PARTIAL FRACTION DECOMPOSITION OF SOME MEROMORPHIC FUNCTIONS

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Dedicated to Professor Bui Minh Phong on his 60th birthday

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Abstract. For positive integers r computable formulas for the partial fraction decomposition of the function $1/\sin^r$ will be presented. Some of the calculations can be found in our attached *Mathematica* notebook.

1. Partial fraction decomposition

Let us denote the set of nonnegative integers, positive integers and complex numbers, by $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{C}$ respectively.

For $r \in \mathbb{N}$ the function $1/\sin^r$ is a meromorphic function on the entire complex plane $\mathbb{C}$, i.e. it is analytic in $\mathbb{C}$ except for isolated singularities that are poles. For the fundamental facts about the complex function theory we refer to [1], [3], [4], [7], [9] or [12]. Here we only mention the following facts.

Let us start with a rational function $f := p/q$, where p and q are polynomials without common factors. Then the poles of f are exactly the zeros of q, and the order of each pole of f is equal to the multiplicity of the corresponding zero

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of q. If $a_1, a_2, \dots, a_n$ are the different roots of q (the poles of f), then there exist uniquely determined polynomials P and G_k $(k = 1, 2, \dots, n)$ such that

$$(1)\quad f(z) = \frac{p(z)}{q(z)} = P(z) + \sum_{k=1}^{n} G_k\left(\frac{1}{z - a_k}\right) \qquad (z \in \mathbb{C} \setminus \{a_1, a_2, \dots, a_n\}).$$

The degree of G_k $(k = 1, 2, \dots, n)$ is equal to the multiplicity of the root a_k. Moreover, $G_k\left(\frac{1}{z-a_k}\right)$ $(k = 1, 2, \dots, n)$ is *the principal part* of f at a_k, consisting of the part of its Laurent expansion which contains the negative powers of $(z - a_k)$. (1) is the *partial fraction decomposition of f*. There are several algorithms for its computation (see [6, §7.1.]).

A similar decomposition is true if we only suppose that the function f is meromorphic in $\mathbb{C}$ and it has only *finitely many* poles $a_1, a_2, \dots, a_n$. In this case the function P is analytic in $\mathbb{C}$.

Let us consider now a function f which is meromorphic in $\mathbb{C}$ with *infinitely many* poles. In this case f has exactly *countable many* poles, since f has only finitely many poles in every bounded subset of $\mathbb{C}$ (see [9, p. 240] or [6, p. 655]). Consequently we can suppose that the poles are a_k $(k \in \mathbb{N})$ and

$$|a_1| \le |a_2| \le |a_3| \le \cdots \quad \text{with} \quad \lim_{k\to+\infty} |a_k| = +\infty.$$

Then the following question arises naturally: Can we get an analogue of (1), if we replace the finite sum by an infinite sum? The problem is that, in this case the series of the principal parts of f at a_k's, i.e. the series $\sum_{k\in\mathbb{N}} G_k(1/(z-a_k))$ does not converge in general.

The main idea of Mittag-Leffler was that convergence is obtained if we subtract an appropriate analytic function g_k from each principal part G_k. *The Mittag-Leffler theorem* asserts that one can arbitrarily prescribe the poles and principal parts of a function meromorphic on the whole complex plane. Furthermore, there is an explicit form describing all these functions (see [1], [3], [7] or [12]). The theorem in this form is mainly of theoretical significance since it is not easy to apply in special cases. It turned out, however, that it is possible to obtain more useful formulas in concrete cases if a further assumption (see (2)) is made on the function.

Theorem A. ([9, p. 243], [7, p. 309]) *Let f be a meromorphic function having poles at the points $a_1, a_2, \dots$ different from ∞, analytic at $z = 0$, and such that*

(2) *f is uniformly bounded on a sequence of circles $C_n := C(0; r_n)$ with radii r_n increasing to $+\infty$.*

Then for every point $z \in \mathbb{C} \setminus \{a_1, a_2, \ldots\}$ we have

$$f(z) = f(0) + \lim_{n \to +\infty} \sum_{C_n} \left[G_k \left(\frac{1}{z - a_k} \right) - G_k \left(-\frac{1}{a_k} \right) \right],$$

where $G_k(1/(z - a_k))$ denotes the principal part of the function f at the point a_k and the index C_n under the summation sign indicates that only the poles lying in the disc $K(0; r_n)$ are considered.

This statement was used to obtain the partial fraction decomposition of the cot function in [9, p. 244] and in [7, p. 309] (see also [12, p. 135]).

2. Partial fraction decomposition of $1/\sin^r$

In this section we shall derive formulas for the partial fraction decomposition of the function

$$f_r(z) := \frac{1}{\sin^r z} \qquad (z \in D),$$

where r is a fixed positive integer and

$$D := \mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\}.$$

The function f_r has a pole of order r at the point $a_k := k\pi$ for every $k \in \mathbb{Z}$, and it is analytic on D. Thus f_r is a meromorphic function on $\mathbb{C}$.

Let us consider first the auxiliary function

$$F_r(z) := \left(\frac{z}{\sin z} \right)^r \qquad (|z| < \pi),$$

where r is a positive integer.* Since $F_r(0) = 1$ and F_r is differentiable at the point $z = 0$ we have that F_r is an even analytic function on the disc $|z| < \pi$. Therefore it has a Taylor series expansion about the point $z = 0$, and the coefficients can be computed.

Theorem 1. *The MacLaurin expansion of the function F_r is of the following form*

$$F_r(z) = \left(\frac{z}{\sin z} \right)^r = \sum_{j=0}^{+\infty} \frac{F_r^{(j)}(0)}{j!} z^j = \sum_{j=0}^{+\infty} \frac{F_r^{(2j)}(0)}{(2j)!} z^{2j} \qquad (|z| < \pi),$$

*Here and below at points $z_0 \in \mathbb{C}$ for which the function is formally undefined but has a finite limit, it is defined to be $f(z_0) := \lim\limits_{z_0} f$, i.e. f is continuously extended.

where

$$F_r^{(2j+1)}(0)=0 \qquad (j\in\mathbb{N}_0),$$

and

$$\text{(3)}\qquad \begin{cases} F_r^{(0)}(0)=F_r(0)=1,\\ F_r^{(2j)}(0)=\dfrac{r}{2j}\displaystyle\sum_{l=0}^{j-1}(-1)^{j+1-l}\binom{2j}{2l}2^{2(j-l)}F_r^{(2l)}(0)B_{2(j-l)}, \quad \textit{if } j\in\mathbb{N}, \end{cases}$$

where the B_{2l}'s ($l\in\mathbb{N}_0$) are the Bernoulli numbers.

We recall that the Bernoulli numbers B_n ($n\in\mathbb{N}_0$) satisfy the recurrence relation

$$B_0=1,$$

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0 \qquad (n=2,3,\ldots)$$

(see [13] or [10, I, p. 682]). The first few Bernoulli numbers B_n are

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30}$$

with $B_{2n+1}=0$ for $n\in\mathbb{N}\setminus\{1\}$.

The first few coefficients obtained from the recurrence formula (3) are as follows

$$F_r^{(0)}(0)=1,\quad F_r^{(2)}(0)=\frac{r}{3},\quad F_r^{(4)}(0)=\frac{5r^2+2r}{15}.$$

From Theorem 1 we obtain that the principal part of f_r at the pole $a_0=0$ of order r has the following form

$$\text{(4)}\qquad G_{0,r}\left(\frac{1}{z}\right)=\sum_{j=0}^{r-1}\frac{F_r^{(j)}(0)}{j!}\frac{1}{z^{r-j}}=\sum_{j=0}^{r'}\frac{F_r^{(2j)}(0)}{(2j)!}\frac{1}{z^{r-2j}} \qquad (z\in\mathbb{C}\setminus\{0\}),$$

where

$$r':=\left[\frac{r-1}{2}\right]$$

($[x]$ denotes the integer part of $x\in\mathbb{R}$).

We say that the doubly (or in two direction) infinite complex series $\sum_{k\in\mathbb{Z}}u_k$ is convergent and its sum is $S\in\mathbb{C}$, if

$$\lim_{m,n\to\infty}\sum_{k=-m}^{n}u_k=S=:\sum_{k=-\infty}^{+\infty}u_k.$$

It is equivalent to the fact that both the (in one direction) infinite series $\sum_{k\in\mathbb{N}_0} u_k$ and $\sum_{-k\in\mathbb{N}} u_{-k}$ are convergent and

$$S=\sum_{k=0}^{+\infty} u_k+\sum_{k=1}^{+\infty} u_{-k}.$$

In this case the doubly infinite sequence of the symmetric partial sums $\sum_{k=-n}^{n} u_k$ of the series $\sum_{k\in\mathbb{Z}}$ is convergent and tends to S if $n\to+\infty$. We remark that from the convergence of the symmetric partial sums of $\sum_{k\in\mathbb{Z}} u_k$ its convergence does not follow, see for example the series $\sum_{k\in\mathbb{Z}} 1/k$ with $1/0 := 1$.

We shall frequently use the notation

$$\sum_{k=-\infty}^{+\infty}{}' u_k,$$

where the prime signifies that the term $k=0$ is omitted in the summation.

The partial fraction decomposition of the function $f_r = 1/\sin^r$ is given in the following statement.

Theorem 2. *Let r be a fixed positive integer, and $r' = [(r-1)/2]$. Then for every point $z\in D$ we have*

$$\begin{aligned}
f_r(z) &= \frac{F_r^{(r)}(0)}{r!}+G_{0,r}\left(\frac{1}{z}\right)+\sum_{k=-\infty}^{+\infty}{}'(-1)^{rk}\left[G_{0,r}\left(\frac{1}{z-k\pi}\right)-G_{0,r}\left(-\frac{1}{k\pi}\right)\right]=\\
&= \frac{F_r^{(r)}(0)}{r!}+\sum_{j=0}^{r'}\frac{F_r^{(2j)}(0)}{(2j)!}\cdot\frac{1}{z^{r-2j}}+\\
&\qquad+\sum_{k=-\infty}^{+\infty}{}'(-1)^{rk}\sum_{j=0}^{r'}\frac{F_r^{(2j)}(0)}{(2j)!}\left[\frac{1}{(z-k\pi)^{r-2j}}+\frac{(-1)^{r-1}}{(k\pi)^{r-2j}}\right],
\end{aligned}$$

where $G_{0,r}$ is given in (4), as the principal part of f_r at the pole 0. The convergence is absolute on the domain $D\subset\mathbb{C}$, and uniform in every compact subset of D.

The formulas in Theorem 2 may be simplified as follows.

Theorem 3. *Let r be a fixed positive integer. Then using the notation of Theorem 2 we have*

$$\begin{aligned}
f_r(z)=\frac{1}{\sin^r z} &= \sum_{k=-\infty}^{+\infty}(-1)^{rk}G_{0,r}\left(\frac{1}{z-k\pi}\right)=\\
&= \sum_{k=-\infty}^{+\infty}(-1)^{rk}\sum_{j=0}^{r'}\frac{F_r^{(2j)}(0)}{(2j)!}\cdot\frac{1}{(z-k\pi)^{r-2j}}.
\end{aligned}$$

The convergence is absolute in every point $z \in D$ and uniform in every compact subset of D.

The formulas are different for even and odd exponents:

$$\frac{1}{\sin^{2r} z} = \sum_{k=-\infty}^{+\infty} G_{0,2r}\left(\frac{1}{z-k\pi}\right) = \sum_{k=-\infty}^{+\infty} \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \cdot \frac{1}{(z-k\pi)^{2(r-j)}},$$

$$\begin{aligned} \frac{1}{\sin^{2r+1} z} &= \sum_{k=-\infty}^{+\infty} (-1)^k G_{0,2r+1}\left(\frac{1}{z-k\pi}\right) = \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k \sum_{j=0}^{r} \frac{F_{2r+1}^{(2j)}(0)}{(2j)!} \cdot \frac{1}{(z-k\pi)^{2(r-j)+1}}. \end{aligned}$$

Remark 1. Let us emphasize that Theorem 3 states that for the computation of the partial fraction decomposition of $1/\sin^r$ it is enough to determine only the polynomial $G_{0,r}$. One can obtain the partial fraction decomposition of similar functions (for example of $1/\cos^r$), too.

Let us see a few special cases of the above theorem. The formulas below are valid at all points $z \in \mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\} = D$. The convergence is absolute in every point $z \in D$ and uniform in every compact subset of D.

$$\frac{1}{\sin z} = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{1}{z-k\pi},$$

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^2},$$

$$\frac{1}{\sin^3 z} = \sum_{k=-\infty}^{+\infty} (-1)^k \left[\frac{1}{(z-k\pi)^3} + \frac{1}{2}\frac{1}{z-k\pi}\right],$$

$$\frac{1}{\sin^4 z} = \sum_{k=-\infty}^{+\infty} \left[\frac{1}{(z-k\pi)^4} + \frac{2}{3}\frac{1}{(z-k\pi)^2}\right],$$

$$\frac{1}{\sin^5 z} = \sum_{k=-\infty}^{+\infty} (-1)^k \left[\frac{1}{(z-k\pi)^5} + \frac{5}{6}\frac{1}{(z-k\pi)^3} + \frac{3}{8}\frac{1}{z-k\pi}\right],$$

$$\frac{1}{\sin^6 z} = \sum_{k=-\infty}^{+\infty} \left[\frac{1}{(z-k\pi)^6} + \frac{1}{(z-k\pi)^4} + \frac{8}{15}\frac{1}{(z-k\pi)^2}\right].$$

Using the attached *Mathematica* notebook [11] further special cases can be calculated.

3. Some other formulas

From Theorem 3 we can obtain closed forms for the sum of the doubly infinite series

$$\sum_{k\in\mathbb{Z}} \frac{1}{(z-k\pi)^{2r}} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(z-k\pi)^{2r-1}}$$
$$(z \in D, \quad r = 1, 2, \ldots),$$

which are absolutely convergent on the domain $D \subset \mathbb{C}$ and uniformly convergent in every compact subset of D. Set

$$A_{2r}(z) := \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{2r}},$$
$$A^{\pm}_{2r-1}(z) := \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^{2r-1}},$$
$$(z \in D, \quad r = 1, 2, \ldots).$$

The following statement is an immediate consequence of Theorem 3.

Corollary 1. *For the functions A_{2r} we have the following recursive relation*

$$A_2(z) = \frac{1}{\sin^2 z},$$
$$A_{2r}(z) = \frac{1}{\sin^{2r} z}\left[1 - \sum_{j=1}^{r-1} \frac{F^{(2j)}_{2r}(0)}{(2j)!}(1-\cos^2 z)^j \cdot \sin^{2r-2j} z \cdot A_{2r-2j}(z)\right]$$
$$(z \in D, \quad r = 2, 3, \ldots),$$

where the coefficients $F^{(2j)}_{2r}(0)$ are given in Theorem 1.

The main advantage of the above representation of A_{2r} is that the functions $\sin^{2j}(z) \cdot A_{2j}(z)$ $(z \in \mathbb{C}, \ \ j = 1, 2, \ldots)$ are algebraic polynomials of the function $\cos^2 z$ $(z \in \mathbb{C})$. Consequently, their exact lower and upper bounds can be seen very easily.

For the first few values of r we get the following formulas, which are valid for every $z \in D$

$$A_2(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^2} = \frac{1}{\sin^2 z},$$

$$A_4(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^4} = \frac{1}{\sin^4 z}\left[\tfrac{1}{3} + \tfrac{2}{3}\cos^2 z\right],$$

$$A_6(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^6} = \frac{1}{\sin^6 z}\left[\tfrac{2}{15} + \tfrac{11}{15}\cos^2 z + \tfrac{2}{15}\cos^4 z\right].$$

Here the convergence is uniform in every compact subset of D.

From Theorem 3 we also immediately obtain closed forms for the functions $A^{\pm}_{2r-1}$, too.

Corollary 2. *For the functions $A^{\pm}_{2r-1}$ we have the following recursive relation*

$$A_1^{\pm}(z) = \frac{1}{\sin z},$$

$$A^{\pm}_{2r-1}(z) = \frac{1}{\sin^{2r-1} z}\left[1 - \sum_{j=1}^{r} \frac{F^{(2j)}_{2r-1}(0)}{(2j)!}(1-\cos^2 z)^j \cdot \sin^{2r-1-2j} z \cdot A^{\pm}_{2r-1-2j}(z)\right]$$

$$(z \in D, \quad r = 2, 3, \ldots),$$

where the coefficients $F^{(2j)}_{2r-1}(0)$ are given in Theorem 1.

It is clear that the functions $\sin^{2j-1}(z) \cdot A^{\pm}_{2j-1}(z)$ $(z \in \mathbb{C}, \quad j = 1, 2, \ldots)$ are odd algebraic polynomials of the function $\cos z$ $(z \in \mathbb{C})$.

For the first few values of r we get the following formulas, which are valid for every $z \in D$ and the convergence is uniform in every compact subset of D.

$$A_1^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)} = \frac{1}{\sin z},$$

$$A_3^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^3} = \frac{1}{\sin^3 z}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos^2 z\right],$$

$$A_5^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^5} = \frac{1}{\sin^5 z}\left[\tfrac{5}{24} + \tfrac{18}{24}\cos^2 z + \tfrac{1}{24}\cos^4 z\right],$$

$$A_7^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^7} = \frac{1}{\sin^7 z}\left[\tfrac{61}{720} + \tfrac{479}{720}\cos^2 z + \tfrac{179}{720}\cos^4 z + \tfrac{1}{720}\cos^6 z\right].$$

Using the attached *Mathematica* notebook [11] further special cases can be calculated.

Remark 2. In the theory of wavelet analysis the exact lower and upper bounds for the functions $\sin^{2r} x \cdot A_{2r}(x)$ ($x \in \mathbb{R}$) have important applications (see [2, p. 90], [8, p. 24]). Using the fact that these functions are algebraic polynomials of $\cos^2$ one can easily obtain the corresponding bounds.

4. Proofs

4.1. Proof of Theorem 1. Let

$$h(z) := \frac{z}{\sin z}, \qquad \text{i.e.} \qquad F_r(z) = h^r(z) \quad (|z| < \pi),$$

and

$$H(z) := \frac{h'(z)}{h(z)} = \frac{1}{z} - \operatorname{ctg} z \qquad (|z| < \pi).$$

The functions h and H are analytic on the disc $|z| < \pi$. It is known that ([10, Volume II, p. 512])

$$h(z) = \frac{z}{\sin z} = 1 + \sum_{j=1}^{+\infty} (-1)^{j-1} \frac{(2^{2j} - 2) B_{2j}}{(2j)!} z^{2j} \qquad (|z| < \pi),$$

and (see [6, p. 111], [10, Volume II, p. 512] or [11])

$$H(z) = \frac{1}{z} - \operatorname{ctg} z = \sum_{j=1}^{+\infty} (-1)^{j+1} \frac{2^{2j} B_{2j}}{(2j)!} z^{2j-1} \qquad (|z| < \pi).$$

Therefore

$$\begin{aligned} H^{(2j)}(0) &= 0 && (j = 0, 1, 2, \ldots), \\ H^{(2j-1)}(0) &= (-1)^{j+1} \frac{2^{2j} B_{2j}}{2j} && (j = 1, 2, 3, \ldots). \end{aligned}$$

Since $F_r = h^r$ we get

$$F_r'(z) = r h^{r-1}(z) h'(z) = r\, h^r(z)\, \frac{h'(z)}{h(z)} = r F_r(z)\, H(z).$$

Using the Leibniz formula we obtain

$$
\begin{aligned}
F_r^{(2j)}(0) &= (rF_rH)^{(2j-1)}(0) = r\sum_{l=0}^{2j-1}\binom{2j-1}{l}F_r^{(l)}(0)H^{(2j-1-l)}(0) = \\
&= r\sum_{l=0}^{j-1}\binom{2j-1}{2l}F_r^{(2l)}(0)H^{(2(j-l)-1)}(0) = \\
&= r\sum_{l=0}^{j-1}(-1)^{j+1-l}\binom{2j-1}{2l}\frac{2^{2(j-l)}}{2(j-l)}F_r^{(2l)}(0)B_{2(j-l)} = \\
&= \frac{r}{2j}\sum_{l=0}^{j-1}(-1)^{j+1-l}\binom{2j}{2l}2^{2(j-l)}F_r^{(2l)}(0)B_{2(j-l)},
\end{aligned}
$$

which proves the statement. ■

4.2. Proof of Theorem 2. For a positive integer r the poles of the function $f_r = 1/\sin^r$ are the zeros of the function sin, i.e. the points $a_k := k\pi$ $(k \in \mathbb{Z})$.

Let us consider the pole $a_0 = 0$ first. From Theorem 1 it follows that its order is r and the principal part of the function f_r at the point a_0 is $G_{0,r}(1/z)$ (see (4)).

Let us take the pole $a_k = k\pi$ for a fixed $k \in \mathbb{Z}\setminus\{0\}$ and determine the principal part of f_r at this pole. Observe that

$$
\frac{1}{\sin^r z} = \left(\frac{(-1)^k}{\sin(z-k\pi)}\right)^r = (-1)^{rk}\frac{1}{\sin^r(z-k\pi)}
$$

$$
(z \in \mathbb{C}\setminus\{k\pi \mid k \in \mathbb{Z}\}).
$$

Consider the function

$$
F_r(z-k\pi) = \left(\frac{z-k\pi}{\sin(z-k\pi)}\right)^r
$$

$$
(|z-k\pi| < \pi).
$$

It is analytic on the whole disc $|z-k\pi| < \pi$. Therefore it has a power series expansion about the point $a_k = k\pi$

$$
F_r(z-k\pi) = \sum_{j=0}^{+\infty}\frac{\frac{d^j}{dz^j}F_r(z-k\pi)_{|z=k\pi}}{j!}(z-k\pi)^j = \sum_{j=0}^{+\infty}\frac{F_r^{(j)}(0)}{j!}(z-k\pi)^j
$$

$$
(|z-k\pi| < \pi),
$$

i.e.

$$\left(\frac{z-k\pi}{\sin(z-k\pi)}\right)^r = \sum_{j=0}^{+\infty} \frac{F_r^{(j)}(0)}{j!}(z-k\pi)^j.$$

From this it follows that on the disc $|z-k\pi| < \pi$ we have

$$\frac{1}{\sin^r z} = (-1)^{rk}\frac{1}{\sin^r(z-k\pi)} = (-1)^{rk}\sum_{j=0}^{r-1}\frac{F_r^{(j)}(0)}{j!}\frac{1}{(z-k\pi)^{r-j}} + U_r(z),$$

where U_r is an analytic function on $\mathbb{C}$. This means that (see (4)) the principal part of the function f_r at the pole $a_k = k\pi$ is

$$(-1)^{rk}G_{0,r}\left(\frac{1}{z-k\pi}\right). \tag{5}$$

Let us define the function

$$\mathfrak{f}_r(z) := \frac{1}{\sin^r z} - G_{0,r}\left(\frac{1}{z}\right) \qquad (z \in D).$$

We verify that the conditions of Theorem A hold for the function $\mathfrak{f}_r$.

First we note that from the Theorem 1 it follows that the function $\mathfrak{f}_r$ is analytic on the disc $|z| < \pi$. Indeed, it has the Taylor series expansion about the point $a_0 = 0$, which means that $a_0 = 0$ is *not a pole* of $\mathfrak{f}_r$, moreover

$$\mathfrak{f}_r(0) = \frac{F_r^{(r)}(0)}{r!}.$$

For the proof of condition (2) of Theorem A a more delicate argument is needed. It can be found in [9, p. 245] or in [7, p. 310].

The poles of $\mathfrak{f}_r$ are exactly the points $a_k = k\pi$ $(k \in \mathbb{Z}\setminus\{0\})$. The function $G_{0,r}(1/z)$ $(z \in D)$ is analytic around the point $a_k = k\pi$ $(k \in \mathbb{Z}\setminus\{0\})$, therefore its principal part at a_k is identically zero. This means that the principal parts of the functions f_r and $\mathfrak{f}_r$ at the poles a_k $(k \in \mathbb{Z}\setminus\{0\})$ are equal. By (5) they can be written in the following form

$$G_{k,r}\left(\frac{1}{z-k\pi}\right) = (-1)^{rk}G_{0,r}\left(\frac{1}{z-k\pi}\right) \qquad (z \in D,\ k \in \mathbb{Z}\setminus\{0\}).$$

Therefore

$$G_{k,r}\left(\frac{1}{z-k\pi}\right) - G_{k,r}\left(-\frac{1}{k\pi}\right) = (-1)^{rk}\left[G_{0,r}\left(\frac{1}{z-k\pi}\right) - G_{0,r}\left(-\frac{1}{k\pi}\right)\right] =$$

$$= (-1)^{rk}\sum_{j=0}^{r'}\frac{F_r^{(2j)}(0)}{(2j)!}\left[\frac{1}{(z-k\pi)^{r-2j}} + \frac{(-1)^{r+1}}{(k\pi)^{r-2j}}\right]$$

$$(z \in D,\ k \in \mathbb{Z}\setminus\{0\}).$$

We shall show that the doubly infinite *series* generated by the above doubly infinite *sequence* is absolutely convergent at every point $z \in D$ and uniformly convergent in every compact subset of D.

The *sequence* in question contains terms which have the form

$$\frac{1}{(z-k\pi)^{r-2j}} + \frac{(-1)^{r+1}}{(k\pi)^{r-2j}}.$$

Now separate the cases when the exponent r is even or odd.

First let us suppose that $r = 2l$ $(l = 1, 2, \dots)$ is an even number, i.e. $r' = \left[\frac{r-1}{2}\right] = l - 1$. The exponents

$$p := r - 2j = 2(l-j) \qquad (j = 0, 1, \dots, l-1; \quad p = 2l, 2l-2, \dots, 2)$$

are also even. Since for every $z \in D$ we have

$$\frac{1}{(z-k\pi)^p} : \frac{1}{(k\pi)^p} = \frac{1}{\left(\frac{z}{k\pi} - 1\right)^p} \to 1 \qquad (k \to +\infty)$$

and $\sum_{k=1} 1/(k\pi)^p < +\infty$ $(p \geq 2)$, thus the series $\sum_{k=1} 1/(z-k\pi)^p$ is absolute convergent for every $z \in D$. Consequently the doubly infinite series

$$\sum_{k=-\infty}^{+\infty}{}' (-1)^{rk} \left[G_{0,r}\left(\frac{1}{z-k\pi}\right) - G_{0,r}\left(-\frac{1}{k\pi}\right)\right] \tag{6}$$

is absolutely convergent on D and uniformly convergent in every compact subset of D.

Now let $r = 2l + 1$ $(l = 0, 1, \dots)$ be an odd number, i.e. $r' = \left[\frac{r-1}{2}\right] = l$. The exponents

$$q := r - 2j = 2(l-j) + 1 \qquad (j = 0, 1, \dots, l; \quad q = 2l+1, \dots, 3, 1)$$

are also odd. The series $\sum_{k=1} 1/(z-k\pi)^q$ $(q \geq 3)$ is absolute convergent for every $z \in D$. If $q = 1$, then using the inequality

$$\left|\frac{1}{z-k\pi} + \frac{1}{k\pi}\right| = \left|\frac{z}{k\pi(z-k\pi)}\right| \leq \frac{1}{(k\pi)^2}\,\frac{|z|}{\left|\frac{z}{k\pi} - 1\right|}$$

we obtain that the doubly infinite series

$$\sum_{k=-\infty}^{+\infty}{}' \left[\frac{1}{z-k\pi} + \frac{1}{k\pi}\right]$$

is absolutely convergent for every $z \in D$, i.e. the series (6) is absolutely convergent on D for every odd exponent r, too.

We have shown that the function $\mathfrak{f}_r$ satisfies the conditions of Theorem A, thus Theorem 2 follows from Theorem A. ■

4.3. Proof of Theorem 3. Let us first consider the case of *even* indices, i.e. consider the functions f_{2r}. Using the fact that $F_{2r}^{(2j+1)}(0) = 0$ ($j \in \mathbb{N}_0$) we have

$$\sum_{j=0}^{2r-1} \frac{F_{2r}^{(j)}(0)}{j!} \frac{1}{z^{2r-j}} = \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{z^{2(r-j)}},$$

$$\sum_{j=0}^{2r-1} \frac{F_{2r}^{(j)}(0)}{j!} \left[\frac{1}{(z-k\pi)^{2r-j}} + \frac{(-1)^{2r-1-j}}{(k\pi)^{2r-j}} \right] =$$

$$= \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(z-k\pi)^{2(r-j)}} - \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(k\pi)^{2(r-j)}}.$$

Since the series in Theorem 2 is absolutely convergent it can be rearranged as

$$\frac{1}{\sin^{2r} z} = \sum_{k=-\infty}^{+\infty} \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(z-k\pi)^{2(r-j)}} +$$
$$+ \left\{ \frac{F_{2r}^{(2r)}(0)}{(2r)!} - {\sum_{k=-\infty}^{+\infty}}' \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(k\pi)^{2(r-j)}} \right\}.$$

Denote by A the part between the brackets $\{\dots\}$. For the proof of the statement it is enough to show that $A = 0$. Using that

$$\sum_{k=1}^{+\infty} \frac{1}{k^{2(r-j)}} = (-1)^{r-j-1} \frac{(2\pi)^{2(r-j)}}{2\,(2(r-j))!} B_{2(r-j)}$$

(see and [10, Vol. I., p. 685]) we get

$${\sum_{k=-\infty}^{+\infty}}' \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(k\pi)^{2(r-j)}} = 2 \sum_{k=1}^{+\infty} \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{1}{(k\pi)^{2(r-j)}} =$$
$$= 2 \sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \sum_{k=1}^{+\infty} \frac{1}{(k\pi)^{2(r-j)}} = \sum_{j=0}^{r-1} (-1)^{r-j-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \frac{2^{2(r-j)}}{(2(r-j))!} B_{2(r-j)} =$$
$$= \frac{1}{(2r)!} \sum_{j=0}^{r-1} (-1)^{r+1-j} \binom{2r}{2j} 2^{2(r-j)} F_{2r}^{(2j)}(0) B_{2(r-j)} = \frac{F_{2r}^{(2r)}(0)}{(2r)!}.$$

Thus $A = 0$, which proves that

$$\frac{1}{\sin^{2r} z} = \sum_{k=-\infty}^{+\infty} G_{0,2r}\left(\frac{1}{z-k\pi}\right) = \sum_{k=-\infty}^{+\infty}\sum_{j=0}^{r-1} \frac{F_{2r}^{(2j)}(0)}{(2j)!} \cdot \frac{1}{(z-k\pi)^{2(r-j)}}.$$

Let us now consider the *odd* cases, i.e. consider the functions f_{2r+1} and the doubly infinite series in Theorem 2. We already know that $F_{2r+1}^{(2r+1)}(0) = 0$, and it is easy to observe that the terms

$$A_k := -(-1)^{(2r+1)k} G_{0,2r+1}\left(-\frac{1}{k\pi}\right),$$
$$B_k := -(-1)^{(2r+1)(-k)} G_{0,2r+1}\left(-\frac{1}{(-k)\pi}\right),$$

have opposite signs for every $k \in \mathbb{Z} \setminus \{0\}$. Indeed,

$$B_k := (-1)^{k+1} G_{0,2r+1}\left(\frac{1}{k\pi}\right),$$

moreover by (4)

$$\begin{aligned} G_{0,2r+1}\left(-\frac{1}{k\pi}\right) &= \sum_{j=0}^{r} \frac{F_{2r+1}^{(2j)}(0)}{(2j)!}\left(-\frac{1}{k\pi}\right)^{2r+1-2j} = \\ &= \sum_{j=0}^{r} \frac{F_{2r+1}^{(2j)}(0)}{(2j)!}\left(\frac{(-1)}{(k\pi)^{2r+1-2j}}\right) = -G_{0,2r+1}\left(\frac{1}{k\pi}\right), \end{aligned}$$

so we obtain that

$$A_k = (-1)^{k+1} G_{0,2r+1}\left(-\frac{1}{k\pi}\right) = (-1)^{k+2} G_{0,2r+1}\left(\frac{1}{k}\right),$$

which shows that $A_k + B_k = 0$ for every $k \in \mathbb{Z} \setminus \{0\}$.

The series in Theorem 2 is absolutely convergent. After sorting its terms we have

$$\begin{aligned} \frac{1}{\sin^{2r+1} z} &= \sum_{k=-\infty}^{+\infty} (-1)^k G_{0,2r+1}\left(\frac{1}{z-k\pi}\right) = \\ &= \sum_{k=-\infty}^{+\infty} (-1)^k \sum_{j=0}^{r} \frac{F_{2r+1}^{(2j)}(0)}{(2j)!} \cdot \frac{1}{(z-k\pi)^{2(r-j)+1}}, \end{aligned}$$

which proves the statement in the odd cases, too. ∎

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