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ON COMPLETELY MULTIPLICATIVE COMPLEX VALUED FUNCTIONS

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Abstract. Here we determine completely multiplicative complex valued functions with nearly Gaussian integer values.

1. Introduction

I. Kátai and B. Kovács [1] determined completely multiplicative real valued functions, $f : \mathbb{N} \to \mathbb{R}$ with nearly integer values, such that $\|f(n)\| \to 0$ as $n \to \infty$, where

$$\|z\| = \min_{n\in\mathbb{N}} |z - n|.$$

In this paper, we determine such completely multiplicative complex valued functions on the set of positive integers with values nearly in $\mathbb{Z}[i]$.

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2. Preliminaries and results

A complex valued function $f(n)$ is said to be completely multiplicative if $f(mn) = f(m)f(n)$ holds for each pair of positive integers. Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function and let, for any complex number z, $\|z\|$ be defined as

$$\|z\| = \min_{\gamma \in \mathbb{Z}[i]} |z - \gamma|.$$

We determine the class of such completely multiplicative functions for which

$$(1) \qquad \|f(n)\| \to 0 \qquad (n \to \infty).$$

Definition 1. We shall say that θ is a generalized Pisot number with respect to Gaussian integers if there exists a polynomial $\phi(z) \in \mathbb{Z}[i][z]$ with leading coefficient 1, and $\phi(z) = \prod_{j=1}^{r} (z - \theta_j)$, $\theta_1 = \theta$, $|\theta| > 1$ and all the conjugates, $\theta_2, \dots, \theta_r$ are in the domain $|z| < 1$.

We call a generalized Pisot number with respect to Gaussian integers as *Gaussian Pisot number*. Naturally, $\theta \in \mathbb{Z}[i]$ is a Gaussian Pisot number, since $\phi(z) = z - \theta \in \mathbb{Z}[i][z]$.

A Gaussian Pisot number θ satisfies the relation

$$(2) \qquad \|\theta^n\| \to 0 \qquad (n \to \infty).$$

Lemma 1. *Let β be an algebraic number, $f(n)$ be completely multiplicative function with values in $\mathbb{Q}(\beta)$. Let $\beta_2, \dots, \beta_r$ be the conjugates of β (with respect to $\mathbb{Z}[i]$). Let $\phi_j(n)$ denote the conjugate of $f(n)$ defined by the substitution $\beta \to \beta_j$. Then ϕ_j are completely multiplicative functions as well.*

Proof. Let $f(n) = r_n(\beta)$. Then $\phi_j(n) = r_n(\beta_j)$. Since $r_{mn}(\beta) = f(mn) = = f(m).f(n) = r_m(\beta).r_n(\beta)$, therefore $\phi_j(mn) = r_{mn}(\beta_j) = r_m(\beta_j)r_n(\beta_j) = = \phi_j(m)\phi_j(n)$.

Lemma 2. *Let β be an algebraic number and $f(n)$ a completely multiplicative function and the values $f(n)$ are in $\mathbb{Z}[i](\beta)$. Assume that $|\phi_j(p)| < 1$ and*

$$(3) \qquad \phi_j(p) \to 0 \quad \textit{as} \quad p \to \infty \qquad (j = 2, \dots, r),$$

where p runs over the set of primes. Then (1) holds.

Proof. From (3) it is obvious that $\phi_j(n) \to 0$ $(n \to \infty)$. Furthermore ($\phi_j(n)$ being algebraic),

$$f(n) + \phi_2(n) + \ldots + \phi_r(n) = E_n = \text{Gaussian integer},$$

which gives

$$\|f(n)\| \to 0 \qquad (n \to \infty). \tag{4}$$

The following Lemma is a generalization of Lemma 3 in [1].

Lemma 3. *Let α be an algebraic number with $|\alpha| > 1$, $\lambda \neq 0$ be a complex number and*

$$\|\lambda\alpha^n\| \to 0 \quad (n \to \infty). \tag{5}$$

Then α is a Gaussian Pisot number and $\lambda \in \mathbb{Q}(\alpha)$.

Proof. The proof follows immediately from a more general result by I. Környei (see Theorem 1 in [2]).

Lemma 4. *Let $f(n)$ be a completely multiplicative function for which (1) holds. If $|f(n_0)| > 1$ for at least one n_0, then either $f(n) = 0$ or $|f(n)| \geq 1$ for each value of n.*

Proof. Assume on the contrary that $0 < |f(m_0)| < 1$. Let $b = |f(n_0)|$, $a = |f(m_0)|$, and $x_0 = [-3\log a] + 1$. For infinitely many k, l pairs of positive integers we have,

$$\frac{-2x_0}{\log a} > k + l\frac{\log b}{\log a} > \frac{-x_0}{\log a},$$

since the length of the interval $\left(\frac{-x_0}{\log a}, \frac{-2x_0}{\log a}\right)$ is at least three. For such pairs k, l we have $2^{-2x_0} < a^k b^l < 2^{-x_0}$. Consequently

$$2^{-2x_0} < |f({m_0}^k {n_0}^l)| = |a^k b^l| < 2^{-x_0}$$

which contradicts (1).

Lemma 5. *Let $f(n)$ be a completely multiplicative function satisfying (1). Assume that there exists an m for which $|f(m)| > 1$. Let $\mathcal{P}_1$ be the set of those primes p for which $f(p) \neq 0$. Then the values $f(p)$ are Gaussian Pisot numbers for each $p \in \mathcal{P}_1$, and for every $p_1, p_2 \in \mathcal{P}$, we have $\mathbb{Q}(\alpha_{p_1}) = \mathbb{Q}(\alpha_{p_2})$, $\alpha_{p_1} = f(p_1), \alpha_{p_2} = f(p_2)$.*

Proof. Let $f(m) = \alpha$. Since $|\alpha| > 1$ and $\|f(m^k)\| = \|\alpha^k\| \to 0$ $(k \to \infty)$, by Lemma 3, we say that $f(m)$ is a Gaussian Pisot number.

Now, let n be an arbitrary natural number for which $f(n) \neq 0$. Since $\|f(nm^k)\| = \|f(n)\alpha^k\| \to 0$ $(k \to \infty)$, from Lemma 3, we deduce that $f(n) \in \in \mathbb{Q}(\alpha)$. Hence, $\beta = f(n) \in \mathbb{Q}(\alpha)$. Since $\beta \neq 0$, from Lemma 4 we get that $|\beta| > 1$, and by repeating the above argument for β, we deduce that β is a Gaussian Pisot number and $\alpha \in \mathbb{Q}(\beta)$. So, $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$ and hence the assertion is proved.

Lemma 6. *Let $f(n)$ be a completely multiplicative function satisfying the relation*

$$\|f(n)\| \leq \varepsilon(n), \tag{6}$$

where $\varepsilon(n)$ is a monotonically decreasing function. Then there are the following possibilities:

a) *f takes values in $\mathbb{Z}[i]$ for every n.*

b) *For a suitable n, $0 < |f(n)| < 1$. Then $|f(n)| \to 0$ as $n \to \infty$.*

c) *For a suitable m, $|f(m)| > 1$. Let $\mathcal{P}_1$ denote the whole set of those primes p for which $f(p) \neq 0$. Then there exists a Gaussian Pisot number Θ such that $\mathbb{Q}(f(p)) = \mathbb{Q}(\Theta)$ for each $p \in \mathcal{P}_1$.*

Proof. The relation (6) involves (5). If $0 < |f(n)| < 1$ then from Lemma 4 we have $|f(m)| \leq 1$ for every m. If $|f(m)| = 1$, then $\|f(nm^k)\| = \|f(n)\|$ as $k \to \infty$, that contradicts (1). Consequently, $|f(m)| < 1$ for each $m > 1$. Assume that there exists a subsequence $n_1 < n_2 < \ldots$ such that $f(n_j) \to 1$. Then $f(nn_j) \to f(n)$ $(j \to \infty)$ which contradicts (1). Consequently $f(m) \to 0$ as $m \to \infty$.

The assertion (c) of the lemma is an immediate consequence of Lemma 5.

Theorem 1. *Let $f(n)$ be a completely multiplicative complex valued function that takes on at least one value n_0 for which $|f(n_0)| > 1$. Let $\mathcal{P}_1$ denote the set of primes p for which $f(p) \neq 0$.*

If (1) holds, then the values $f(p) = \alpha_p$ are Gaussian Pisot numbers, for each $p_1, p_2 \in \mathcal{P}_1$ we have $\mathbb{Q}(\alpha_{p_1}) = \mathbb{Q}(\alpha_{p_2})$. Let Θ denote one of the values α_p $(p \in \mathcal{P}_1)$, $\Theta_2, \ldots, \Theta_r$ its conjugates $(i = 2, \ldots, r)$, $\phi_2(n), \ldots, \phi_r(n)$ be defined as in Lemma 1. Then

$$\phi_j(n) \to 0 \quad \textit{as} \quad n \to \infty, \quad j = 2, \ldots, r. \tag{7}$$

Conversely, let us assume that the values $f(p)$ are zeros or Gaussian Pisot numbers from a given algebraic number field $\Omega(\Theta)$. If

$$\phi_j(p) \to 0 \quad \textit{as} \quad p \to \infty, \quad j = 2, \ldots, r, \tag{8}$$

then (1) holds.

Proof. Let us assume that (1) holds. From Lemma (5) we get that the non-zero values of $f(n)$ are Gaussian Pisot numbers from a given algebraic number field $\Omega(\Theta)$. Then $|\phi_j(n)| < 1$ $(j = 2, \dots, r)$. Let us consider the vector

$$\psi(n) = (\phi_2(n), \dots, \phi_r(n)),$$

and denote by X the set of the limit points of $\psi(n)$ $(n \to \infty)$. Let

$$(x_2, \dots, x_r) \in X.$$

Since

$$f(n) + \phi_2(n) + \dots + \phi_r(n) = \text{Gaussian integer},$$

and $\|f(n)\| \to 0$, as $n \to \infty$, we get that $x_2 + \dots + x_r =$ Gaussian integer and $|x_j| \le 1$.

Let m_j be such a sequence for which

$$\psi(m_j) \to (x_2, \dots, x_r).$$

Then $\psi({m_j}^k) \to ({x_2}^k, \dots, {x_r}^k)$, ${x_2}^k + \dots + {x_r}^k =$ Gaussian integer for every $k = 1, 2, \dots$. This can happen only in the case when all x_j's are Gaussian integers. Since $|x_j| \le 1$ for every $j = 2, \dots, r$, therefore $x_j \in \{0, 1, -1, i, -i\}$. Hence, for all j, either $x_j = 0$ or $|x_j| = 1$. Now, let n be fixed such that $f(n) \ne 0$. Then $\phi_j(n) \ne 0$ and $|\phi_j(n)| < 1$. Consequently,

$$\psi(nm_j) \to (\phi_2(n)x_2, \dots, \phi_r(n)x_r) \in X.$$

Let $y_j = x_j\phi_j(n)$ $(j = 2, \dots, r)$. If $x_{j_0} \ne 0$ for some j_0, then $0 < |y_{j_0}| = |\phi_{j_0}(n)| < 1$ and $(y_2, \dots, y_r)$ is an element of X for which there is a component y_{j_0} such that $|y_{j_0}| < 1$ and $y_{j_0} \ne 0$, which is not possible. Consequently, we have (7).

The converse assertion is an immediate consequence of Lemma 2.

Theorem 2. *Let $f(n)$ be a completely multiplicative complex valued function satisfying the condition (6). Let us assume that $f(n) \not\to 0$, and that $f(n)$ takes on at least one value other than Gaussian integer. Then the first assertion in Theorem 1 holds.*

Proof. The proof follows immediately from Lemma 5 and Theorem 1.

Remark 1. *A similar result can be obtained for completely multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ with values that are nearly integers in imaginary quadratic fields, in which case*

$$\|z\| = \min_{\gamma \in \mathcal{O}_K} |z - \gamma|,$$

where $\mathcal{O}_K$ be the ring of integers of $K = \mathbb{Q}(\sqrt{d})$, d square free, $d < 0$.

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