Annales Univ. Sci. Budapest., Sect. Comp. 37 (2012) 339–354

APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY STEP FRACTIONAL BROWNIAN MOTION

Judit Robu and Anna Soós

(Cluj - Napoca, Romania)

Communicated by Imre Kátai

(Received January 15, 2011; revised February 20, 2012;
accepted February 29, 2012)

Abstract. The aim of this paper is to approximate the solution of a stochastic differential equation driven by step fractional Brownian motion using a series expansion for the noise. We prove that the solution of the approximating equations converges in probability to the solution of the given equation. We illustrate the approximation through the model for the price of risky assets from mathematical finance. The figures are generated in GeoGebra.

1. Introduction

The notion of fractional Brownian motion (FBM) was introduced by Kolmogorov in 1940, but Mandelbrot and Van Ness's paper [10] emphasizes the relevance of FBM to modelling natural phenomena. Long-range dependence, self-similarity and smoothness of the sample paths make FBM a useful tool in different areas like internet traffic modelling, turbulence, image processing.

Key words and phrases: Stochastic differential equations, approximation, step fractional Brownian motion.
2010 Mathematics Subject Classification: 60H10, 60G22, 60H05.
https://doi.org/10.71352/ac.37.339

A constant Hurst parameter is too rigid for some applications, for example in finance and turbulence. Therefore different generalizations of FBM have been introduced. The multifractional Brownian motion (MFBM) was proposed in [11] and [1] replacing the Hurst parameter H of FBM by a scaling function $t \to H(t)$. In some fields (image analysis and control of the internet traffic) the interesting information is the location of change point of the function H. In [2] the step fractional Brownian motion (SFBM) was defined as the generalization of FBM with a piecewise constant function H.

Investigations concerning stochastic differential equations driven by a fractional Brownian motion or a more general fractional process have been done by L. Coutin and L. Decreusefond [3], L. Coutin and Z. Qian [4], M.L. Kleptsyna, P.E. Kloeden and V.V. Anh [7], F. Klingenhöfer and M. Zähle [8], M. Zähle [16], [17] and many others. The main difficulty raised by the fractional Brownian motion and the processes related to it, is that they are not Markovian, even more, they are not even semimartingales. Therefore a new approach to stochastic fractional calculus was developed. There exist several ways to define the stochastic integral pathwise and related techniques, Dirichlet forms, anticipating techniques using Malliavin calculus and Skorohod integration (e.g. [15], [5]). In this paper we use the approach of M. Zähle [15], based on the ideas of Lebesgue-Stieltjes integrals and fractional calculus [12] for step fractional Brownian motion.

We consider the following stochastic differential equation driven by step fractional Brownian motion

$$(1.1)\quad X(t) = X_0 + \int_0^t F(X(s), s)ds + \int_0^t G(X(s), s)dB(s),\ \ t \in [0, T].$$

We assume that $F \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$, $G \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ and for each $t \in [0, T]$ the functions $F(\cdot, t)$, $\dfrac{\partial G(\cdot, t)}{\partial x}$, $\dfrac{\partial G(\cdot, t)}{\partial t}$ are locally Lipschitz with probability 1. F and G can be random.

The step fractional Brownian motion $\Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index H can be approximated by using the series expansion given in [13].

Let J_ν be the Bessel function of first type of order ν and let $x_1(t) < x_2(t) < < \ldots$ be the positive, real zeros of $J_{-H(t)}$, while $y_1(t) < y_2(t) < \ldots$ are the positive, real zeros of $J_{1-H(t)}$. We consider two independent sequences of centered Gaussian random variables $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ we have

$$\mathrm{Var} X_n = \frac{2c^2_{H(t)}}{x_n^{2H(t)} J^2_{1-H(t)}(x_n)}, \quad \mathrm{Var} Y_n = \frac{2c^2_{H(t)}}{y_n^{2H(t)} J^2_{-H(t)}(y_n)},$$

where

$$c_{H(t)}^2 = \frac{\sin(\pi H(t))}{\pi}\Gamma(1+2H(t)).$$

The SFBM $\Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index H can be written as

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1].$$

We approximate equation (1.1) for each $N \in \mathbb{N}$ by

$$X_N(t) = X_0 + \int_0^t F(X_N(s), s)ds + \int_0^t G(X_N(s), s)dB_N(s), \tag{1.2}$$

where

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1], N \in \mathbb{N}.$$

We will show that equation (1.2) has a local solution, which converges in probability to the solution of (1.1) in the interval where the solutions exist. We illustrate the approximation through a model for the price of risky assets from mathematical finance. The figures are generated in GeoGebra.

2. Series expansion for step fractional Brownian motion $\boldsymbol{B}$

A Gaussian random process $\big(B(t)\big)_{t\geq 0}$ is called *fractional Brownian motion with Hurst index* $H \in]0,1[$, if it has zero mean, continuous sample paths and covariance function

$$E\Big(B(s)B(t)\Big) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |s-t|^{2H}\Big).$$

Note that if $H = \frac{1}{2}$, then the fractional Brownian motion is the ordinary standard Brownian motion.

The step fractional Brownian motion is an extension of FBM where the Hurst parameter H is replaced by a piecewise constant function $H : R \rightarrow]0,1[$

$$H(t) = \sum_{i=0}^{K} a_i \mathbf{1}_{[\tau_i, \tau_{i+1}[}(t),$$

where $\tau_0 = -\infty, \tau_{K+1} = \infty$ and $\tau_1, \tau_2, ...\tau_K$ is an increasing finite sequence of real numbers.

The step fractional Brownian motion B has Hölder continuous paths on any finite interval $[0, T]$ (see [5]). If $H(t) \neq \frac{1}{2}$, then B is not a semimartingale, so the classical stochastic integration does not work. But the Hölder continuity of B will ensure the existence of integrals

$$\int_0^T G(u)dB(u),$$

defined in terms of fractional integration ([15] and [17]).

For $\nu \neq -1, -2, \ldots$ *the Bessel function* J_ν *of the first type of order* ν is defined on the region $\{z \in \mathbb{C} : |\arg z| < \pi\}$ as the absolutely convergent sum

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$

It is known that for $\nu > -1$ the function J_ν has a countable number of real, positive simple zeros (see [14]). Let $x_1(t) < x_2(t) < \ldots$ be the positive, real zeros of $J_{-H(t)}$ and let $y_1(t) < y_2(t) < \ldots$ be the positive, real zeros of $J_{1-H(t)}$.

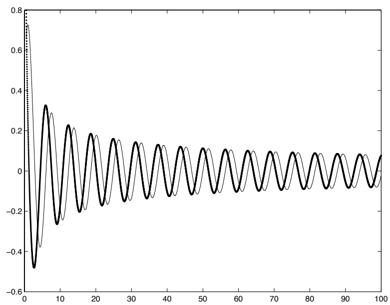


Figure 1. Bessel functions: J_{-H} (with '·'), J_{1-H} (with '-'), $H = 0.65$

Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two independent sequences of independent Gaussian random variables such that for each $n \in \mathbb{N}$ we have

$$E(X_n) = E(Y_n) = 0$$

and

$$\mathrm{Var} X_n = \frac{2c_{H(t)}^2}{x_n^{2H(t)} J_{1-H(t)}^2(x_n)}, \quad \mathrm{Var} Y_n = \frac{2c_{H(t)}^2}{y_n^{2H(t)} J_{-H(t)}^2(y_n)},$$

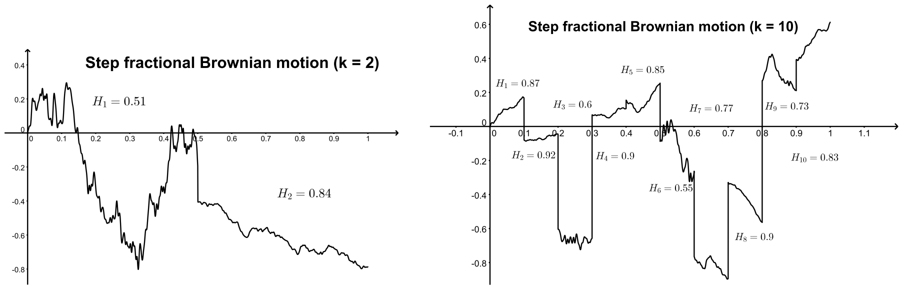
where

$$c^2_{H(t)} = \frac{\sin(\pi H(t))}{\pi}\Gamma(1 + 2H(t)).$$

It is proved in [13] that the random process $\Big(B(t)\Big)_{t\in[0,1]}$ given by

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1]$$

is well defined and both series converge absolutely and uniformly in $t \in [0,1]$with probability 1. The process B is a step fractional Brownian motion with Hurst index H.



a) $K = 2$ b) $K = 10$

Figure 2. Approximation B_N of step fractional Brownian motion

For each $N \in \mathbb{N}$ we define the process

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1]. \tag{2.1}$$

Then using the above mentioned result from [6] we have

$$P(\lim_{N\to\infty} \sup_{t\in[0,1]} |B(t) - B_N(t)| = 0) = 1. \tag{2.2}$$

We will use the following result:

Theorem 2.1. *For all $N \in \mathbb{N}$ the approximating processes $(B_N(t))_{t\in[0,1]}$ are Lipschitz continuous with probability* 1 .

Proof. Let $N \in \mathbb{N}$ be fixed. We write

$$|B_N(t) - B_N(s)| \le \sum_{n=1}^{N} \left| \frac{\sin(x_n t) - \sin(x_n s)}{x_n} X_n \right| + \sum_{n=1}^{N} \left| \frac{\cos(y_n s) - \cos(y_n t)}{y_n} Y_n \right|.$$

The functions sin and cos are Lipschitz continuous. Therefore

$$|B_N(t) - B_N(s)| \le |t-s| \sum_{n=1}^{N} \Big(|X_n| + |Y_n|\Big) = C_N |t-s| \text{ for all } s,t \in [0,1],$$

where $C_N = \sum_{n=1}^{N} \Big(|X_n| + |Y_n|\Big) < \infty$ is a random variable. ■

3. Fractional integrals and derivatives

Let $a,b \in \mathbb{R}$, $a < b$ and $f,g : \mathbb{R} \to \mathbb{R}$. We use notions and results about fractional calculus from [12] and [15]:

$$f(a+) := \lim_{\delta \searrow 0} f(a+\delta), \quad f(b-) := \lim_{\delta \searrow 0} f(b-\delta),$$

$$f_{a+}(x) = \mathbb{I}_{(a,b)}(x)(f(x) - f(a+)), \quad g_{b-}(x) = \mathbb{I}_{(a,b)}(x)(g(x) - g(b-)).$$

Note that for $\alpha > 0$ we have $(-1)^\alpha = e^{i\pi\alpha}$.

For $f \in L_1(a,b)$ and $\alpha > 0$ the *left-* and *right-sided fractional Rieman-Liouville integral of f of order* α on (a,b) is given for a.e. x by

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy$$

and

$$I^\alpha_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy.$$

For $p > 1$ let $I^\alpha_{a+}(L_p(a,b))$, be the class of functions f which have the representation $f = I^\alpha_{a+}\Phi$, where $\Phi \in L_p(a,b)$, and let $I^\alpha_{b-}(L_p(a,b))$ be the class of functions g which have the representation $g = I^\alpha_{b-}\varphi$, where $\varphi \in L_p(a,b)$. If $0 < \alpha < 1$, then the functions Φ, respectively φ, in the above representations agree a.s. with the *left-sided* and respectively *right-sided fractional derivative of f of order* α (in the Weyl representation)

$$\Phi(x) = D^\alpha_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbb{I}_{(a,b)}(x)$$

and

$$\varphi(x) = D^{\alpha}_{b-}g(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{g(x)-g(y)}{(y-x)^{\alpha+1}}dy\right)\mathbb{I}_{(a,b)}(x).$$

The convergence at the singularity $y = x$ holds in the L_p-sense. Recall that

$$I^{\alpha}_{a+}(D^{\alpha}_{a+}f) = f \text{ for } f \in I^{\alpha}_{a+}(L_p(a,b)), \quad I^{\alpha}_{b-}(D^{\alpha}_{b-}g) = g \text{ for } g \in I^{\alpha}_{b-}(L_p(a,b))$$

and

$$D^{\alpha}_{a+}(I^{\alpha}_{a+}f) = f, \quad D^{\alpha}_{b-}(I^{\alpha}_{b-}g) = g \text{ for } f, g \in L_1(a,b).$$

For completeness we denote

$$D^0_{a+}f(x) = f(x), D^0_{b-}g(x) = g(x), D^1_{a+}f(x) = f'(x), D^1_{b-}g(x) = g'(x).$$

Let $0 \le \alpha \le 1$. The *fractional integral* of f with respect to g is defined as

$$\begin{aligned}\int_a^b f(x)dg(x) \quad &= \quad (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)dx + \\ &\quad +f(a+)(g(b-)-g(a+))\end{aligned} \tag{3.1}$$

if $f_{a+} \in I^{\alpha}_{a+}(L_p(a,b)), g_{b-} \in I^{1-\alpha}_{b-}(L_q(a,b))$ for $\frac{1}{p} + \frac{1}{q} \le 1$.

In our investigations we will take $p = q = 2$. If $0 \le \alpha < \frac{1}{2}$, then the integral in (3.1) can be written as

$$\int_a^b f(x)dg(x) = (-1)^{\alpha}\int_a^b D^{\alpha}_{a+}f(x)D^{1-\alpha}_{b-}g_{b-}(x)dx \tag{3.2}$$

if $f \in I^{\alpha}_{a+}(L_2(a,b))$, $f(a+)$ exists, $g_{b-} \in I^{1-\alpha}_{b-}(L_2(a,b))$ (see [15]).

4. The stochastic integral

Without loss of generality we may assume $0 < T \le 1$, because for arbitrary $T > 0$ we can rescale the time variable using the H-self-similar property of the fractional Brownian motion meaning that $\big(B(ct)\big)_{t\ge 0}$ and $\big(c^H B(t)\big)_{t\ge 0}$ are equal in distribution for every $c > 0$.

We will define the Itô integral $\int_0^T G(u)dB(u)$ instead of $\int_0^t G(u)dB(u)$ and use

$$\int_0^t G(u)dB(u) = \int_0^T \mathbb{I}_{[0,t]}(u)G(u)dB(u) \text{ for } t \in [0,T]$$

(see [15]).

We consider $\alpha > 1-H$. It follows by (3.2) that

$$\int_0^T G(u)dB(u) = (-1)^\alpha \int_0^T D^\alpha_{0+}G(u)D^{1-\alpha}_{T-}B_{T-}(u)du \tag{4.1}$$

for $G \in I^\alpha_{0+}(L_2(0,T))$, where $G(0+)$ exists and $B_{T-} \in I^{1-\alpha}_{T-}(L_2(0,T))$.

The condition $G \in I^\alpha_{0+}(L_2(0,T))$ (with probability 1) means that $G \in$ $\in L_2(0,T)$ and

$$\mathcal{I}_\varepsilon(x) = \int_0^{x-\varepsilon} \frac{G(x)-G(y)}{(x-y)^{\alpha+1}}dy \text{ for } x \in (0,T)$$

converges in $L^2(0,T)$ as $\varepsilon \searrow 0$.

The condition $B_{T-} \in I^{1-\alpha}_{T-}(L_2(0,T))$ means $B_{T-} \in L_2(0,T)$ and

$$\mathcal{J}_\varepsilon(x) = \int_{x+\varepsilon}^{T} \frac{B(x)-B(y)}{(y-x)^{2-\alpha}}dy \text{ for } x \in (0,T)$$

converges in $L_2(0,T)$ as $\varepsilon \searrow 0$ This condition for B is fulfilled for $\alpha > 1-H$, since the fractional Brownian motion B is a.s. Hölder continuous with exponent $\gamma \in (0,H)$ (see [5]).

We will use (3.2) for the integrals with respect to the approximating processes $\big(B_N(t)\big)_{t\in[0,T]}$. Observe that $B_{N,T-} \in I^{1-\alpha}_{T-}(L_2(0,T))$, which follows from the Lipschitz continuity property in Theorem 2.1. We have

$$\int_0^T G(u)dB_N(u) = (-1)^\alpha \int_0^T D^\alpha_{0+}G(u)D^{1-\alpha}_{T-}B_{N,T-}(u)du \tag{4.2}$$

for $G \in I^\alpha_{0+}(L_2(0,T))$, where $G(0+)$ exists.

Let $\big(Z(t)\big)_{t\in[0,T]}$ be a cádlág process. Its *generalized quadratic variation process* $\big([Z](t)\big)_{t\in[0,T]}$ is defined as

$$[Z](t)=\lim_{\varepsilon\searrow 0}\varepsilon\int_0^1 u^{\varepsilon-1}\int_0^t \frac{1}{u}(Z_{t-}(s+u)-Z_{t-}(s))^2 ds du+(Z(t)-Z(t-))^2,$$

if the limit exists uniformly in probability (see [17]).

In particular, if B is a fractional Brownian motion with Hurst index $H\in$ $\in(\frac{1}{2},1)$ and B_N is an approximation of B as given in (2.1) then it is easy to verify that

$$(4.3)\qquad [B](t)=0 \quad\text{and}\quad [B_N](t)=0 \quad\text{for each } \ t\in[0,T],$$

because B is locally Hölder continuous with exponent $>\frac{1}{2}$ and B_N is Lipschitz continuous. The *Ito formula* for change of variable for fractional integrals is given in the next theorem.

Theorem 4.1 ([17]). *Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a continuous process with generalized quadratic variation $[Z]$. Let $Q:\mathbb{R}\times[0,T]\to\mathbb{R}$ be a random function such that a.s. we have $Q\in\mathcal{C}^1(\mathbb{R}\times[0,T])$ and $\frac{\partial^2 Q}{\partial x^2}\in\mathcal{C}(\mathbb{R}\times[0,T])$. Then, for $t_0,t\in[0,T]$ we have*

$$\begin{aligned}Q(Z(t),t)-Q(Z(t_0),t_0) &= \int_{t_0}^t \frac{\partial Q}{\partial x}(Z(s),s)dZ(s)+\int_{t_0}^t \frac{\partial Q}{\partial t}(Z(s),s)ds+\\ &\quad +\frac{1}{2}\int_{t_0}^t \frac{\partial^2 Q}{\partial^2 x}(Z(s),s)d[Z]s.\end{aligned}$$

Let $1-H<\alpha<\frac{1}{2}$ and let $G\in I^{\alpha}_{0+}(L_2(0,T))$ such that $G(0+)$ exists. We define the processes

$$Z(t)=\int_0^t G(s)dB(s) \text{ and } Z_N(t)=\int_0^t G(s)dB_N(s),\quad t\in]0,T].$$

Then by [17] it follows that

$$[Z](t)=0 \text{ and } [Z_N](t)=0.$$

So, if $Q:\mathbb{R}\times[0,T]\to\mathbb{R}$ is a random function such that a.s. $Q\in\mathcal{C}^1(\mathbb{R}\times[0,T])$ and $\frac{\partial^2 Q}{\partial x^2}\in\mathcal{C}(\mathbb{R}\times[0,T])$, then for $t_0, t\in[0,T]$ we have

$$Q(Z(t),t)-Q(Z(t_0),t_0) = \int_{t_0}^{t}\frac{\partial Q}{\partial x}(Z(s),s)G(s)dB(s) + \int_{t_0}^{t}\frac{\partial Q}{\partial t}(Z(s),s)ds, \tag{4.4}$$

and

$$Q(Z_N(t),t)-Q(Z_N(t_0),t_0) = \int_{t_0}^{t}\frac{\partial Q}{\partial x}(Z_N(s),s)G(s)dB_N(s) + \int_{t_0}^{t}\frac{\partial Q}{\partial t}(Z_N(s),s)ds. \tag{4.5}$$

5. Stochastic differential equations driven by step fractional Brownian motion

Let $\big(B(t)\big)_{t\geq 0}$ be a step fractional Brownian motion with Hurst parameter H such that $H(t)>\frac{1}{2}$. We investigate stochastic differential equations of the form

$$\begin{aligned} dX(t) &= F(X(t),t)dt+G(X(t),t)dB(t),\\ X(t_0) &= X_0, \end{aligned} \tag{5.1}$$

where $t_0\in]0,T]$, X_0 is a random vector in $\mathbb{R}^n$ and the random functions F and G satisfy the following conditions with probability 1:

(C1) $F\in C(\mathbb{R}^n\times[0,T],\mathbb{R}^n), G\in C^1(\mathbb{R}^n\times[0,T],\mathbb{R}^n)$;

(C2) for each $t\in[0,T]$ the functions $F(\cdot,t), \dfrac{\partial G(\cdot,t)}{\partial x^i}, \dfrac{\partial G(\cdot,t)}{\partial t}$ are locally Lipschitz for each $i\in\{1,\ldots,n\}$.

We consider the pathwise auxiliary partial differential equation on $\mathbb{R}^n \times \mathbb{R} \times [0,T]$

$$
\begin{aligned}
\frac{\partial K}{\partial z}(y,z,t) &= G(K(y,z,t),t), \\
K(Y_0,Z_0,t_0) &= X_0,
\end{aligned}
\tag{5.2}
$$

where Y_0 is an arbitrary random vector in $\mathbb{R}^n$ and Z_0 is an arbitrary random variable in $\mathbb{R}$. From the theory of differential equations it follows that with probability 1 there exists a local solution $K \in C^1(\mathbb{R}^n \times \mathbb{R} \times [0,T], \mathbb{R}^n)$ in a neighbourhood V of (Y_0, Z_0, t_0) with partial derivatives being Lipschitz in the variable y and

$$
\det\left(\frac{\partial K^i}{\partial y^j}(y,z,t)\right)_{1\le i,j\le n} \neq 0.
$$

For $(x,y,t) \in V$ we have

$$
\frac{\partial^2 K}{\partial z^2}(y,z,t) = \sum_{j=1}^{n} \frac{\partial G}{\partial x^j}(K(y,z,t),t)G^j(K(y,z,t),t).
$$

We also consider the pathwise differential equation (in matrix representation) on $[0,T]$

$$
\begin{aligned}
\frac{\partial K}{\partial y}(Y(t),B(t),t)dY(t) + \frac{\partial K}{\partial t}(Y(t),B(t),t)dt &= F(K(Y(t),B(t),t),t)dt \\
Y(t_0) &= Y_0,
\end{aligned}
$$

or

$$
\begin{aligned}
dY(t) &= \left(\frac{\partial K}{\partial y}(Y(t),B(t),t)\right)^{-1}\left[F(K(Y(t),B(t),t),t) - \frac{\partial K}{\partial t}(Y(t),B(t),t)\right]dt \\
Y(t_0) &= Y_0,
\end{aligned}
$$

which has a unique local solution on a maximal interval $]t_0^1, t_0^2[\subseteq [0,T]$ with $t_0 \in]t_0^1, t_0^2[$ (see [9]).

Applying the Ito formula, and relation (4.4) to the random function $Q(z,t) = K(Y(t),z,t)$ (in fact, successively for $K^1,\ldots,K^n$) and the step fractional Brownian motion B we obtain

$$
\begin{aligned}
&K(Y(t),B(t),t)-K(Y(t_0),B(t_0),t_0)=\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s)+\int_{t_0}^{t}\frac{\partial K}{\partial z}(Y(s),B(s),s)dB(s)+\\
&\quad+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y(s),B(s),s)ds=\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s)+\int_{t_0}^{t}G(K(Y(s),B(s),s),s)dB(s)+\\
&\quad+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y(s),B(s),s)ds=\\
&=\int_{t_0}^{t}F(K(Y(s),B(s),s),s)ds+\int_{t_0}^{t}G(K(Y(s),B(s),s),s)dB(s).
\end{aligned}
$$

Therefore,

$$X(t):=K(Y(t),B(t),t)$$

satisfies

$$X(t)=X_0+\int_{t_0}^{t}F(X(s),s)ds+\int_{t_0}^{t}G(X(s),s)dB(s).$$

Instead of the process $\big(B(t)\big)_{t\in[0,1]}$ we consider its approximations $\big(B_N(t)\big)_{t\in[0,1]}$ given in (2.1). For each $N\in\mathbb{N}$ we consider the pathwise differential equation (in matrix representation)

$$
\begin{aligned}
dY_N(t) &= \left(\frac{\partial K}{\partial y}(Y_N(t),B_N(t),t)\right)^{-1}\Big[F(K(Y_N(t),B_N(t),t),t)-\\
&\quad -\frac{\partial K}{\partial t}(Y_N(t),B_N(t),t)\Big]dt\\
Y_N(t_0) &= Y_0,
\end{aligned}
$$

which has a unique local solution Y_N on a maximal interval $(t^1,t^2)\subset(t_0^1,t_0^2)$ of existence which contains t_0 ([13]). Applying the Ito formula to the random function $Q(z,t)=K(Y_N(t),z,t)$ (in fact, successively for $K^1,\ldots,K^n$) and the process B_N we obtain

$$\begin{aligned}
&K(Y_N(t),B_N(t),t)-K(Y_N(t_0),B_N(t_0),t_0)=\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y_N(s),B_N(s),s)dY_N^j(s)+\int_{t_0}^{t}\frac{\partial K}{\partial z}(Y_N(s),B_N(s),s)dB_N(s)+\\
&\qquad+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y_N(s),B_N(s),s)ds=\\
&=\sum_{j=1}^{n}\int_{t_0}^{t}\frac{\partial K}{\partial y^j}(Y_N(s),B_N(s),s)dY_N^j(s)+\int_{t_0}^{t}G(K(Y_N(s),B_N(s),s),s)dB_N(s)+\\
&\qquad+\int_{t_0}^{t}\frac{\partial K}{\partial t}(Y_N(s),B_N(s),s)ds=\\
&=\int_{t_0}^{t}F(K(Y_N(s),B_N(s),s),s)ds+\int_{t_0}^{t}G(K(Y_N(s),B_N(s),s),s)dB_N(s).
\end{aligned}$$

Therefore,

$$X_N(t):=K(Y_N(t),B_N(t),t)$$

satisfies

$$X_N(t)=X_0+\int_{t_0}^{t}F(X_N(s),s)ds+\int_{t_0}^{t}G(X_N(s),s)dB_N(s),\quad t\in]t_1,t_2[.$$

So we have the following pathwise property

$$\lim_{N\to\infty}\sup_{t\in]t_1,t_2[}\|Y_N(t)-Y(t)\|=0.$$

Then the continuity properties of K and (2.2) imply that for a.e. $\omega\in\Omega$ it holds

$$\lim_{N\to\infty}\sup_{t\in]t_1,t_2[}\|X_N(t)-X(t)\|=0.$$

By this we have proved the main result of our paper:

Theorem 5.1. *Let B be a fractional Brownian motion approximated by the processes B_N given in* (2.1) *and* (2.2)*. Let $F,G:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ be random* *functions satisfying conditions* (C1) *and* (C2) *with probability* 1*. Let* $t_0 \in]0,T]$ *be fixed. Then each of the stochastic equations*

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s),s)ds + \int_{t_0}^{t} G(X(s),s)dB(s),$$

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s),s)ds + \int_{t_0}^{t} G(X_N(s),s)dB_N(s), \quad N \in \mathbb{N}$$

admits almost surely a unique local solution on a common interval $]t_1,t_2[$ *(which is independent of* N *and contains* t_0*). Moreover, we have the following approximation result*

$$P\Big(\lim_{N\to\infty} \sup_{t\in]t_1,t_2[} \|X_N(t) - X(t)\| = 0\Big) = 1.$$

6. Application

We consider the one dimensional stochastic linear equation from financial mathematics, modeling the price S of a stock

$$S(t) = S_0 + \int_0^t \mu(s)S(s)ds + \int_0^t \sigma(s)S(s)dB(s),$$

where $(B(t))_{t\in[0,T]}$ is a step fractional Brownian motion with Hurst index $H(t) > \frac{1}{2}$, μ is the interest rate and σ the volatility function.

It is known (see [8]) that this equation has the following unique solution

$$S(t) = S_0 \exp\left\{\int_0^t \mu(u)du + \int_0^t \sigma(u)dB(u)\right\} \text{ for all } t \in [0,T].$$

By the methods of the above section we approximate B through the processes B_N, via (2.1) and (2.2) and consider

$$S_N(t) = S_0 \exp\left\{\int_0^t \mu(u)du + \int_0^t \sigma(u)dB_N(u)\right\} \text{ for all } t \in [0,T].$$

Using Theorem 5.1 it follows that

$$P(\lim_{N\to\infty} \sup_{t\in[0,T]} \|S_N(t) - S(t)\| = 0) = 1.$$

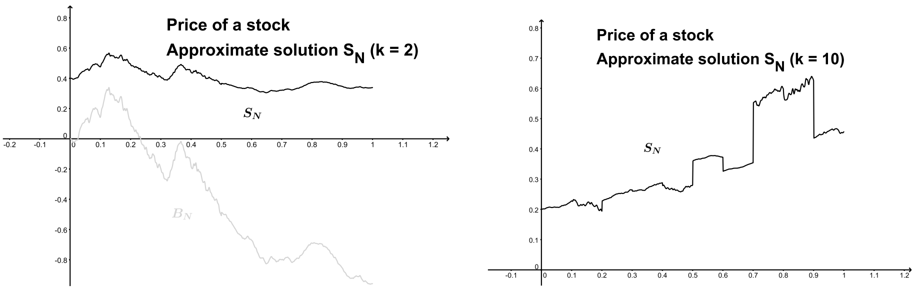
In the special case when μ and σ are constants, we obtain that the price of a stock is

$$S(t) = S_0 e^{\mu t + \sigma B(t)}$$

and we can simulate it by computer using

$$S_N(t) = S_0 e^{\mu t + \sigma B_N(t)}$$

as given in Figure 3, where K is the number of constant pieces of H.



a) $K = 2$ b) $K = 10$

Figure 3. Approximate solution S_N

References

[1] **Benassi, A., S. Jaffard and D. Roux,** Elliptic Gaussian random processes, *Rev. Mathematica Iberoamericana*, **13**(1997), no.1, 19–90.

[2] **Benassi, A., P. Bertrand, S. Cohen and J. Istas,** Identification of the Hurst index of a Step Fractional Brownian Motion, *Statistical Inference for Stochastic Processes*, **3** (2000), 101–111.

[3] **Coutin, L. and L. Decreusefond,** *Stochastic Differential Equations Driven by a Fractional Brownian Motion*, Tech. Report 97C004, ENST Paris (1997).

[4] **Coutin, L. and Z. Qian,** Stochastic differential equations for fractional Brownian motions, *C. R. Acad. SCI. Paris*, **331**, Série I (2000), 75–80.

[5] **Decreusefond, L. and A.S. Üstünel,** Stochastic analysis of the fractional Brownian motion, *Potential Analysis*, **10** (1999), 177–214.

[6] **Dzhaparidze, D. and H. van Zanten,** A series expansion of fractional Brownian motion, *Probab. Theory Relat. Fields*, **130** (2004), 39–55.

[7] **Kleptsyna, M.L., P.E. Kloeden and V.V. Anh,** Existence and uniqueness theorems for FBM stochastic differential equations, *Information Transmission Problems*, **34** (1998), 41–51.

[8] **Klingenhöfer, F. and M. Zähle,** Ordinary differential equations with fractal noise, *Proc. Amer. Math. Soc.*, **127** (1999), 1021–1028.

[9] **Lisei, H. and A. Soós,** Approximation of stochastic differential equations driven by fractional Brownian motion, in: *Seminar on Stochastic Analysis, Random Fields and Applications V, Progress in Probability*, **59**, Birkhuser Verlag 2007, pp. 229–244.

[10] **Mandelbrot, B. and Van Ness,** Fractional Brownian motion, fractional noises and applications, *SIAM Rev.*, **10** (1968), 422–437.

[11] **Peltier, R. and J. Lévy Vehel,** Multifractional Brownian motion: definition and preliminary results, Rapport technique INRIA, 1996.

[12] **Samko, S.G., A.A. Kilbas and O.I. Marichev,** *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, 1993.

[13] **Soós, A.,** Series expansion of step fractional Brownian motion (to appear).

[14] **Watson, G.N.,** *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1944.

[15] **Zähle, M.,** Integration with respect to fractal functions and stochastic calculus I., *Probab.Theory Relat. Fields*, **111** (1998), 333–374.

[16] **Zähle, M.,** Ordinary differential equations with fractal noise, *Proc. Am. Math. Soc.*, **127** (1999), 333–374.

[17] **Zähle, M.,** Integration with respect to fractal functions and stochastic calculus II., *Math. Nachr.*, **225** (2001), 145–183.

J. Robu and A. Soós
Babeş - Bolyai University
Faculty of Mathematics and Computer Science
Cluj - Napoca
Romania
robu@cs.ubbcluj.ro
asoos@math.ubbcluj.ro