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A NOVEL POLYCONVOLUTION FOR THE FOURIER SINE, FOURIER COSINE AND THE KONTOROVICH–LEBEDEV INTEGRAL TRANSFORMS AND APPLICATIONS

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

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Abstract. The polyconvolution $\underset{1}{*}(f,g,h)(x)$ of three functions $f,\ g,\ h$ is constructed for the integral transforms Fourier sine (F_s), Fourier cosine (F_c) and Kontorovich-Lebedev (K_{iy}), whose factorization equality is of the form

$$F_s(\underset{1}{*}(f,g,h))(y) = (F_sf)(y).(F_cg)(y).(K_{iy}h),\ \forall y > 0.$$

Relations of this polyconvolution to the Fourier convolution and the Fourier cosine convolution and also relations between the new polyconvolution product and other known convolution products are established. Then we give applications to solving a class of generalized integral equations with Toeplitz plus Hankel kernel.

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1. Introduction

Recently, many authors are interested in the theory of convolution for integral transforms and gave several interesting applications (see [2, 8, 9, 13, 15, 16, 17, 18]). In particular, they consider integral equations with the Toeplitz plus Hankel kernel (see [3, 6, 14])

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)]f(y)dy = g(x), \quad x > 0, \tag{1.1}$$

where k_1, k_2, g are known functions, and f is unknow function. Various special partial cases of this equation can be solved in closed form with the help of convolutions and generalized convolutions. However, for general case of the Hankel kernel k_1 and the Toeplitz kernel k_2, solving this equation is still open.

In [5] V.A. Kakichev introduced a constructive method for defining a polyconvolution $\overset{\gamma}{*}(f_1, f_2, \cdots, f_n)(x)$ of functions $f_1, f_2, \cdots, f_n$ with a weight function γ for the integral transforms $K, K_1, K_2, \cdots, K_n$, for which the following factorization property holds

$$K[\overset{\gamma}{*}(f_1, f_2, \cdots, f_n)](y) = \gamma(y)\prod_{i=1}^{n}(K_i f_i)(y), \ \ n \geq 3.$$

Basing on Kakichev's method, several polyconvolutions for integral transforms have been constructed and studied (see [10]).

It is worth noting that this polyconvolution convolution is quite different from the ones studied in [15, 17]. We will see in Theorem 2.1 the difficulty in proof of this theorem. Therefore, we obtain quite interesting different applications to solving integral equations and systems of integral equations with Toeplitz plus Hankel kernel, which seems difficult to be solved by using the results studied in [15, 17].

In this paper, we construct a new polyconvolution for the Fourier sine, Fourier cosine and the Kontorovich-Lebedev transforms. The existence and the factorization equality of this new polyconvolution is proved (Theorem 2.1). In Section 3, we obtain several relations of this new polyconvolution with other known convolutions (Theorem 3.1). Finally, in Section 4, we apply the new polyconvolution (2.1) to solve a class of generalized integral equations with Toeplitz plus Hankel kernel.

2. Polyconvolution

Let F_c, F_s and K_{iy} denote the Fourier cosine, the Fourier sine and the Kontorovich-Lebedev transforms, respectively (see [8]) defined by

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos yx.f(x)dx, \quad y > 0,$$

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin yx.f(x)dx, \quad y > 0,$$

$$K_{ix}[f] = \int_0^\infty K_{ix}(t)f(t)dt,$$

here $K_{ix}(t)$ is the Macdonald function (see [1])

$$K_{ix}(t) = \int_0^\infty e^{-t\cosh u} \cos xu\, du, \quad x \geq 0, \ t > 0.$$

Definition 1. *The polyconvolution of functions f, g and h for the Fourier sine, Fourier cosine and the Kontorovich-Lebedev integral transforms is defined as follows*

$$\underset{1}{*}(f,g,h)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty \theta(x,u,v,w) f(u)g(v)h(w)dudvdw, \ x > 0, \tag{2.1}$$

where

$$\theta(x,u,v,w) =$$

$$= e^{-w\cosh(x-u+v)} + e^{-w\cosh(x-u-v)} - e^{-w\cosh(x+u+v)} - e^{-w\cosh(x+u-v)}.$$

Definition 2. *Denote by $L_1(\mathbb{R}_+)$ and $L_1(\beta(x), \mathbb{R}_+)$ respectively the set of all functions f and g defined on $(0, +\infty)$ such that*

$$\int_0^{+\infty} |g(x)|dx < +\infty \quad \text{and} \quad \int_0^{+\infty} \beta(x)|f(x)|dx < +\infty.$$

The norm of a function f in $L_1(\mathbb{R}_+)$ and a function g in $L_1(\beta(x),\mathbb{R}_+)$ are respectively defined as follow

$$\|f\|_{L_1(\mathbb{R}_+)} = \int\limits_0^\infty |f(x)|dx \ \ \textit{and} \ \ \|g\|_{L_1(\beta,\mathbb{R}_+)} = \int\limits_0^\infty \beta(x)|g(x)|dx.$$

Theorem 2.1. *Let f, g be functions in $L_1(\mathbb{R}_+)$, and h a function in $L_1\left(\frac{1}{\sqrt{w}},\mathbb{R}_+\right)$, then the polyconvolution (2.1) belongs to $L_1(\mathbb{R}_+)$ and satisfies the following factorization equality*

$$F_s(\underset{1}{*}(f,g,h))(y) = (F_s f)(y).(F_c g)(y).(K_{iy}h), \quad \forall y > 0. \tag{2.2}$$

Proof. Since $e^{-w\cosh x} \le 1$ for all $w > 0$ and for all $x \in \mathbb{R}_+$ and $L_1\left(\frac{1}{\sqrt{w}},\mathbb{R}_+\right)$ is a subspace of $L_1(\mathbb{R}_+)$, we have

$$|\underset{1}{*}(f,g,h)(x)| \le$$

$$\frac{1}{2\sqrt{2\pi}}\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty |f(u)||g(v)||h(w)| \Big[e^{-w\cosh(x-u+v)} + e^{-w\cosh(x-u-v)} +$$

$$+e^{-w\cosh(x+u+v)} + e^{-w\cosh(x+u-v)}\Big]\, dudvdw \le$$

$$\le \sqrt{\frac{2}{\pi}}\int\limits_0^\infty |f(u)|du.\int\limits_0^\infty |g(v)|dv.\int\limits_0^\infty |h(w)|dw < +\infty.$$

On the other hand, note that $\cosh(x-u+v) \ge \frac{(x-u+v)^2}{2}$, therefore

$$e^{-w\cosh(x-u+v)} \le e^{-w\frac{(x-u+v)^2}{2}}.$$

Using formular 3.321.3 in [1], p.321, we have

$$\begin{aligned}\int\limits_0^\infty e^{-w\cosh(x-u+v)}dx \le & \sqrt{\frac{2}{w}}\int\limits_0^\infty e^{-\left(\sqrt{\frac{w}{2}}(x-u+v)\right)^2} d\left(\sqrt{\frac{w}{2}}(x-u+v)\right) \le \\ & \le 2\sqrt{\frac{2}{w}}\int\limits_0^\infty e^{-s^2}ds = \sqrt{\frac{2\pi}{w}}.\end{aligned} \tag{2.3}$$

It implies that

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-w\cosh(x-u+v)}|f(u)||g(v)||h(w)|dudvdwdx \leq$$

$$\leq \int_0^\infty\int_0^\infty\int_0^\infty \sqrt{\frac{2\pi}{w}}|h(w)||f(u)||g(v)|dudvdw =$$

$$= \sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw \int_0^\infty |f(u)|du. \int_0^\infty |g(v)|dv < +\infty.$$

Similarly,

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-w\cosh(x-u-v)}|f(u)||g(v)||h(w)|dudvdwdx < +\infty; \tag{2.4}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-w\cosh(x+u+v)}|f(u)||g(v)||h(w)|dudvdwdx < +\infty; \tag{2.5}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-w\cosh(x+u-v)}|f(u)||g(v)||h(w)|dudvdwdx < +\infty. \tag{2.6}$$

From formulae (2.1), (2.3), (2.4), (2.5) and (2.6) we get

$$\int_0^\infty |\underset{1}{*}(f,g,h)(x)|dx < +\infty.$$

This shows that the polyconvolution (2.1) belongs to $L_1(\mathbb{R}_+)$. We now prove the factorization equality (2.2). Indeed, we have

$$(F_s f)(y)(F_c g)(y)(K_{iy}h) = \int_0^\infty\int_0^\infty\int_0^\infty \sin(yu)\cos(yv)K_{iy}(w)f(u)g(v)h(w)dudvdw.$$

Using formula 2 in [1], p .130, we get

(2.7)

$$
\begin{aligned}
&(F_s f)(y)(F_c g)(y)(K_{iy} h) = \\
&= \frac{2}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \sin(yu)\cos(yv)\cos(y\alpha) e^{-w\cosh\alpha} f(u)g(v)h(w)\,dudvdwd\alpha = \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-w\cosh\alpha}[\sin y(u+v+\alpha) + \sin y(u+v-\alpha) + \\
&\quad + \sin y(u-v+\alpha) + \sin y(u-v-\alpha)] f(u)g(v)h(w)\,dudvdwd\alpha.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^\infty e^{-w\cosh\alpha}[\sin y(u+v+\alpha) + \sin y(u-v+\alpha)]d\alpha = \\
&= \int_0^\infty \sin xy[e^{-w\cosh(x-u-v)} + e^{-w\cosh(x-u+v)}]dx,
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
&\int_0^\infty e^{-w\cosh\alpha}[\sin y(u+v-\alpha) + \sin y(u-v-\alpha)]d\alpha = \\
&= -\int_0^\infty \sin xy[e^{-w\cosh(x+u+v)} + e^{-w\cosh(x+u-v)}]dx.
\end{aligned} \tag{2.9}
$$

From formulaes (2.7), (2.8), (2.9) and (2.1) we obtain

$$(F_s f)(y)(F_c g)(y)(K_{iy} h) = F_s(\underset{1}{*}(f,g,h))(y).$$

The proof is completed.

Corollary 2.1. *Let f, g be functions in $L_1(\mathbb{R}_+)$, and h a function in $L_1(\beta, \mathbb{R}_+)$. Then the following estimation holds*

$$\|\underset{1}{*}(f,g,h)\|_{L_1(\mathbb{R}_+)} \leq \|f\|_{L_1(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_1(\beta,\mathbb{R}_+)}.$$

Proof. From formulas (2.1), (2.3)-(2.6) we have

$$\int |\underset{1}{*}(f,g,h)(x)|dx \leq 2\int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw. \int_0^\infty |f(u)|du. \int_0^\infty |g(v)|dv.$$

Therefore, by the Definition 2, the proof is completed.

Remark 1. *In view of Corollary 2.1 if we fixed function h, the space* $L_1(\mathbb{R}_+)$ *equipped with polyconvolution multiplication, namely* $f*g := \underset{1}{*}(f,g,h)$, *is obviously a noncommutative Banach algebra.*

3. Relations with known convolutions

In order to construct other properties for the polyconvolution (2.1), we recall the following known convolutions and generalized convolutions. The convolution of two functions f and g for the Fourier transform is given by (see [8]):

$$(3.1)\qquad (f\underset{F}{*}g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x\in\mathbb{R}.$$

This convolution satisfies the so-called factorization equality

$$F(f\underset{F}{*}g)(y) = (Ff)(y)(Fg)(y), \quad \forall y\in\mathbb{R}.$$

The convolution of two functions f and g for the Fourier cosine is of the form (see [8])

$$(3.2)\qquad (f\underset{1}{*}g)(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(y)[g(|x-y|)+g(x+y)]dy, \quad x>0,$$

which satisfies the following factorization equality

$$F_c(f\underset{1}{*}g)(y) = (F_cf)(y)(F_cg)(y), \quad \forall y>0,\ f,g\in L_1(\mathbb{R}_+).$$

The convolution with a weight function $\gamma(x) = \sin x$ of two functions f and g for the Fourier sine transform was introduced in [4] as

$$(3.3)\qquad (f \overset{\gamma}{*} g)(x) =$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\mathrm{sign}(x+y-1)g(|x+y-1|)+$$
$$+ \mathrm{sign}(x-y+1)g(|x-y+1|)-$$
$$- g(x+y+1) - \mathrm{sign}(x-y-1)g(|x-y-1|)]dy, \quad x>0,$$

and the following factorization identity holds

$$F_s(f \overset{\gamma}{*} g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y>0,\ f,g \in L_1(\mathbb{R}_+).$$

The generalized convolution of two functions f, g for the Fourier sine and Fourier cosine transforms was studied in [8] as

$$(3.4)\qquad (f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[g(|x-u|) - g(x+u)]du, \quad x>0,$$

and the respective factorization identity is (see [8]):

$$F_s(f \underset{2}{*} g)(y) = (F_s f)(y).(F_c g)(y), \quad \forall y>0,\ f,g \in L_1(\mathbb{R}_+).$$

The generalized convolution of two functions f and g for the Fourier cosine and the Fourier sine transforms is defined in [12] as

$$(3.5)\quad (f \underset{3}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(u)[\mathrm{sign}(u-x)g(|u-x|) + g(u+x)]du,\ x>0.$$

For this generalized convolution the following factorization equality holds (see [12])

$$F_c(f \underset{3}{*} g)(y) = (F_s f)(y)(F_s g)(y),\ \forall y>0,\ f,g \in L_1(\mathbb{R}_+).$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ for the Fourier cosine and the Fourier sine transforms of f and g has the form

$$(3.6)\qquad (f \overset{\gamma}{\underset{1}{*}} g)(x) =$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-g(|x-u-1|)]du, x > 0.$$

It satisfies the factorization property

$$F_c(f \underset{1}{\overset{\gamma}{*}} g)(y) = \sin y\,(F_s f)(y)(F_c g)(y), \;\; \forall y > 0, \; f, g \in L_1(\mathbb{R}_+).$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ of f and g for the Fourier sine and Fourier cosine was studied in [11] as

$$(f \underset{2}{\overset{\gamma}{*}} g)(x) = \tag{3.7}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(u)[g(|x+u-1|)+g(|x-u-1|)-g(x+u+1)-g(|x-u+1|)]du, \; x > 0,$$

and satisfies the following factorization identity

$$F_s(f \underset{2}{\overset{\gamma}{*}} g)(y) = \sin y\,(F_c f)(y)(F_c g)(y), \;\; \forall y > 0, \; f, g \in L_1(\mathbb{R}_+).$$

Next, we consider the relations of the polyconvolution (2.1) with well-known convolutions.

Theorem 3.1. *Let f, g, h, l be functions in $L_1(\mathbb{R}_+)$, and p, k functions in $L\big(\frac{1}{\sqrt{w}}, \mathbb{R}_+\big)$. Then the following properties hold*

a) $\underset{1}{*}(\underset{1}{*}(f, g, p), l, k) = \underset{1}{*}(\underset{1}{*}(f, l, k), g, p)$;

b) $\underset{1}{*}(f \underset{2}{*} g, h, k) = \underset{1}{*}(f \underset{2}{*} h, g, k)$;

c) $\underset{1}{*}(f, g \underset{1}{\overset{\gamma}{*}} h, k) = \underset{1}{*}(g, f \underset{1}{\overset{\gamma}{*}} h, k)$;

d) $\underset{1}{*}(f \underset{2}{\overset{\gamma}{*}} g, h, k) = \underset{1}{*}(f \underset{2}{\overset{\gamma}{*}} h, g, k)$;

e) $\underset{1}{*}(f, g \underset{3}{*} h, k) = \underset{1}{*}(g, f \underset{3}{*} h, k)$;

f) $\underset{1}{*}(f, g \underset{1}{*} h, k) = \underset{1}{*}(f \underset{2}{*} *g, h, k)$;

g) $\underset{1}{*}(f \overset{\gamma}{*} g, h, k) = \underset{1}{*}(f, g \underset{1}{\overset{\gamma}{*}} h, k)$.

Here, the polyconvolution $\underset{1}{*}(\cdot, \; \cdot, \; \cdot)$ is defined by (2.1), convolutions $(\cdot \underset{2}{*} \cdot)$, $(\cdot \underset{1}{\overset{\gamma}{*}} \cdot)$, $(\cdot \underset{2}{\overset{\gamma}{*}} \cdot)$, $(\cdot \underset{3}{*} \cdot)$, $(\cdot \overset{\gamma}{*} \cdot)$ are respectively defined by (2.1), (3.4), (3.6), (3.7), (3.5) and (3.3). Moreover, the polyconvolution (2.1) can be presented by means *of the Fourier convolution (3.1) and the Fourier cosine convolution (3.2) as follow*

$$(3.8) \qquad \underset{1}{*}(f,g,p) = \sqrt{\frac{\pi}{2}} \int\limits_0^\infty p(w)\,\big(\,(g \underset{1}{*} e^{-w\cosh t}) \underset{F}{*} (f(|t|)\text{sign } t)\,\big)\,(x)dw.$$

Proof. First, we prove the assertion a). From Theorem 2.1 we have

$$\begin{aligned} F_s(\underset{1}{*}(\underset{1}{*}(f,g,p),l,k)(y) =& F_s(\underset{1}{*}(f,g,p))(y)(F_c l)(y)(K_{iy}k) = \\ =& (F_s f)(y)(F_c g)(y)(K_{iy}p)(F_c l)(y)(K_{iy}k). \end{aligned}$$

Again, by virtue of Theorem 2.1 we get

$$F_s(\underset{1}{*}(\underset{1}{*}(f,g,p),l,k))(y) = F_s(\underset{1}{*}(\underset{1}{*}(f,l,k),g,p))(y).$$

This shows the assertion a). The assertions b)- f) can be proved similarly.

We now prove the assertion g). From Theorem 2.1 and the factorization equality of the convolution (3.3) we obtain

$$\begin{aligned} F_s(\underset{1}{*}(f \overset{\gamma}{*} g,h,k))(y) =& F_s(f \overset{\gamma}{*} g)(y)(F_c h)(y)(K_{iy}k) = \\ =& \sin y.(F_s f)(y)(F_s g)(y)(F_c h)(y)(K_{iy}k). \end{aligned}$$

Therefore, by using the generalized convolution (3.6) and the Theorem 2.1 we have

$$\begin{aligned} F_s(\underset{1}{*}(f \overset{\gamma}{*} g,h,k))(y) =& (F_s f)(y)F_c(g \overset{\gamma}{\underset{1}{*}} h)(y)(K_{iy}k) = \\ =& F_s(\underset{1}{*}(f,g \overset{\gamma}{\underset{1}{*}} h,k))(y), \end{aligned}$$

which implies g).

Finally, we prove the relation (3.8), from the definition (2.1) of the polyconvolution and the convolution (3.2) we have

(3.9)

$$\underset{1}{*}(f,g,p)(x) = \frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty f(u)p(w)[(g \underset{1}{*} e^{-w\cosh t})(x-u)-(g \underset{1}{*} e^{-\cosh t})(x+u)]dudw.$$

From (3.9) and changing variables we obtain (3.8). The proof is complete.

4. Applications to solving generalized integral equations

Not many integral equations and systems of integral equations of the second kind can be solved in closed form. The polyconvolution (2.1) introduced in this paper allows us to get the solutions in closed form for a class of integral equations with the Toeplitz plus Hankel kernel (1.1) and also a class of systems of generalized integral equations.

a) First, consider the integral equation with the Toeplitz plus Hankel kernel (1.1)

$$f(x) + \int_0^\infty [k_1(x+y) + k_2(x-y)]f(y)dy = g(x), \quad x > 0,$$

in case of the Hankel kernel k_1 and the Toeplitz kernel k_2 are defined as follow

$$k_1(t) = \tag{4.1}$$

$$= -\frac{\lambda_2 k(t)}{\sqrt{2\pi}} - \frac{\lambda_1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty g(v)h(w)[e^{-w\cosh(t+v)} + e^{-w\cosh(t-v)}]dvdw;$$

$$k_2(t) = \tag{4.2}$$

$$= \frac{\lambda_2 k(|t|)}{\sqrt{2\pi}} + \frac{\lambda_1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty g(v)h(w)[e^{-w\cosh(t+v)} + e^{-w\cosh(t-v)}]dvdw,$$

where g, h, k, p are known functions, λ_1, λ_2 are given constants. This particular case of equation (1.1) can be solved in closed form with the help of polyconvolution (2.1), which seems to be difficult to solve by using other known generalized convolutions.

Lemma 1. *For $f \in L_1(\mathbb{R}_+)$ and $g \in L_1\big(\frac{1}{\sqrt{v}}, \mathbb{R}_+\big)$, the generalized convolution $(f*g)(x)$ belongs to $L_1(\mathbb{R}_+)$ and the respective factorization equality is*

$$F_c(f*g)(y) = (F_cf)(y)(K_{iy}g), \ \ \forall y > 0, \tag{4.3}$$

where

$$(f*g)(x) = \frac{1}{2}\int_0^\infty \int_0^\infty [e^{-v\cosh(x+u)} + e^{-v\cosh(x-u)}]f(u)g(v)dudv, \ x > 0. \tag{4.4}$$

The proof of this lemma is similar to that of Theorem 2.1, so we omit it.

Theorem 4.1. *Suppose that* $p, g, k \in L_1(\mathbb{R}_+)$ *and* $h \in L_1\left(\frac{1}{\sqrt{v}}, \mathbb{R}_+\right)$ *satisfy the condition*

$$1 + \lambda_1 (F_c g)(y)(K_{iy} h) + \lambda_2 (F_c k)(y) \neq 0.$$

Then the Toeplitz plus Hankel integral equation (1.1) with Toeplitz and Hankel kernels defined by (4.1), (4.2) has a unique solution in $L_1(\mathbb{R}_+)$ *whose closed form is*

$$f(x) = p(x) - (p \underset{2}{*} l)(x).$$

Here, $l \in L_1(\mathbb{R}_+)$ *is defined uniquely by*

$$(F_c l)(y) = \frac{F_c(\lambda_1 (g * h) + \lambda_2 k)(y)}{1 + F_c(\lambda_1 (g * h) + \lambda_2 k)(y)},$$

where the convolutions $(\cdot \underset{2}{*} \cdot)$ *and* $(\cdot * \cdot)$ *are defined by (3.4) and (3.5), respectively.*

Proof. The Toeplitz plus Hankel integral equation (1.1) whose Toeplitz and Hankel kernels defined by (4.1), (4.2) can be rewritten in the form

$$f(x) + \lambda_1 (\underset{1}{*}(f, g, h)(x)) + \lambda_2 (f \underset{2}{*} h)(x) = p(x), \ x > 0. \tag{4.5}$$

By Theorem 2.1 and the generalized convolution (3.4), we have

$$(F_s f)(y) + \lambda_1 (F_s f)(y).(F_c g)(y).(K_{iy} h) + \lambda_2 (F_s f)(y).(F_c k)(y) = (F_s p)(y).$$

Therefore, by the given condition

$$(F_s f)(y) = (F_s p)(y) \left(1 - \frac{\lambda_1 (F_c g)(y)(K_{iy} h) + \lambda_2 (F_c k)(y)}{1 + \lambda_1 (F_c g)(y)(K_{iy} h) + \lambda_2 (F_c k)(y)}\right).$$

Using the Lemma 1 we obtain

$$(F_s f)(y) = (F_s p)(y) \left(1 - \frac{F_c(\lambda_1 (g * h) + \lambda_2 k)(y)}{1 + F_c(\lambda_1 (g * h) + \lambda_2 k)(y)}\right). \tag{4.6}$$

Recall that the Wiener-Levy theorem (see [7], p. 63) states that if f is the Fourier transform of an $L_1(\mathbb{R})$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier transform of an $L_1(\mathbb{R})$ function. For the Fourier cosine transform it means that if f is the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y), \forall y \in \mathbb{R}_+\}$, and $\varphi(0) = 0$, then $\varphi(f)$ is also the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function.

By the given condition, the function $\varphi(z) = \frac{z}{1+z}$ satisfies conditions of the Wiener-Levy theorem, and therefore, there exists a unique function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{F_c(\lambda_1(g * h) + \lambda_2 k)(y)}{1 + F_c(\lambda_1(g * h) + \lambda_2 k)(y)}. \tag{4.7}$$

From (4.6), (4.7) and the generalized convolution (3.4) we get

$$f(x) = p(x) - (p \underset{2}{*} l)(x) \in L_1(\mathbb{R}_+).$$

b) Next, we consider the following system of two integral equations for $x > 0$.

$$f(x) + \frac{\lambda_1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty \theta(x,u,v,w) h_1(u) g(v) h(w) dudvdw +$$

$$+ \frac{\lambda_2}{\sqrt{2\pi}} \int_0^\infty \theta_1(x,u) g(u) du = p(x), \tag{4.8}$$

$$\lambda_3 \int_0^\infty \int_0^\infty \int_0^\infty \theta_2(x,u,v,w) f(u) k_2(v) h_2(w) dudvdw +$$

$$+ \lambda_4 \int_0^\infty \theta_3(x,u) f(u) du + g(x) = q(x).$$

Here $\theta(x,u,v,w)$ is defined as in Definition 1, and

$$\theta_1(x,u) = \frac{1}{\sqrt{2\pi}} [k(|x-u|) - k(x+u)],$$

$$\theta_2(x,u) = \frac{1}{2\pi} \int_0^\infty k_2(v)[h_2(|x+u-v|) + h_2(|x-u+v|) - h_2(|x-u-v|) -$$

$$- h_2(x+u+v)]dv,$$

$$\theta_3(x,u) = \frac{1}{\sqrt{2\pi}} [\mathrm{sign}(u-x) k_1(|u-x|) + k_1(u+x)].$$

$h, h_1, h_2, k, k_1, k_2, p, q$ are known functions, λ_i $(i = \overline{1, 4})$ are constants, and f, g are unknown functions.

Recall that the polyconvolution for the Fourier cosine and the Fourier sine integral transforms has the form (see [11])

$$\begin{aligned} *(f,g,h)(x) = \frac{1}{2\pi}\int_0^\infty\int_0^\infty f(u)g(v)[h(|x+u-v|)+h(x-u+v|)-\\ - h(|x-u-v|)-h(x+u+v)]dudv,\\ x>0. \end{aligned} \tag{4.9}$$

This polyconvolution satisfies the following factorization property

$$F_c(*(f,g,h))(y) = (F_s f)(y).(F_s g)(y).(F_c h)(y), \quad \forall y > 0.$$

Theorem 4.2. *Given $h_1, h_2, k, k_1, k_2, p, q \in L_1(\mathbb{R}_+)$ and $h \in L_1\big(\frac{1}{\sqrt{v}}, \mathbb{R}_+\big)$ so that*

$$1 - (F_c\xi)(y) \neq 0, \qquad \forall y > 0,$$

where

$$\begin{aligned} \xi(x) = \lambda_1\lambda_3(\underset{1}{*}(k_2,h_2,h)\underset{3}{*}h_1)(x) + \lambda_2\lambda_3(*(k_2,k,h_2))(x)+\\ + \lambda_1\lambda_4((k_1\underset{3}{*}h_1)*h)(x) + \lambda_2\lambda_4(k\underset{3}{*}h)(x). \end{aligned}$$

Then the system (4.8) has a unique solution in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ whose closed forms are given as follow

$$\begin{aligned} f(x) = p(x) - \underset{1}{*}(h_1,q,h)(x) - \lambda_2(k\underset{2}{*}q)(x) + (p\underset{2}{*}l)(x) - \lambda_1(\underset{1}{*}(h_1,q,h)\underset{2}{*}l)(x)-\\ - \lambda_2((k\underset{2}{*}q)\underset{2}{*}l)(x),\\ g(x) = q(x) - \underset{1}{*}(k_2,p,h_2)(x) - \lambda_4(p\underset{3}{*}k_1)(x) + (q\underset{1}{*}l)(x)-\\ - \lambda_3(*(k_2,p,h_2)\underset{1}{*}l)(x) - \lambda_4((p\underset{3}{*}k_1)\underset{1}{*}l)(x). \end{aligned}$$

Here, $l \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c l)(y) = \frac{(F_c\xi)(y)}{1-(F_c\xi)(y)}, \quad y > 0,$$

where the convolutions $(\cdot\underset{2}{*}\cdot)$, $(\cdot\underset{1}{*}\cdot)$, $(\cdot\underset{3}{*}\cdot)$ are defined by (3.4), (3.2), (3.5), the polyconvolutions $\underset{1}{*}(\cdot,\ \cdot,\ \cdot)$ and $*(\cdot,\ \cdot,\ \cdot)$ are defined by (2.1) and (4.9).

Proof. The system (4.8) can be rewritten in the form

$$f(x) + \lambda_1\big(\underset{1}{*}(h_1, g, h)(x)\big) + \lambda_2(k_2 \underset{2}{*} g)(x) = p(x),$$
$$\lambda_3\big(*\,(k_2, f, h_2)(x)\big) + \lambda_4(k_1 \underset{3}{*} f)(x) + g(x) = q(x).$$

Using Theorem 2.1, generalized convolutions (3.4), (3.5) and the polyconvolution (4.9) we obtained

$$(F_s f)(y) + \big(\lambda_1(F_s h_1)(y)K_{iy}h + \lambda_2(F_s k)(y)\big)(F_c g)(y) =(F_s p)(y),$$
$$\big(\lambda_3(F_s k_2)(y)(F_c h_2)(y) + \lambda_4(F_s k_1)(y)\big)(F_s f)(y) + (F_c g)(y) =(F_c q)(y).$$

By Theorem 2.1, Lemma 1 and the generalized convolution (3.5) we have

$$\Delta = \tag{4.10}$$

$$= \begin{vmatrix} 1 & \lambda_1(F_s h_1)(y)K_{iy}h + \lambda_2(F_s k)(y) \\ \lambda_3(F_s k_2)(y)(F_c h_2)(y) + \lambda_4(F_s k_1)(y) & 1 \end{vmatrix} =$$
$$= 1 - (F_c \xi)(y).$$

In view of the Wiener-Levy Theorem (see [7]), and the condition of the theorem, there is a unique function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{(F_c \xi)(y)}{1 - (F_c \xi)(y)}. \tag{4.11}$$

From (4.10) and (4.11) we get

$$\frac{1}{\Delta} = 1 + (F_c l)(y). \tag{4.12}$$

On the other hand, using Theorem 2.1 and the generalized convolution (3.4) gives

$$\Delta_1 = \begin{vmatrix} (F_s p)(y) & \lambda_1(F_s h_1)(y)K_{iy}h + \lambda_2(F_s k)(y) \\ (F_c q)(y) & 1 \end{vmatrix} =$$

$$=(F_s p)(y) - \lambda_1 F_s(\underset{1}{*}(h_1, q, h))(y) - \lambda_2 F_s(k \underset{2}{*} q)(y).$$

Hence, from (4.12) it follows that

$$(F_s f)(y) = \frac{\Delta_1}{\Delta} =(F_s p)(y) - \lambda_1 F_s(\underset{1}{*}(h_1, q, h))(y) - \lambda_2 F_s(k \underset{2}{*} q)(y) +$$
$$+ F_s(p \underset{2}{*} l)(y) - \lambda_1 F_s(\underset{1}{*}(h_1, q, h) \underset{2}{*} l)(y) - \lambda_2 F_s((k \underset{2}{*} q) \underset{2}{*} l)(y).$$

This implies
(4.13)
$$f(x) = p(x) - (\underset{1}{*}(h_1, q, h))(x) - \lambda_2(k \underset{2}{*} q)(x) + (p \underset{2}{*} l)(x) -$$
$$-\lambda_1(\underset{1}{*}(h_1, q, h) \underset{2}{*} l)(x) - \lambda_2((k \underset{2}{*} q) \underset{2}{*} l)(x).$$

Similarly, from convolution (3.2) and the generalized convolution (3.5) we have

$$\Delta_2 = \begin{vmatrix} 1 & (F_s p)(x) \\ \lambda_3 (F_s k_2)(x)(F_c h_2)(x) + \lambda_4 (F_s k_1)(x) & (F_c q)(x) \end{vmatrix} =$$
$$= (F_c q)(x) - \lambda_3 F_c(*(k_2, p, h_2))(x) - \lambda_4 F_c(p \underset{3}{*} k_1)(x).$$

Using formula (4.12) and the generalized convolution (3.2) we get

$$F_c g = \frac{\Delta_2}{\Delta} =$$
$$= F_c q - \lambda_3 F_c(*(k_2, p, h_2)) - \lambda_4 F_c(p \underset{3}{*} k_1) + F_c(q \underset{1}{*} l) - \lambda_3 F_c(*(k_2, p, h_2) \underset{1}{*} l) -$$
$$- \lambda_4 F_c((p \underset{3}{*} k_1) \underset{1}{*} l).$$

This shows that

$$g(x) = q(x) - \underset{1}{*}(k_2, p, h_2)(x) - \lambda_4(p \underset{3}{*} k_1)(x) + (q \underset{1}{*} l)(x) -$$
$$- \lambda_3(*(k_2, p, h_2) \underset{1}{*} l)(x) - \lambda_4((p \underset{3}{*} k_1) \underset{1}{*} l)(x). \tag{4.14}$$

The pair (f, g), defined by fomulas (4.13) and (4.14), is a closed form solution of the system (4.8) in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$. The proof is complete.

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