Putting (4.10) and (4.11) into (4.9), we obtain (4.8). The induction is complete. Hence Lemma 4.1 follows.

Lemma 4.2. (under the hypothesis of Lemma 4.1) *We have*

$$\begin{aligned}&L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right) = \\ &= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right) \mu_\psi\left(\prod_{j=1}^{r} p_j^{x_j}\right) = \\ &= \sum_{k=0}^{r-1}(-1)^k \sum_{1\le i_1<i_2<\ldots<i_{r-k}\le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) + (-1)^r h(1).\end{aligned} \tag{4.12}$$

Proof. We shall prove Lemma 4.2 by induction on r. The identity in (4.12) is true when $r = 1$. We assume (4.12) for some positive integer r. We consider the left hand side of (4.12) when r is replaced by $r+1$. Suppose that

$p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}, p_{r+1}^{\alpha_{r+1}}$ are any $r+1$ distinct ψ-primitive elements. Since μ_ψ is multiplicative, we have

$$L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}}\right) = \tag{4.13}$$

$$= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r+1}} h\left(\prod_{j=1}^{r+1} p_j^{x_j}\right) \mu_\psi\left(\prod_{j=1}^{r+1} p_j^{y_j}\right) =$$

$$= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} \mu_\psi\left(\prod_{j=1}^{r} p_j^{y_j}\right) \sum_{\theta(x_{r+1},y_{r+1})=\alpha_{r+1}} h\left(\prod_{j=1}^{r+1} p_j^{x_j}\right) \mu_\psi\left(p_{r+1}^{y_{r+1}}\right) =$$

$$= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} \mu_\psi\left(\prod_{j=1}^{r} p_j^{y_j}\right) \left\{-h\left(\prod_{j=1}^{r} p_j^{x_j}\right) + h\left(p_{r+1}^{\alpha_{r+1}} \prod_{j=1}^{r} p_j^{x_j}\right)\right\} =$$

$$= -\Sigma_8 + \Sigma_9,$$

where

$$\Sigma_8 = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right) \mu_\psi\left(\prod_{j=1}^{r} p_j^{y_j}\right), \tag{4.14}$$

and

$$\Sigma_9 = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(p_{r+1}^{\alpha_{r+1}} \prod_{j=1}^{r} p_j^{x_j}\right) \mu_\psi\left(\prod_{j=1}^{r} p_j^{y_j}\right). \tag{4.15}$$

We can directly apply our induction hypothesis to the sum Σ_8.

As in the proof of Lemma 4.1 let $g\left(\prod_{j=1}^{r} p_j^{x_j}\right) = h\left(p_{r+1}^{\alpha_{r+1}} \prod_{j=1}^{r} p_j^{x_j}\right)$. We replace h by g in the sum Σ_9 and apply (4.12). Substituting these results in (4.13), we obtain

$$L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1}}\right) = \Sigma_4 + (-1)^{r+1} h(1), \tag{4.16}$$

where Σ_4 is given in (4.7). By (4.8), the left hand side of (4.16) is precisely the left hand side of (4.12) when r is replaced by $r+1$. This completes the induction and the proof of Lemma 4.2.

Definition 4.1. Let h be an arithmetic function with $h(1) = 0$. Let t be a fixed positive integer. We say that h is *additive of order* t, if $h\left(\prod_{i=1}^{t} p_i^{x_i}\right) = \sum_{i=1}^{t} h\left(p_i^{x_i}\right)$, for all distinct primes $p_1, p_2, \ldots, p_t$ and non-negative integers $x_1, x_2, \ldots, x_t$ such that $p_j^{x_j}$ is ψ-primitive if $x_j > 0$.

Lemma 4.3. *Let $r \geq 2$ and let h be an additive arithmetic function of order $r-1$. If $p_1^{\alpha_1}, p_2^{\alpha_2} ..., p_r^{\alpha_r}$ are any r distinct ψ primitive elements, then*

(4.17)
$$L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right) = \sum_{k=0}^{r-1} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-k} \leq r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) =$$
$$= h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right) - \sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right).$$

Proof. The first equality in (4.17) is (4.12) since $h(1) = 0$. For each integer $t \geq 2$, let $P(t)$ denote the proposition that (4.17) holds (when r is replaced by t) for any additive arithmetic function h of order $t-1$. Clearly $P(2)$ is true. We assume $P(t)$ for $2 \leq t \leq r$. Let h be an additive function of order r. We have

(4.18)
$$\Sigma = \sum_{k=0}^{r} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_{r+1-k} \leq r+1} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right) =$$
$$= \sum_{k=0}^{r} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-k} \leq r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}} p_{r+1}^{\alpha_{r+1}}\right) +$$
$$+ \sum_{k=0}^{r} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_{r+1-k} \leq r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right) =$$
$$= \Sigma_{10} + \Sigma_{11},$$

say. The inner sum of Σ_{11} is empty for $k = 0$. Also, the term corresponding to $k = 1$ in Σ_{11} is $-h\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right)$. Hence

(4.19)
$$\Sigma_{11} =$$

$$= -h\left(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}\right) + \sum_{k=2}^{r} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r+1-k} \le r} h\left(p_{i_1}^{\alpha_{i_1}} \dots p_{i_{r+1-k}}^{\alpha_{i_{r+1-k}}}\right) =$$

$$= -h\left(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}\right) - \sum_{k=1}^{r-1} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r-k} \le r} h\left(p_{i_1}^{\alpha_{i_1}} \dots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) =$$

$$= -\sum_{k=0}^{r-1} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r-k} \le r} h\left(p_{i_1}^{\alpha_{i_1}} \dots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) =$$

$$= -\left\{ h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right) - \sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right) \right\} = 0,$$

by our induction hypothesis.

In Σ_{10}, the term corresponding to $k = 0$ is $h\left(\prod_{j=1}^{r+1} p_j^{\alpha_j}\right)$. Also, since h is additive function of order r, we obtain

(4.20)

$$\Sigma_{10} = h\left(\prod_{j=1}^{r+1} p_j^{\alpha_j}\right) +$$

$$+ \sum_{k=1}^{r} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r-k} \le r} \left\{ h\left(p_{i_1}^{\alpha_{i_1}} \dots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) + h\left(p_{r+1}^{\alpha_{r+1}}\right) \right\} =$$

$$= h\left(\prod_{j=1}^{r+1} p_j^{\alpha_j}\right) + \sum_{k=1}^{r} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r-k} \le r} h\left(p_{i_1}^{\alpha_{i_1}} \dots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) +$$

$$+ h\left(p_{r+1}^{\alpha_{r+1}}\right) \sum_{k=1}^{r} (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_{r-k} \le r} 1 =$$

$$= h\left(\prod_{j=1}^{r+1} p_j^{\alpha_j}\right) + \Sigma_{12} + h\left(p_{r+1}^{\alpha_{r+1}}\right) \Sigma_{13},$$

say. The inner sum in Σ_{12} is empty when $k = r$. Further, the term corresponding to $k = 0$ in Σ_{12} is $h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right)$. Hence by (4.17),

$$\Sigma_{12} = -\sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right). \tag{4.21}$$

From (4.20) we have
(4.22)

$$\Sigma_{13} = \sum_{k=1}^{r} (-1)^k \sum_{1 \le i_1 < i_2 < \ldots < i_{r-k} \le r} 1 = \sum_{k=1}^{r} (-1)^k \binom{r}{r-k} = -1 + (1-1)^r = -1.$$

Lemma 4.3 now follows from (4.22), (4.21), (4.20), (4.19) and (4.18).

Lemma 4.4. *Assume that the logarithmic operator* $L : (P', \psi) \to (\mathbb{A}_1, +)$ *defined in (1.13)-(1.14) is a homomorphism and let* $h \in \mathbb{A}$ *and* $h(1) = 0$. *If* $p_1^{\alpha_1}, p_2^{\alpha_2} ..., p_r^{\alpha_r}$ *are any* r *distinct* $\psi-$*primitive elements, then*

$$h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right) = \sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right). \tag{4.23}$$

Proof. Since L is a homomorphism we have

$$L(d_\psi) = L(1\,\psi\,1) = L(1) + L(1) = 2L(1). \tag{4.24}$$

We can assume that $r \geq 2$. Since $h(1) = 0$, by Lemmas 4.1, 4.2 and (4.24), we obtain

$$\sum_{k=0}^{r-1} (-1)^k \sum_{1 \le i_1 < i_2 < \ldots < i_{r-k} \le r} h\left(p_{i_1}^{\alpha_{i_1}} \ldots p_{i_{r-k}}^{\alpha_{i_{r-k}}}\right) = 0. \tag{4.25}$$

We now show that h is additive of order t for $1 \leq t \leq r$ and this proves (4.23). Clearly h is additive of order 1. Suppose that h is additive of order $r-1$. It follows from Lemma 4.3 and (4.25) that (4.23) holds. This completes the induction and the proof of Lemma 4.4.

Theorem 4.1. *Let* $h \in \mathbb{A}$ *and* $h(1) = 0$. *Then we have the following :*

(a) If $L : (P', \psi) \to (\mathbb{A}_1, +)$ *is a homomorphism, then* h *is* ψ*-additive.*

(b) If L *is an injection, then* $h(n) \neq 0$ *for all* $n > 1$.

Proof. The proof of (b) is not difficult. Suppose that $h(k) = 0$ for some $k > 1$. We define $f \in P'$ by

$$f(n) = \begin{cases} 0, & \text{if } n \neq k, \\ 1, & \text{if } n = 1 \text{ or } k. \end{cases}$$

Since $h(1) = 0$, it follows that $f(x)h(x) = 0$ for all $x \in \mathbb{Z}^+$. Hence

$$(Lf)(1) = \log f(1) = 0,$$

and

$$(Lf)(n) = \sum_{\psi(x,y)=n} f(x)h(x)f^{-1}(y) = 0,$$

for all $n > 1$. Thus $Lf \equiv 0$. Since $Le \equiv 0$, where e is as given in (1.5) and L is an injection, we must have $f = e$. But $f \neq e$. This contradiction proves that $h(n) \neq 0$ for all $n > 1$. Hence (b) follows.

Proof of (a). We assume that L is a homomorphism. First we prove that h is additive. For each non-negative integer m, let $P(m)$ denote the proposition that

$$h\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = \sum_{i=1}^{r} h\left(p_i^{\alpha_i}\right), \tag{4.26}$$

whenever $p_1, p_2, \ldots, p_r$ are r distinct primes where $r \geq 2$ and $\alpha_1, \alpha_2, \ldots \alpha_r$ are non-negative integers with $\alpha_1 + \alpha_2 + \ldots + \alpha_r = m$.

Clearly $P(0)$ is true. We assume $P(t)$ for $0 \leq t < m$. We prove $P(m)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be non-negative integers such that $\alpha_1 + \alpha_2 + \ldots + \alpha_r = m$. First we prove that

$$\Sigma = \sum_{\substack{x_1+x_2+\ldots+x_r<\alpha_1+\alpha_2+\ldots+\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r \\ x_r<\alpha_r}} \left(\prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right)\right) = 0. \tag{4.27}$$

Indeed,

$$\Sigma = \sum_{\substack{x_r<\alpha_r \\ \theta(x_r,y_r)=\alpha_r}} \mu_\psi\left(p_r^{y_r}\right)\Sigma_1\Sigma_2\ldots\Sigma_{r-1}, \tag{4.28}$$

where for $j = 1, 2, \ldots, r-1$,

$$\Sigma_j = \sum_{\substack{x_j<m_j \\ \theta(x_j,y_j)=\alpha_j}} \mu_\psi\left(p_j^{y_j}\right) \tag{4.29}$$

and

$$m_j = \sum_{k=1}^{j-1} (\alpha_k - x_k) + \sum_{k=j}^{r-1} \alpha_k + (\alpha_r - x_r). \tag{4.30}$$

We show that $\Sigma_{r-1} = 0$, from which (4.27) follows by (4.28). By (4.29) ($j = r-1$) we have

$$\Sigma_{r-1} = \sum_{\substack{x_{r-1} < m_{r-1} \\ \theta(x_{r-1}, y_{r-1}) = \alpha_{r-1}}} \mu_\psi \left(p_{r-1}^{y_{r-1}}\right), \tag{4.31}$$

where

$$m_{r-1} = \sum_{k=1}^{r-2} (\alpha_k - x_k) + \alpha_{r-1} + (\alpha_r - x_r).$$

The conditions under the sum on the right hand side of (4.27) imply that $x_k \le \alpha_k$ for $k = 1, 2, \ldots, r$ and $x_r < \alpha_r$. Hence $m_{r-1} > 0$. Let

$$S_{p_{r-1}, \alpha_{r-1}} = S_{\alpha_{r-1}} = \{0 < a_1 < a_2 < \ldots a_t = \alpha_{r-1}\}. \tag{4.32}$$

In the sum Σ_{r-1} given in (4.31) the possible choices of y_{r-1} are $y_{r-1} = 0$ and $y_{r-1} = a_1$ (for the other choices of y_{r-1}, $\mu_\psi \left(p_{r-1}^{y_{r-1}}\right) = 0$). For these choices of y_{r-1} the corresponding choices of x_{r-1} are $x_{r-1} = \alpha_{r-1} < m_{r-1}$ and $x_{r-1} = a_{t-1} \le \alpha_{r-1} < m_{r-1}$. Hence in the sum Σ_{r-1} both the choices, namely, $y_{r-1} = 0$ and $y_{r-1} = a_1$ are admissible. Then

$$\Sigma_{r-1} = 1 + \mu_\psi \left(p_{r-1}^{a_1}\right) = 1 - 1 = 0,$$

since $p_{r-1}^{a_1}$ is ψ-primitive. Thus (4.27) follows.

We now prove that

$$\Sigma'_r = \sum_{\substack{x_1 + x_2 + \ldots + x_r < \alpha_1 + \alpha_2 + \ldots + \alpha_r \\ \theta(x_i, y_i) = \alpha_i \\ 1 \le i \le r \\ x_r = \alpha_r}} \left(\prod_{j=1}^{r} \mu_\psi \left(p_j^{y_j}\right) \right) = -1. \tag{4.33}$$

Let $r = 2$. We have

$$\Sigma'_2 = \sum_{\substack{x_1 + x_2 < \alpha_1 + \alpha_2 \\ \theta(x_1, y_1) = \alpha_1 \\ \theta(x_2, y_2) = \alpha_2 \\ x_2 = \alpha_2}} \mu_\psi \left(p_1^{y_1}\right) \mu_\psi \left(p_2^{y_2}\right). \tag{4.34}$$

Let S_{α_1} be as given in (4.32) ($r = 2$). In (4.34), the conditions $x_2 = \alpha_2$ and $\theta(x_2, y_2) = \alpha_2$ imply that $y_2 = 0$. Hence

$$\Sigma'_2 = \sum_{\substack{x_1 < \alpha_1 \\ \theta(x_1, y_1) = \alpha_1}} \mu_\psi \left(p_1^{y_1}\right). \tag{4.35}$$

In the sum in (4.35), the possible choices of y_1 for which $\mu_\psi\left(p_1^{y_1}\right)) \neq 0$ are $y_1 = 0$ and a_1. The choice $y_1 = 0$ is forbidden since this implies $x_1 = \alpha_1$. The choice $y_1 = a_1$ implies that $x_1 = a_{t-1} < a_t = \alpha_1$. Hence $y_1 = a_1$ is admissible. Then from (4.35), we obtain

$$\Sigma_2' = \mu_\psi\left(p_1^{a_1}\right) = -1.$$

Thus (4.33) is true when $r = 2$. We assume (4.33). We have

$$\Sigma_{r+1}' = \sum_{\substack{x_1+x_2+\ldots+x_r+x_{r+1}<\alpha_1+\alpha_2+\ldots+\alpha_r+\alpha_{r+1} \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r+1 \\ x_{r+1}=\alpha_{r+1}}} \left(\prod_{j=1}^{r+1} \mu_\psi\left(p_j^{y_j}\right)\right) =$$

$$=\mu_\psi\left(p_{r+1}^0\right) \sum_{\substack{x_1+x_2+\ldots+x_r<\alpha_1+\alpha_2+\ldots+\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r}} \left(\prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right)\right) = \tag{4.36}$$

$$= \sum_{\substack{x_1+x_2+\ldots+x_r<\alpha_1+\alpha_2+\ldots+\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r \\ x_r=\alpha_r}} \left(\prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right)\right) +$$

$$+ \sum_{\substack{x_1+x_2+\ldots+x_r<\alpha_1+\alpha_2+\ldots+\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r \\ x_r<\alpha_r}} \left(\prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right)\right) = \tag{4.37}$$

$$= -1 + 0 = -1, \tag{4.38}$$

by our induction hypothesis and (4.27).

The passage from (4.36)-(4.38) also proves that

$$\sum_{\substack{x_1+x_2+\ldots+x_r<\alpha_1+\alpha_2+\ldots+\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r}} \left(\prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right)\right) = -1. \tag{4.39}$$

We shall now evaluate $L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right)$. We have

$$\begin{aligned} L(1)\left(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}\right) &= \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\leq i\leq r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right) \prod_{j=1}^{r} \mu_\psi\left(p_j^{y_j}\right) = \\ &= \Sigma_1' + \Sigma_2', \end{aligned} \tag{4.40}$$

where

$$\Sigma_1' = \sum_{\substack{x_1+x_2+\ldots x_r=\alpha_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right)\prod_{j=1}^{r}\mu_\psi\left(p_j^{y_j}\right) = \tag{4.41}$$

$$= h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right),$$

since the conditions in the sum Σ_1' imply $x_j = \alpha_j$ and consequently $y_j = 0$ for $j = 1, 2, 3, \ldots, r$; also,

$$\Sigma_2' = \sum_{\substack{x_1+x_2+\ldots x_r<\alpha_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right)\prod_{j=1}^{r}\mu_\psi\left(p_j^{y_j}\right). \tag{4.42}$$

The conditions under the sum Σ_2' are favourable to apply the induction hypothesis (4.26). By doing so we obtain,

$$\Sigma_2' = \sum_{j=1}^{r}\sum_{\substack{x_1+x_2+\ldots x_r<\alpha_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(p_j^{x_j}\right)\prod_{j=1}^{r}\mu_\psi\left(p_j^{y_j}\right) = \tag{4.43}$$

$$= \sum_{j=1}^{r}\Sigma_j'',$$

say. We now evaluate Σ_1''. The same procedure is applicable for the general sum Σ_j''. We have

$$\Sigma_1'' = \sum_{\substack{x_1+x_2+\ldots x_r<\alpha_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r \\ x_1=\alpha_1}} h\left(p_1^{x_1}\right)\prod_{k=1}^{r}\mu_\psi\left(p_k^{y_k}\right) + \tag{4.44}$$

$$+ \sum_{\substack{x_1+x_2+\ldots x_r<\alpha_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r \\ x_1<\alpha_1}} h\left(p_1^{x_1}\right)\prod_{k=1}^{r}\mu_\psi\left(p_k^{y_k}\right) =$$

$$=\Sigma_1 + \Sigma_2,$$

say. We have

$$\Sigma_1 = h\left(p_1^{\alpha_1}\right)\mu_\psi\left(p_1^0\right) \sum_{\substack{x_2+\ldots x_r<\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} \prod_{k=2}^{r} \mu_\psi\left(p_k^{y_k}\right) = = -h\left(p_1^{\alpha_1}\right), \tag{4.45}$$

by (4.39). From (4.44), we have

$$\Sigma_2 = \sum_{\substack{x_1<\alpha_1 \\ \theta(x_1,y_1)=\alpha_1}} h\left(p_1^{x_1}\right)\mu_\psi\left(p_1^{y_1}\right) \sum_{\substack{x_2+\ldots x_r<\alpha_1-x_1+\alpha_2+\ldots\alpha_r \\ \theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} \prod_{k=2}^{r}\mu_\psi\left(p_k^{y_k}\right) = 0, \tag{4.46}$$

since the inner sum vanishes as in the proof of (4.27).

It follows from (4.46), (4.45) and (4.44) that $\Sigma_1'' = -h\left(p_1^{\alpha_1}\right)$. In a similar way, we can show that $\Sigma_j'' = -h\left(p_j^{\alpha_j}\right)$, for $j = 2, 3, \ldots, r$. Hence from (4.43) it follows that

$$\Sigma_2' = -\sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right). \tag{4.47}$$

Substituting the results in (4.47) and (4.41) into (4.40), we obtain

$$L(1)\left(p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}\right) = h\left(\prod_{j=1}^{r} p_j^{\alpha_j}\right) - \sum_{j=1}^{r} h\left(p_j^{\alpha_j}\right). \tag{4.48}$$

We now prove that for $r \ge 2$,

$$L(\mu_\psi)\left(p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}\right) = 0. \tag{4.49}$$

For $i = 1, 2, \ldots, r$, let $\beta_i = \tau_\psi(p_i^{\alpha_i})$. Since $p_i^{\beta_i}$ is ψ-primitive for $i = 1, 2 \ldots, r$, by Lemma 4.4 and $h(1) = 0$, we have

$$h\left(\prod_{i=1}^{r} p_i^{x_i}\right) = \sum_{i=1}^{r} h\left(p_i^{x_i}\right), \tag{4.50}$$

if each $x_i = 0$ or β_i. We have

$$L(\mu_\psi)\left(p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}\right) = \sum_{\substack{\theta(x_i,y_i)=\alpha_i \\ 1\le i\le r}} h\left(\prod_{j=1}^{r} p_j^{x_j}\right)\prod_{j=1}^{r}\mu_\psi\left(p_j^{x_j}\right). \tag{4.51}$$

Since $\mu_\psi\left(p^\alpha\right)=0$ if p^α is not ψ-primitive, in (4.51) we can assume that each $x_i=0$ or β_i for $i=1,2,\ldots r$. Hence from (4.50), we obtain

$$L(\mu_\psi)\left(p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}\right)=$$

$$=\sum_{j=1}^{r}\sum_{\theta(x_j,y_j)=\alpha_j} h\left(p_j^{x_j}\right)\mu_\psi\left(p_j^{x_j}\right)\sum_{\substack{\theta(x_i,y_i)=\alpha_i\\1\le i\le r\\ i\ne j}}\mu_\psi\left(\prod_{\substack{k=1\\k\ne j}}^{r}p_k^{x_k}\right)=$$

$$=\sum_{j=1}^{r}\sum_{\theta(x_j,y_j)=\alpha_j} h\left(p_j^{x_j}\right)\mu_\psi\left(p_j^{x_j}\right)\prod_{\substack{i=1\\i\ne j}}^{r}\left(\sum_{\theta(x_i,y_i)=\alpha_i}\mu_\psi\left(p_i^{x_i}\right)\right)=$$

$$=0,$$

since for $\alpha>0$,

$$\sum_{\theta(a,b)=\alpha}\mu_\psi\left(p^a\right)=0.$$

$\mu_\psi=1^{-1}$ and L is a homomorphism, therefore

$$L(\mu_\psi)=L(1^{-1})=-L(1). \tag{4.52}$$

Now (4.26) follows from (4.52), (4.48) and (4.49). The induction is complete. Hence h is additive.

We now prove that h is ψ-additive. Fix a prime p and a positive integer α. Let

$$S_{p,\alpha}=\{0<a_1<a_2<\ldots<a_r=\alpha\}.$$

Following the discussion in Example 3.3, to prove that h is ψ-additive, it is enough to show that

$$h\left(p^\alpha\right)=rh\left(p^{a_1}\right). \tag{4.53}$$

In fact we prove that

$$h\left(p^{a_k}\right)=kh\left(p^{a_1}\right) \tag{4.54}$$

for $1\le k\le r$. From this (4.53) follows by taking $k=r$.

Clearly (4.54) is true when $k=1$. We assume (4.54) for $1 \le k < t$ where $t \le r$. We have

$$\begin{aligned} L(\mu_\psi)\left(p^{a_t}\right) &= \sum_{\psi(x,y)=p^{a_t}} \mu_\psi(x)h(x) = \\ &= \sum_{\theta(u,v)=a_t} \mu_\psi\left(p^u\right) h\left(p^u\right) = \\ &= \sum_{u \in S_{p,a_t}} \mu_\psi\left(p^u\right) h\left(p^u\right) = \\ &= -h\left(p^{a_1}\right). \end{aligned} \tag{4.55}$$

On the other hand,

$$\begin{aligned} L(1)\left(p^{a_t}\right) &= \sum_{\psi(x,y)=p^{a_t}} h(x)\mu_\psi(y) = \\ &= \sum_{\theta(u,v)=a_t} h\left(p^u\right) \mu_\psi\left(p^v\right) = \\ &= h\left(p^{a_t}\right) - h\left(p^{a_{t-1}}\right) = \\ &= h\left(p^{a_t}\right) - (t-1)h\left(p^{a_1}\right). \end{aligned} \tag{4.56}$$

Evaluating both sides of (4.52) at p^{a_t}, and making use of (4.54), (4.55), we obtain that $h\left(p^{a_t}\right) = th\left(p^{a_1}\right)$. This completes the proof of Theorem 4.1.

In connection with Theorem 4.1 we note that ψ that the condition that ψ is an L-N-convolution is only a sufficient condition but not a necessary one.

Indeed, let

$$F_1 = \{f \in \mathbb{A} : f(1) = 1\}, \quad F_0 = \{f \in \mathbb{A} : f(1) = 0\}$$

and let β be a basic sequence (see Example 3.4). Let $T = \beta$ and $\psi(x,y) = xy$ on T. Let $h \in \mathbb{A}$ with $h(1) = 0$ and L be defined as in (1.13) and (1.14). If $L : (P', \psi) \to (\mathbb{A}_1, +)$ is a homomorphism then L is also a homomorphism from (F_1, ψ) to $(F_0, +)$; now, a result of K.P.R. Sastry and P. Suvarna Kumari (Characterization of certain homomorphisms on groups of arithmetic functions, *Bull. Calcutta Math. Soc.*, **90** (5) (1998), 319-324) extended to complex-valued functions, implies that h is $\psi - additive$. If $L : (P', \psi) \to (\mathbb{A}_1, +)$ is an injection, by using the same proof as in (a) of Theorem 4.1, it follows that $h(n) \neq 0$ for $n > 1$. Thus Theorem 4.1 is valid when ψ is a basic convolution. If $\beta = \{(1,n), (n,1) : n \in \mathbb{Z}\}$, then the corresponding basic convolution ψ ($T = \beta$

and $\psi(x,y) = xy$ on T) is not a multiplicativity preserving convolution and hence is not an L-N-convolution. Also, for each prime p, if

$$\pi_p : \{0,2,3\}\,;\{0,4,5\}\,;\{0,6\}\,;\{0,7\}\,;\{0,8\};....$$

then the corresponding ψ convolution (see Theorem 2.2) is an L-N convolution but not a basic convolution since $\psi(x,y) \neq xy$ for at least one pair $(x,y) \in T$. For instance $\psi(p^2,p^2) = p^3$, for each prime p.

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