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THE THREE–SERIES THEOREM IN ADDITIVE ARITHMETICAL SEMIGROUPS

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

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Abstract. In this paper, we embed the additive arithmetical semigroup in a probability space $\Omega := (\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ where βG denote the Stone-Čech compactification of G. We show that every additive function g on G, $g(a) = \sum\limits_{p^k \| a} g(p^k)$ $(a \in G)$, can be identified with a sum $\bar{g} = \sum\limits_{p} \bar{X}_p$ of independent random variables on Ω. The main result will be that the existence of the limit distribution of a real-valued additive function g is equivalent to the a.e. convergence of $\bar{g}$.

1. Introduction

Let (G, ∂) be an additive arithmetical semigroup. By definition, G is a free commutative semigroup with identity element 1, generated by a countable set P of primes and admitting an integer valued degree mapping $\partial : G \to \mathbb{N} \cup \{0\}$ with the properties

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(i) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in P$,
(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,
(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \geq 0$.

Obviously, $G(0) = 1$ and G is countable. Let

$$\pi(n) := \#\{p \in P : \partial(p) = n\}$$

denote the total number of primes of degree n in G. We obtain the identity, at least in the formal sense,

$$Z(y) := 1 + \sum_{n=1}^{\infty} G(n) y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-\pi(n)}.$$

In a monograph [9], Knopfmacher, motivated by earlier work of Fogels [3] on polynomial rings and algebraic function fields, developed the concept of an additive arithmetical semigroup satisfying the following axiom.

Axiom $A^{\#}$. *There exist constants* $A > 0$, $q > 1$ and ν with $0 \leq \nu < 1$ (all depending on G), such that

$$G(n) = Aq^n + O(q^{\nu n}), \quad \text{as } n \to \infty.$$

If G satisfies Axiom $A^{\#}$, then the generating function

$$Z(y) = \sum_{n=0}^{\infty} G(n) y^n, \tag{1.1}$$

is holomorphic in the disc $|y| < q^{-\nu}$ up to a pole of order one at $y = q^{-1}$, and we get

$$Z(y) = \frac{A}{1 - qy} + H_1(y),$$

where

$$H_1(y) = \sum_{n=0}^{\infty} r_n y^n$$

with

$$r_n := G(n) - Aq^n.$$

Putting

$$H(y) := A + (1 - qy) H_1(y)$$

gives

$$Z(y) = \frac{H(y)}{1 - qy}$$

with $H(0) = 1$ and $H(q^{-1}) = A$. H and H_1 are holomorphic for $|y| < q^{-\nu}$.

Z can be considered as the zeta-function associated with the semigroup (G, ∂), and it has an Euler-product representation (cf. [9], Chapter 2):

$$Z(y) = \prod_{n=1}^{\infty} (1 - y^n)^{-\pi(n)}, \qquad |y| < q^{-1}.$$

Obviously

$$\log \prod_{m=1}^{\infty} (1 - y^m)^{-\pi(m)} =$$

$$= \sum_{m=1}^{\infty} \pi(m) \sum_{j=1}^{\infty} j^{-1} z^{jm} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{d|m} d\pi(d) y^m = \sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m,$$

where

$$\lambda(m) = \sum_{d|m} d\pi(d)$$

denotes the *von Mangoldt* coefficients. Then, because of the Möbius inversion formula

$$\pi(n) = \frac{1}{n} \sum_{d|n} \lambda(d) \mu\left(\frac{n}{d}\right).$$

Defining the *von Mangoldt* function $\Lambda : G \to \mathbb{R}$ by

$$\Lambda(a) = \begin{cases} \partial(p), & \text{if } a \text{ is a prime power } p^r \neq 1, \\ 0, & \text{otherwise,} \end{cases}$$

we see

$$\lambda(n) = \sum_{\substack{a \in G \\ \partial(a) = n}} \Lambda(a)$$

and

$$\partial(a) = \sum_{\substack{b \in G \\ b|a}} \Lambda(b) \qquad (a \in G).$$

Further

$$\Lambda^{\#}(y) := \sum_{n=1}^{\infty} \lambda(n) y^n = y \frac{Z'(y)}{Z(y)}.$$

Chapter 8 of [9] deals with a theorem called the *abstract prime number theorem*:

If the additive arithmetical semigroup G satisfies Axiom $A^{\#}$, then

$$\pi(n) = \frac{q^n}{n} + O\left(\frac{q^n}{n^\alpha}\right), \quad n \to \infty,$$

or equivalently,

$$\lambda(n) = q^n + O\left(\frac{q^n}{n^{\alpha-1}}\right), \quad n \to \infty,$$

is true for any $\alpha > 1$.

But this result is only valid if $Z(-q^{-1}) \neq 0$.

In [8], Indlekofer, Manstavičius and Warlimont gave (in a more general setting) much sharper results valid also in the case $Z(-q^{-1}) = 0$. For instance, if $Z(-q^{-1}) = 0$ and Axiom $A^{\#}$ holds, then there exists some θ, $\max\{\frac{1}{2}, v\} <$ $< \theta < 1$ such that

$$\frac{\lambda(n)}{q^n} = 1 - (-1)^n + O\left(q^{(\theta-1)n}\right).$$

In both cases, the Chebyshev inequality

$$\lambda(n) \ll q^n \quad \text{or equivalently} \quad \pi(n) \ll \frac{q^n}{n}$$

holds.

Let us move to the investigation of the mean-value properties of complex valued multiplicative functions $\tilde{f}$ satisfying $|\tilde{f}(a)| \leq 1$ for all $a \in G$. Indlekofer and Manstavičius [7] proved analogues of the results of Delange, Wirsing and Halász (in the case of multiplicative function on the natural numbers $\mathbb{N}$) for arithmetical semigroups satisfying Axiom A$^{\#}$.

Here, as in the classical case, an arithmetical function $\tilde{f}$ on G is called *additive* if $\tilde{f}(ab) = \tilde{f}(a) + \tilde{(}b)$ for all coprime $a, b \in G$ and $\tilde{f}$ is called *completely additive* if $\tilde{f}(ab) = \tilde{f}(a) + \tilde{f}(b)$ for all $a, b \in G$. An arithmetical function $\tilde{f}$ is *multiplicative* if $\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b)$ whenever $a, b \in G$ are coprime and $\tilde{f}$ is *completely multiplicative* if $\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b)$ for all $a, b \in G$. The general aim is to characterize the asymptotic behaviour of the summatory function

$$M(n, \tilde{f}) := \begin{cases} \frac{1}{G(n)} \sum\limits_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a), & \text{if } G(n) \neq 0, \\ 0, & \text{if } G(n) = 0. \end{cases}$$

We say that the function $\tilde{f}$ possesses an *(arithmetical) mean-value* $M(\tilde{f})$, if the limit

$$M(\tilde{f}) := \lim_{n \to \infty} M(n, \tilde{f})$$

exists.

The simple and seemingly appropriate choice of Axiom $A^{\#}$ as a basic assumption has been regarded as an incomplete encoding of the fundamental situation and rather loose (and appropriate) conditions have been introduced (cf. Knopfmacher-Zhang [10], Indlekofer [6], Barát-Indlekofer [1]) so as to include the results about the asymptotic behaviour of the summatory function $M(n, \tilde{f})$ $(n \to \infty)$ for multiplicative functions f of modulus ≤ 1. (see [6]). The main consequence of these weak conditions is that the circle $\{y \in \mathbb{C} : |y| = q^{-1}\}$ may be a natural boundary for $Z(y)$.

At present the weakest hypothesis can be summarized as follows. If the function Z in (1.1) can be represented in the form

$$Z(y) = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right)$$

for $|y| < q^{-1}$, then the basic conditions will be

$$0 \leq \lambda(m) = O(q^m) \quad (m \in \mathbb{N}) \tag{1.2}$$

and

$$|Z(y)| \ll Z(|y|) \left|\frac{1 - q|y|}{1 - qy}\right|^{\varepsilon} \qquad (|y| < q^{-1}) \tag{1.3}$$

for some $\varepsilon > 0$. Further, let

$$B(n) = \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} q^{-m}\right).$$

Then we assume that

$$nG(n) \asymp q^n B(n) \tag{1.4}$$

and

$$B(m) = o(B(n)) \quad \text{if} \ \ m = o(n) \ \ (n \to \infty). \tag{1.5}$$

Definition 1. We say that the function Z in (1.1) belongs to the exp-log class $\mathcal{F}$ in case (1.2), (1.3), (1.4) and (1.5) hold.

Example 1. Let $Z(y)$, defined in (1.1), have the form

$$Z(y) = \sum_{n=0}^{\infty} G(n) y^n = \frac{H(y)}{(1-qy)^\tau} \quad (|y| < q^{-1}), \tag{1.6}$$

where $\tau > 0$ and $H(y) = O(1)$ for $|y| < q^{-1}$ and

$$H(r) \asymp 1 \quad \text{for} \quad 0 < r < q^{-1}. \tag{1.7}$$

Further, we assume

$$G(n) \asymp q^n n^{\tau - 1}.$$

Observe, that if $Z(y)$ is defined by (1.6) with (1.7), respectively, and $0 \leq \leq \lambda(m) \ll q^m$ then, for $r = q^{-1} - \frac{1}{n}$,

$$B(n) = \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} q^{-m} \right) \asymp$$

$$\asymp \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} q^{-m} r^m \right) \asymp Z(r) \asymp (1-qr)^{-\tau} = n^\tau,$$

which implies (1.4) and

$$\frac{B(m)}{B(n)} \ll \left(\frac{m}{n}\right)^\tau = o(1) \quad \text{if} \ \ m = o(n)$$

as $n \to \infty$ and (1.5) is satisfied.

Example 2. Assume that

$$0 < c_1 q^m \leq \lambda(m) \leq c_2 q^m < \infty$$

holds for all $m \in \mathbb{N}$. Then, obviously

$$
\begin{aligned}
|Z(y)| &= Z(|y|)\exp\left(\sum_{m=1}^{\infty}\frac{\lambda(m)}{m}|y|^m(\cos(mt)-1)\right) \leq \\
&\leq Z(|y|)\exp\left(c_1\sum_{m=1}^{\infty}\frac{q^m|y|^m}{m}(\cos(mt)-1)\right) = \\
&= Z(|y|)\left|\frac{1-q|y|}{1-qy}\right|^{c_1}
\end{aligned}
$$

and

$$
\frac{B(m)}{B(n)} = \exp\left(-\sum_{m<l\leq n}\frac{\lambda(l)}{l}q^{-l}\right) \ll \exp\left(c_1\log\frac{m}{n}\right) = o(1)
$$

if $m = o(n)$ $(n \to \infty)$. Elementary estimates immediately yield

$$
q^n G(n) \asymp \frac{B(n)}{n},
$$

where the constants involved in $\asymp$ only depend on c_1 and c_2 (see Manstavičius [11], Lemma 3.1).

In this paper we deal with real-valued additive functions defined on arithmetical semigroups G which are described in Example 1 and Example 2, respectively. We show how G can be embedded in a probability space $(\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta})$ such that $\tilde{g}$ can be identified with a sum of independent random variables on βG. Here βG denotes the Stone-Čech compactification of G.

To be more specific, for each $n \in \mathbb{N}$ we define the distribution function

$$
D_n(y) := \frac{1}{G(n)}\#\{a \in G : \partial(a) = n, \ \ \tilde{g}(a) \leq y\},
$$

where

$$
\tilde{g} = \sum_{p \in \mathcal{P}} X_p
$$

with

$$
X_p(a) := \begin{cases} \tilde{g}(p^k), & \text{if } p^k \| a, \\ 0, & \text{otherwise.} \end{cases}
$$

(Here $b\|a$ means that $b|a$ and $a = b.c$ implies that $(b,c) = 1$.)

Extending, for each $p \in \mathcal{P}$ the function X_p uniquely to a function $\overline{X_p}$ on βG we show that the $\{\overline{X_p}\}$ are independent and

$$\bar{\tilde{g}} := \sum_{p \in \mathcal{P}} \overline{X_p}$$

converges a.e. if and only if D_n converge weakly to some limit distribution D. To ease notational difficulties we restrict ourselves to completely additive functions $\tilde{g}$.

2. Lemmata

Let us assume $f : \mathbb{N}_0 \to \mathbb{C}$ with $f(0) = 1$. Further, we assume that the generating function

$$F(y) := \sum_{n=0}^{\infty} f(n) y^n$$

can be written in the form

$$F(y) := \sum_{n=0}^{\infty} f(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m} y^m \right) \tag{2.1}$$

for $|y| < q^{-1}$, and in addition, we assume $|\lambda_f(m)| = O(1)$ for all $m \in \mathbb{N}$. With these notations we have

Proposition. (Cf. [6], Theorem 2) *Let Z be an element of the* exp-*log class $\mathcal{F}$ and let the coefficients in (2.1) satisfy*

$$\lambda_f(m) = \lambda_{f,1}(m) + \lambda_{f,2}(m), \quad m \in \mathbb{N},$$

with

$$|\lambda_{f,1}(m)| \le \lambda(m) \quad for \;\; all \;\; m \in \mathbb{N}$$

and

$$\sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} q^{-m} < \infty.$$

Put

$$F(y) = F_I(y) \cdot F_{II}(y),$$

where

$$F_I(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,1}(m)}{m} y^m\right),$$

$$F_{II}(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m\right)$$

for $|y| < q^{-1}$. *Then the following two assertions hold.*

(i) Let

$$\sum_{m=1}^{\infty} \frac{\lambda(m) - \mathrm{Re}\lambda_{f,1}(m)e^{ima}}{m} \cdot q^{-m} \tag{2.2}$$

converge for some $a \in \mathbb{R}$. *Put*

$$A_n = \exp\left(-ina + \sum_{m\le n} \frac{\lambda_{f,1}(m)e^{ima} - \lambda(m)}{m} \cdot q^{-m}\right) F_{II}(q^{-1}).$$

Then

$$f(n) = A_n G(n) + o(G(n)) \;\; as \;\; n \to \infty.$$

(ii) Let (2.2) diverge for all $a \in \mathbb{R}$. *Then*

$$f(n) = o(G(n)) \;\; as \;\; n \to \infty.$$

An application to completely multiplicative function on G is contained in Lemma 1.

Lemma 1. *Let* (G, ∂) *be an additive arithmetical semigroup such that*

$$Z(y) = \sum_{n=0}^{\infty} G(n)y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right) = \frac{H(y)}{(1-qy)^{\tau}}, \qquad \tau > 0 \tag{2.3}$$

where $H(y) = O(1)$ *for* $|y| < q^{-1}$, $H(r) \asymp 1$ *for* $0 < r < q^{-1}$. *Assume that*

$$\lambda(m) = O(q^m) \quad and \quad G(n) \asymp q^n n^{\tau-1}. \tag{2.4}$$

Suppose $|\tilde{f}(a)| \le 1$ *for all* $a \in G$ *and* $\tilde{f}$ *is a completely multiplicative function on* G.

If there exists a real number a such that

$$\sum_{p\in P} q^{-\partial(p)}\left(1-\mathrm{Re}(\tilde{f}(p)q^{-i\vartheta\partial(p)})\right) \tag{2.5}$$

converges for $\vartheta = a$, then

$$\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) =$$

$$= q^{ina}\prod_{\partial(p)\le n}(1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)(1+ia)}\right)G(n)+o(G(n)).$$

If (2.5) diverges for all $\vartheta\in\mathbb{R}$ then

$$\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) = o(G(n)).$$

Remark 1. Obviously

$$M(\tilde{f}) \text{ exists and is } \neq 0$$

if and only if

$$\sum_{p\in P} q^{-\partial(p)}(1-\tilde{f}(p)) \text{ converges.}$$

A further consequence of proposition leads to Lemma 2.

Lemma 2. *Let (G,∂) be an additive arithmetical semigroup satisfying the condition of Lemma 1. Further, let $\{p_1^{k_1},...,p_r^{k_r}\}$ be a finite set of prime powers such that $p_i\neq p_j$ if $i\neq j$. Suppose $\partial(p_i)\ge \frac{\log 2}{\log q}$ for all $p_i\in\mathcal{P}$, $i=1,...,r$ and define a multiplicative function $\tilde{f}$ by*

$$\tilde{f}(p^j) := \begin{cases} 0, & \text{if } p^j\in\{p_1^{k_1},...,p_r^{k_r}\},\\ 1, & \text{otherwise.}\end{cases}$$

Then $\tilde{f}$ possesses the mean-value $M(\tilde{f}) = \prod_{i=1}^{r}\left(1-q^{-\partial(p_i^{k_i})}(1-q^{-\partial(p_i)})\right)$.

Proof. We use Proposition. If $\tilde{f}$ is described in the above form we write

$$F(y) = \prod_p \left(1 + \sum_{j=1}^{\infty} \tilde{f}(p^j) y^{j\partial(p)}\right) =$$

$$= \prod_{i=1}^{r} \left(1 + \sum_{\substack{j=1\\ j\neq k_i}}^{\infty} y^{j\partial(p_i)}\right) \prod_{\substack{p\neq p_i\\ i=1,\dots,r}} \left(1 - y^{\partial(p)}\right)^{-1} =$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n} y^n\right).$$

Since

$$1 + \sum_{\substack{j=1\\ j\neq k_i}}^{\infty} y^{j\partial(p_i)} = \frac{1 - y^{k_i\partial(p_i)}(1 - y^{\partial(p_i)})}{1 - y^{\partial(p_i)}},$$

we get

$$F(y) = \prod_{i=1}^{r} \left(1 - y^{k_i\partial(p_i)}(1 - y^{\partial(p_i)})\right) \cdot Z(y).$$

Because of

$$|y^{\partial(p_i)k_i}\left(1 - y^{\partial(p_i)}\right)| \leq q^{-\partial(p_i)k_i}|1 + q^{-\partial(p_i)}| \leq \frac{3}{4}$$

for all $\partial(p_i) \geq \frac{\log 2}{\log q}$ the function $F(y)$ is non-zero in $|y| < q^{-1}$. So

$$F(y) = Z(y) \prod_{i=1}^{r} \left(1 + \sum_{\substack{j=1\\ j\neq k_i}}^{\infty} y^{j\partial(p_i)}\right) =$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{\tilde{f}}(m)}{m}\right),$$

where

$$\lambda_{\tilde{f}} = \lambda_{\tilde{f},1} + \lambda_{\tilde{f},2}$$

such that

$$\lambda_{\tilde{f},1} = \lambda \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{|\lambda_{\tilde{f},2}(m)|}{m} q^{-m} < \infty.$$

Applying Proposition gives Lemma 2.

In the case of completely multiplicative functions Lemma 2 reads as

Lemma 2$'$. *Let (G, ∂) be an additive arithmetical semigroup satisfying the condition of Lemma 1. Further, let $\{p_1, ..., p_r\}$ be a finite set of different primes. If we define a completely multiplicative function $\tilde{f}$ by*

$$\tilde{f}(p) := \begin{cases} 0, & \text{if } p \in \{p_1, ..., p_r\}, \\ 1, & \text{otherwise,} \end{cases}$$

then $\tilde{f}$ possesses the mean-value $M(\tilde{f}) = \prod_{i=1}^{r} \left(1 - q^{-\partial(p_i)}\right)$.

3. The Stone-Čech compactification of G

Suppose that $\mathcal{A}$ is an *algebra of subsets of* G, i.e.

(i) $G \in \mathcal{A}$,

(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$,

(iii) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

Embedding G, endowed with the discrete topology, in the compact space βG, the Stone-Čech compactification of G. This implies

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

is an algebra in βG, where $\bar{A} := clos_{\beta G} A$ (for details see K.-H. Indlekofer [4], [5]).

Let $\delta(A)$ be a content on $\mathcal{A}$ and define $\bar{\delta}$ on $\bar{\mathcal{A}}$ by

$$\bar{\delta}(\bar{A}) = \delta(A), \quad \bar{A} \in \bar{\mathcal{A}},$$

then $\bar{\delta}$ is a pseudo-measure in $\bar{\mathcal{A}}$ and measure in $\sigma(\bar{\mathcal{A}})$. We have then the measure (probability) space $(\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta})$.

Let us consider the following examples. For primes $p \in P$ and $k \in \mathbb{N}_0$ let

$$A_{p^k} := \{a \in G : p^k \mid a\}$$

be the set of all elements of G divisible by p^k. Let $\mathcal{A}$ be the algebra generated by the sets $\{A_{p^k}\}$.

We assume that (1.4) holds, i.e.

$$G(n) \asymp q^n \frac{B(n)}{n} \tag{3.1}$$

and we consider for $A \in \mathcal{A}$ the means

$$M(n, 1_A) = \frac{\sum\limits_{\substack{a \in A \\ \partial(a)=n}} 1}{\sum\limits_{\substack{a \in G \\ \partial(a)=n}} 1}$$

where the characteristic function 1_A of A is defined by

$$1_A(a) := \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{A}$.

Note that the following relation of the characteristic functions

$$\begin{aligned} 1_{A \cap B} &= 1_A \cdot 1_B, \\ 1_{A \setminus B} &= 1_A - 1_A \cdot 1_B \\ 1_{A \cup B} &= 1_A + 1_B - 1_A \cdot 1_B \end{aligned}$$

implies that the characteristic function of a set $A \in \mathcal{A}$ is a finite linear combination of products of $1_{A_{p_1^{k_1}}} \cdots 1_{A_{p_r^{k_r}}}$. Let $a_1, a_2, \ldots$ be the sequence of the elements from G, such that $\partial(a_1) \leq \partial(a_2) \leq \cdots$. If we consider $M(n, 1_{A_{a_j}})$ where $A_{a_j} := \{a \in G : a_j \mid a\}$, then we obtain by (3.1) that

$$\begin{aligned} M(n, 1_{A_{a_j}}) &\asymp \frac{q^{n-\partial(a_j)}}{q^n} \cdot \left(\frac{n}{n-\partial(a_j)}\right) \cdot \frac{B(n-\partial(a_j))}{B(n)} \asymp \\ &\asymp q^{-\partial(a_j)} \qquad \text{for } \; n \geq 2\partial(a_j). \end{aligned}$$

This means that there exists a subsequence $\{n_k\}$ such that

$$\lim_{k \to \infty} M(n_k, 1_{A_{a_j}}) =: \delta_1(A_{a_j}) \tag{3.2}$$

exists for all A_{a_j}. If we define $\delta_1(A)$ for every $A \in \mathcal{A}$ by

$$\delta_1(A) := \lim_{k \to \infty} M(n_k, 1_A),$$

then we have a content on $\mathcal{A}$ which is well-defined for all sets $A \in \mathcal{A}$. The above construction leads to the probability space $(\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta_1})$.

Now, in addition, we assume that $Z(y)$ belongs to the exp-log class $\mathcal{F}$. Let $p \in P$, $k \in \mathbb{N}_0$ and let

$$A^{'}_{p^k} := \{a \in G : p^k \mid\mid a\}$$

be the set of all elements of G divisible exactly by p^k, i.e. a can be written in the form $a = p^k.b$ where $p \not| b$. Further, let $\mathcal{A}^{'}$ be the algebra generated by the sets $\{A^{'}_{p^k}\}$. Because of $1_{A^{'}_{p^k}} = 1_{A_{p^k}} - 1_{A_{p^{k+1}}}$ it follows $\mathcal{A}^{'} \subset \mathcal{A}$. If we define a multiplicative function $\tilde{f}_{p^k}$ for all $p' \in P$ and $j \in \mathbb{N}$ by

$$\tilde{f}_{p^k}(p'^j) := \begin{cases} 0, & \text{if } p = p',\ j = k, \\ 1, & \text{otherwise,} \end{cases}$$

then $1_{A^{'}_{p^k}} = 1 - \tilde{f}_{p^k}$. By Lemma 2 we obtain for all $p \in P$ with $\partial(p) \geq \frac{\log 2}{\log q}$ that

$$M(\tilde{f}_{p^k}) = 1 + q^{-\partial(p)(k+1)} - q^{-\partial(p)k}.$$

For this p we put

$$\begin{aligned} \delta_2(A'_{p^k}) :=& 1 - M(\tilde{f}_{p^k}) = \\ =& q^{-k\partial(p)} - q^{-(k+1)\partial(p)}. \end{aligned}$$

Let $a_1, a_2, \ldots$ be the sequence of the elements from G, such that $\partial(a_1) \leq$ $\leq \partial(a_2) \leq \cdots$. If we consider $M(n, 1_{A'_{a_j}})$ where $A'_{a_j} := \{a \in G : a_j \mid\mid a\}$, then

$$\begin{aligned} M(n, 1_{A'_{a_j}}) &= \frac{1}{G(n)} \sum_{\substack{a \in A'_{a_j} \\ \partial(a)=n}} 1 = \\ &= \frac{1}{G(n)} \sum_{\substack{ba_j \in G \\ (b,a_j)=1,\ \partial(ba_j)=n}} 1 = \\ &= \frac{G(n-\partial(a_j))}{G(n)} \cdot \frac{1}{G(n-\partial(a_j))} \sum_{\substack{b \in G \\ \partial(b)=n-\partial(a_j),\ (b,a_j)=1}} 1. \end{aligned}$$

Now define the completely multiplicative function $\tilde{f}$ for all $p \in P$ by

$$\tilde{f}(p) := \begin{cases} 1, & p \not| a_j, \\ 0, & p \mid a_j, \end{cases}$$

then

$$M(n, 1_{A'_{a_j}}) = \underbrace{\frac{G(n-\partial(a_j))}{G(n)}}_{I} \cdot \underbrace{\frac{1}{G(n-\partial(a_j))} \sum_{\substack{b\in G \\ \partial(b)=n-\partial(a_j)}} \tilde{f}(b)}_{II}.$$

By Lemma $2^{'}$ we obtain that II tends to $\prod\limits_{p|a_j} \left(1-q^{-\partial(p)}\right)$ as $n\to\infty$. Since I equals $M(n, 1_{A_{a_j}})$, we observe (see (3.2)) that there exists a subsequence n_k such that

$$\lim_{k\to\infty} M(n_k, 1_{A'_{a_j}}) = \delta_1(A_{a_j}) \prod_{p|a_j} (1-q^{-\partial(p)}) =: \delta_2(A'_j)$$

exists for all $j\in\mathbb{N}$. If we define $\delta_2(A')$ for every $A'\in\mathcal{A}^{'}$ by

$$\delta_2(A') = \lim_{k\to\infty} M(n_k, 1_{A'}),$$

then we have a content on $\mathcal{A}^{'}$. Thus $(\beta G, \sigma(\bar{\mathcal{A}}'), \bar{\delta}_2)$ is a probability space. Since

$$A_{p^k} = \bigcup_{j=k}^{\infty} A^{'}_{p^j}$$

where $A^{'}_{p^i}\cap A^{'}_{p^j} = \emptyset \quad (i\neq j)$, we obtain $\sigma(\bar{\mathcal{A}}^{'}) = \sigma(\bar{\mathcal{A}})$, which implies $(\beta G, \sigma(\bar{\mathcal{A}}), \bar{\delta}_1) = (\beta G, \sigma(\bar{\mathcal{A}}'), \bar{\delta}_2)$.

Now we can formulate the three series theorem for additive arithmetical semigroups.

4. The Three Series Theorem

Theorem. *Let G be an additive arithmetical semigroup such that $Z(y)$ belongs to* exp-*log class $\mathcal{F}$. Assume $\tilde{g}: G\to\mathbb{R}$ is completely additive. Then the following assertions are equivalent:*

(i) $\tilde{g} = \sum\limits_p X_p$ *possesses a limit distribution,*

(ii) $\bar{\tilde{g}} = \sum\limits_p \overline{X_p}$ *converges $\bar{\delta}$-almost everywhere,*

(iii) the series

$$\sum_{\substack{p \\ |\tilde{g}(p)|\geq 1}} \bar{\delta}(\bar{A}_p), \quad \sum_{\substack{p \\ |\tilde{g}(p)|<1}} \mathbb{E}[\overline{X_p}], \quad \sum_{\substack{p \\ |\tilde{g}(p)|<1}} Var[\overline{X_p}]$$

converge,

(iv) the series

$$\sum_{|\tilde{g}(p)|\geq 1} q^{-\partial(p)}, \quad \sum_{|\tilde{g}(p)|<1} \tilde{g}(p)q^{-\partial(p)}, \quad \sum_{|\tilde{g}(p)|<1} \tilde{g}^2(p)q^{-\partial(p)}$$

converge.

Remark 2. The equivalence of (i) and (iv) has been proven by W.-B. Zhang [12], in the case when the zeta function of G has the form

$$Z(y) = \frac{A}{1-qy} + \sum_{n=0}^{\infty} r(n)y^n$$

with $\sum\limits_{n=0}^{\infty} |r(n)|q^{-n} < \infty$ and the inequality $\lambda(n) = O(q^n)$ holds.

Proof of Theorem.

(i) $\Rightarrow$ *(iv)*

Suppose that the function

$$\tilde{g} = \sum_p X_p$$

has a limit distribution function $D(x)$. By the continuity theorem of Levy (see Billingsley, [2]) there exists a function $\varphi(t)$ which is the characteristic function of $D(x)$ and is continuous at $t=0$ such that

$$\int_{-\infty}^{\infty} e^{itx} d(D_n(x)) = \frac{1}{G(n)} \sum_{\partial(a)=n} e^{it\tilde{g}(a)} \to \varphi(t) \tag{4.1}$$

$n \to \infty$ for $-\infty < t < \infty$. Since $\varphi(t)$ is continuous at $t=0$ and $\varphi(0)=1$, there exist constants $T>0$ and $C \in (0,1)$ such that

$$|\varphi(t)| > C \quad \text{for} \quad |t| \leq T.$$

Put

$$\tilde{f}_t(a) := e^{it\tilde{g}(a)},$$

then the limit in (4.1) exists. This limit is equal to $M(\tilde{f}_t)$ and is non-zero for $|t| \le T$. By Remark 1 the series

$$\sum_p q^{-\partial(p)}(1-\tilde{f}_t(p)) \tag{4.2}$$

converges for $|t| \le T$. Now we show that the three series in (iv) are convergent.

We write (4.2) in the following form

$$\sum_p q^{-\partial(p)}(1-\tilde{f}_t(p)) = \sum_p q^{-\partial(p)}(1-\cos(t\tilde{g}(p)))-i\sum_p q^{-\partial(p)}\sin(t\tilde{g}(p)). \tag{4.3}$$

The convergence of (4.3) implies

$$\sum_p q^{-\partial(p)}(1-\cos(t\tilde{g}(p))) = 2\sum_p q^{-\partial(p)}\sin^2\left(\frac{t\tilde{g}(p)}{2}\right) \le K \tag{4.4}$$

for $|t| \le T$. We note that

$$\frac{2}{\pi}t \le \sin t$$

holds for $t \in [0, \frac{\pi}{2}]$. Therefore, by (4.4),

$$2\sum_{T|\tilde{g}(p)|\le\frac{\pi}{2}} q^{-\partial(p)}\left(\frac{2}{\pi}\right)^2\frac{T^2\tilde{g}^2(p)}{4} \le K,$$

and

$$\sum_{|\tilde{g}(p)|<1} q^{-\partial(p)}\tilde{g}^2(p) \ll \frac{1}{T^2}.$$

Thus, the convergence of the third series in (iv) is proved.

Integrating (4.4) from 0 to T gives

$$\sum_p q^{-\partial(p)}\int_0^T(1-\cos(t\tilde{g}(p))dt \le KT,$$

and we obtain

$$\sum_p q^{-\partial(p)}\left(T-\frac{\sin(T\tilde{g}(p))}{\tilde{g}(p)}\right) \le KT.$$

We observe that

$$\sin t \leq \frac{2}{\pi} t$$

holds for $t \in [\frac{\pi}{2}, \infty)$. Then

$$\sum_{|\tilde{g}(p)|\geq 1} q^{-\partial(p)} T\left(1 - \frac{2}{\pi}\right) \leq KT,$$

and

$$\sum_{|\tilde{g}(p)|\geq 1} q^{-\partial(p)} \ll 1,$$

which proves the convergence of the first series in (iv).

The last (convergent) series in (4.3) can be written as

$$\sum_{p} q^{-\partial(p)} \sin(t\tilde{g}(p)) = \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} \sin(t\tilde{g}(p)) + \sum_{\substack{p \\ |\tilde{g}(p)|\geq 1}} q^{-\partial(p)} \sin(t\tilde{g}(p)). \tag{4.5}$$

The estimate $|\sin(t\tilde{g}(p))| \leq 1$ and the convergence of the first series in (iv) imply that the last series in (4.5) converges for $|t| < T$. Therefore

$$\sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} \sin(t\tilde{g}(p)) \tag{4.6}$$

must be convergent for $|t| \leq T$. We note that

$$|\sin(t\tilde{g}(p)) - t\tilde{g}(p)| \leq \frac{|t\tilde{g}(p)|^3}{3!} \leq \frac{t^2\tilde{g}^2(p)}{3},$$

for $|t\tilde{g}(p)| \leq 2$. The series in (4.6) can be written as

$$\sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} \sin(t\tilde{g}(p)) = \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} (\sin(t\tilde{g}(p)) - t\tilde{g}(p)) - \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} t\tilde{g}(p). \tag{4.7}$$

Since the second series in (4.7) can be estimated by

$$\ll t^2 \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)} \tilde{g}^2(p),$$

the last sum in (4.7) must converge, too. This ends the proof of the implication $(i) \Rightarrow (iv)$.

$(iv) \Rightarrow (i)$

We assume that the three series in (iv) converge and show that the series in (4.2) is convergent, too, for all $t \in \mathbb{R}$. This implies, by Lemma 1 and Remark 1, that

$$M(\tilde{f}_t) = \prod_p (1 - q^{-\partial(p)}) \left(1 + \sum_{k=1}^{\infty} q^{-k\partial(p)} \tilde{f}_t(p^k) \right), \tag{4.8}$$

where $M(\tilde{f}_t)$ is non-zero and continuous at $t = 0$, and which is equivalent to the existence of the limit distribution.

We write the series (4.2) in the form

$$\sum_p q^{-\partial(p)}(1 - \tilde{f}_t(p)) = \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)}(1 - \tilde{f}_t(p)) + \sum_{\substack{p \\ |\tilde{g}(p)|\geq 1}} q^{-\partial(p)}(1 - \tilde{f}_t(p)) =$$

$$\begin{aligned} &= \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)}(1 - \tilde{f}_t(p) + it\tilde{g}(p)) - \\ &\qquad - \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)}(it\tilde{g}(p)) + \sum_{\substack{p \\ |\tilde{g}(p)|\geq 1}} q^{-\partial(p)}(1 - \tilde{f}_t(p)). \end{aligned} \tag{4.9}$$

and observe that

$$|e^{it\tilde{g}(p)} - 1 - it\tilde{g}(p)| \leq \frac{t^2\tilde{g}^2(p)}{2}$$

holds for each real number t. The absolute value of the first series on the right side of (4.9) is smaller than

$$\frac{T^2}{2} \sum_{\substack{p \\ |\tilde{g}(p)|<1}} q^{-\partial(p)}\tilde{g}^2(p)$$

which is convergent by assumption. The second series on the right side of (4.9) converges by assumption. The absolute value of the last series in (4.9) is obviously smaller than

$$2 \sum_{\substack{p \\ |\tilde{g}(p)|\geq 1}} q^{-\partial(p)},$$

and therefore the series (4.2) is convergent.

The equivalence of (iv) and (iii) is obvious. The assertions (ii) and (iii) are equivalent by the three series theorem. This ends the proof of the theorem.

References

[1] **Barát, A. and Indlekofer, K.-H.,** On mean-value theorems for multiplicative functions in additive arithmetical semigroups, *Annales Univ. Sci. Budapest. Sect. Comp.*, **33** (2010), 49-72.

[2] **Billingsley, P.,** *Probability and measure. Chapter 5, Section 26.,* John Wiley and Sons, New York, 1986.

[3] **Fogels, E.,** On the abstract theory of primes I-III, *Acta Arith.*, **10** (1964), 137-182.

[4] **Indlekofer, K.-H.,** A new method in probabilistic number theory, *Probability theory and applications. Essays to the memory of József Mogyoródi,* eds. J.Galambos and I.Kátai, Kluwer, 1992, 299-308.

[5] **Indlekofer, K.-H.,** New approach to probabilistic number theory - compactifications and integration, *Advenced Studies in Pure Mathematics*, **49** (2005), 133-170.

[6] **Indlekofer, K.-H.,** Tauberian theorems with applications to arithmetical semigroups and probabilistic combinatorics, *Annales Univ. Sci. Budapest. Sect. Comp.*, **34** (2011), 135-177.

[7] **Indlekofer, K.-H. and Manstavičius, E.,** Additive und multiplicative functions on arithmetical semigroups, *Publ. Math. Debrecen*, **45** (1994), 1-17.

[8] **Indlekofer, K.-H., Manstavičius, E. and Warlimont, R.,** On a certain class of infinite products with an application to aritmetical semigroups, *Arch. Math.,* **56** (1991), 446-453.

[9] **Knopfmacher, J.,** *Analytic arithmetic of algebraic function fields,* Lecture Notes in Pure and Appl. Math. **50**, Marcel Decker, New York, 1979.

[10] **Knopfmacher, J. and Zhang, W.-B.,** *Number theory arising from finite fields,* Analytic and probabilistic theory **241**, Marcel Decker, New York, 2001.

[11] **Manstavičius, E.,** An analytic method in probabilistic combinatorics, *Osaka J. Math.,* **46** (2009), 273-290.

[12] **Zhang, W.-B.,** Probabilistic number theory in additive arithmetic semigroups I., *Analytic number theory,* eds. B.C. Berndt et al., Prog. Math. **139**, Birkhäuser, 1996, 839-884.

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