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GENERALIZED HAAR–FOURIER TRANSFORM

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Abstract. We give a new generalization for Haar functions. The generalization starts from the Walsh-like functions and based on the connection between the original Walsh and Haar systems. We generalize the Haar–Fourier Transform too.

1. Introduction

The Haar-system introduced in [3] has several generalizations (see [1], [7], [8]). The most commonly used generalizations are the wavelets. In this case the functions are derived from the basic function $\psi \in L^2(\mathbb{R})$ (from the mother wavelet) by translation and dilation: $\psi_{n,k}(x) := 2^{n/2}\psi(2^n x - k)$ $(x \in \mathbb{R},\ 0 \leq \leq k < 2^n,\ n \in \mathbb{N})$. In the case of the Haar system the mother wavelet is

$$h(x) := \begin{cases} 1, & (0 \leq x < 1/2), \\ -1, & (1/2 \leq x < 1), \\ 0, & (1 \leq x < \infty). \end{cases} \tag{1.1}$$

We will use the $\chi_{n,k}$ characteristic functions of dyadic intervals which are also known as the Haar scaling functions. These functions can be derived

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similarly to Haar-functions by means of $\chi_{[0,1]}$, the characteristic function of the unit interval:

$$\chi_{n,k}(x) = \chi(2^n x - k) \; (x \in \mathbb{R},\; 0 \le k < 2^n, n \in \mathbb{N}).$$

For these systems the following relations hold true

$$\begin{aligned}
(1.2) \qquad \chi_{n,k} &= \chi_{n+1,2k} + \chi_{n+1,2k+1}, \quad (0 \le k < 2^n, n \in \mathbb{N}),\\
(1.3) \qquad h_{n,k} &= 2^{n/2}(\chi_{n+1,2k} - \chi_{n+1,2k+1}), \quad (0 \le k < 2^n, n \in \mathbb{N}).
\end{aligned}$$

Equation (1.2) is called scaling equation.

Let $f : [0,1) \to \mathbb{R}$ be a discrete function with constant values on the intervals $\big[k2^{-N}, (k+1)2^{-N}\big)$ $\big(k = 0, 1, \cdots, 2^N - 1\big)$ and denote

$$\langle f, g\rangle := 2^{-N} \sum_{k=0}^{2^N-1} f(k2^{-N})g(k2^{-N})$$

the discrete scalar product of the functions f and g. Then for the Haar–Fourier coefficients we have

$$(1.4) \qquad \langle f, h_{n,k}\rangle := 2^{n/2}(\langle f, \chi_{n+1,2k}\rangle - \langle f, \chi_{n+1,2k+1}\rangle),$$

and

$$(1.5) \quad \langle f, \chi_{n,k}\rangle := (\langle f, \chi_{n+1,2k}\rangle + \langle f, \chi_{n+1,2k+1}\rangle), \; (0 \le k < 2^n, 0 \le n < N).$$

The Haar–Fourier analysis and synthesis are based on these equations and these equations imply that the coefficients can be computed with $O(2^N)$ operations. For the reconstruction of the discrete function from its Haar–Fourier coefficients we need the same number of operations.

The analogues of the equations (1.2) and (1.3) for wavelets are the base of the effective use of wavelet analysis.

In this paper we investigate another generalization of Haar system. Our examination is based on the connections between Haar, Walsh and Rademacher systems. This kind of generalization was introduced by Alexits [1], [2], but it was mentioned by Kaczmarz too [4]. The construction of Haar system was started from the Rademacher system. When we substitute the Rademacher system with different types of multiplicative systems we obtain Haar-like systems. It was proved in [7], [8] that these systems have as good properties of convergence as Haar system.

In our constructions we substituted the dilation with an iterated map of an appropriate two-hold map. The Haar-like functions generated in this way will be investigated in this paper. It will be proved that equations like (1.2) and (1.3) are satisfied in the general case.

2. Twofold maps, Rademacher-like functions

In this section we generalize the Rademacher system by replacing the dilation with the iterated map of two-fold maps.

Let us fix the set $X \neq \emptyset$ and let us denote a two-fold map by $A : X \to X$ i.e.

$$\forall x \in X \ \exists | x'', x' \in X, x' \neq x'' : A(x') = A(x'') = x. \tag{2.1}$$

We can define the iterated map of A by

$$A^0(x) = x, \quad A^n := A^{n-1} \circ A \quad (x \in X, n \in \mathbb{N}^* := \{1, 2, \cdots\}). \tag{2.2}$$

It is easy to see that A^n is a 2^n-fold map on the set X. Starting from a fixed element $x_0^0 \in X$ we can define the preimage of this element with respect to A^n, this discrete set is denoted by

$$X_n := \{x \in X : A^n(x) = x_0^0\} = A^{-n}(\{x_0^0\}) \ \ (n \in \mathbb{N}). \tag{2.3}$$

It is easy to see that the set X_n has 2^n elements. The elements of $X_n = \{x_k^n : 0 \leq k < 2^n\}$ can be indexed such that $A^{-1}(\{x_k^n\}) = \{x_{2k+1}^{n+1}, x_{2k}^{n+1}\}$, and consequently $A(x_\ell^{n+1}) = x_{[\ell/2]}^n$, where $[s]$ stands for the integer part of the real number s. From this we get by induction

$$A^j(x_\ell^n) = x_{[\ell 2^{-j}]}^{n-j} \ \ (0 \leq \ell < 2^n, 0 \leq j \leq n). \tag{2.4}$$

Let us fix $N \in \mathbb{N}^*$ and define the following subsets of the set X_N:

$$I_{n,k} := A^{n-N}(\{x_k^n\}) \quad (0 \leq k < 2^n, 0 \leq n \leq N). \tag{2.5}$$

$I_{n,k}$ has 2^{N-n} elements and is called a dyadic interval of the set X_N. These subsets have similar properties as intervals $[k2^{-n}, (k+1)2^{-n})$:

$$\begin{aligned} &I_{0,0} = A^{-N}(\{x_0^0\}) = X_N, \ I_{N,k} = A^0(\{x_k^N\}) = \{x_k^N\}, \\ &I_{n+1,2k+1} \cup I_{n+1,2k} = I_{n,k} \ \ (0 \leq k < 2^n, 0 \leq n < N). \end{aligned} \tag{2.6}$$

It is important to highlight that these subsets are similar to real dyadic intervals. These subsets are in a very special connection with each other. Two of them are always disjoint or one of them includes the other.

Starting from the function $\varphi : X \to \mathbb{C}$ and using the iterated map of the two-fold map $A : X \to X$ we can define a sequence of functions:

$$\varphi_n(x) := \varphi(A^n(x)) \ \ (x \in X, n \in N). \tag{2.7}$$

In the special case when $X=[0,1)$, $A(x)=2x \bmod 1$, and

$$\varphi(x)=h(x)=\begin{cases} 1, & 0\le x<\frac{1}{2}, \\ -1, & \frac{1}{2}, \le x<1, \\ 0, & \text{otherwise}, \end{cases}$$

we get the Rademacher system. Generally the system $(\varphi_n, n\in\mathbb{N})$ is called Rademacher-like system.

The systems $\varphi^j,\ \psi^j : X\to\mathbb{C}$ $(j=0,1)$ are called biorthogonal with respect to the two-fold map A, if for $A(x')=A(x'')=x, x'\ne x''$ and for $i,j=0,1$ the following holds

$$\frac{1}{2}\sum_{z\in A^{-1}(x)}\varphi^i(z)\overline{\psi}^j(z)=\frac{1}{2}(\varphi^i(x')\overline{\psi}^j(x')+\varphi^i(x'')\overline{\psi}^j(x''))=\delta_{ij}. \tag{2.8}$$

If $\varphi^j=\psi^j$ then φ^j $(j=0,1)$ is called orthonormal.

In the mentioned special case the functions 1 and h are orthonormal with respect to the two-fold map A.

3. Walsh-like and Haar-like systems

Let us fix $N\in\mathbb{N}^*$. The product system of systems $\{\varphi_j^0,\varphi_j^1\}$ $(j\in\mathbb{N})$ and $\{\psi_j^0,\psi_j^1\}$ $(j\in\mathbb{N})$ can be defined as:

$$\Phi_m^N:=\Phi_m:=\prod_{j=0}^{N-1}\varphi_j^{m_j},\ \Psi_m^N:=\Psi_m:=\prod_{j=0}^{N-1}\psi_j^{m_j},\ \ (m=\sum_{j=0}^{N-1}m_j2^j). \tag{3.1}$$

The product system of a Rademacher-like system is called Walsh-like system.

In the special case when $\varphi_j^0=1$ and $\varphi_j^1=r_j$ $(j\in\mathbb{N})$ the system $\{\Phi_m,\ m\in\in\mathbb{N}\}$ is the Walsh system.

It is easy to see that the mixed Dirichlet-kernel of these systems can be written in the product form:

$$D_{2^N}(x,y):=\sum_{m=0}^{2^N-1}\Phi_m(x)\overline{\Psi}_m(y)=\prod_{j=0}^{N-1}(\varphi_j^0(x)\overline{\psi}_j^0(y)+\varphi_j^1(x)\overline{\psi}_j^1(y)). \tag{3.2}$$

Using the function

$$L(x,y)=\varphi^0(x)\overline{\psi}^0(y)+\varphi^1(x)\overline{\psi}^1(y)\quad (x,y\in X)$$

the Dirichlet-kernel D_{2^N} can be written as

$$D_{2^N}(x,y) := \prod_{j=0}^{N-1} L(A^{N-1-j}(x), A^{N-1-j}(y)) \quad (x,y \in X).$$

Let us introduce the following analogues of the scaling functions $\chi_{n,k}$:

$$\mathcal{I}_{n,k}(x) := 2^{-n} \prod_{j=0}^{n-1} L(A^{N-1-j}(x), A^{N-1-j}(x_k^n)), \tag{3.3}$$
$$(x \in X, 0 \le k < 2^n, n = 1, 2, \cdots, N).$$

In the special case when $\varphi^0 = \psi^0 = 1$ and $\varphi^1 = \psi^1 = h$ we get the Haar scaling functions.

It will be proved that similarly to Haar scaling functions $\mathcal{I}_{n,k}$ is the characteristic function of the set $I_{n,k}$.

Theorem 3.1. *If the biorthogonal relation* (2.8) *is satisfied then for the scaling functions $\mathcal{I}_{n,k}$ on the set X_N we have*

$$\mathcal{I}_{n,k} = \chi_{I_{n,k}} \quad (0 \le k < 2^n, n = 1, 2, \cdots, N), \tag{3.4}$$

thus in the points of the set X_N the following scaling equation is true

$$\mathcal{I}_{n+1,2k}(x) + \mathcal{I}_{n+1,2k+1}(x) = \mathcal{I}_{n,k}(x), \tag{3.5}$$
$$(x \in X_N, 0 \le k < 2^n, n = 1, 2, \cdots, N).$$

Proof. For any $x \in I_{n,\ell}$, $A^{N-n}(x) = x_\ell^n$ we have

$$A^{N-j-1}(x) = A^{n-j-1}(A^{N-n}(x)) = A^{n-j-1}(x_\ell^n) = x^{j+1}_{[\ell 2^{j+1-n}]}$$
$$(0 \le j < n, 0 \le \ell < 2^n).$$

Thus

$$\mathcal{I}_{n,k}(x) := 2^{-n} \prod_{j=0}^{n-1} L(x^{1+j}_{[\ell 2^{j+1-n}]}, x^{1+j}_{[k 2^{j+1-n}]}) \quad (n = 0, 1, \cdots, N, x \in I_{n,\ell}).$$

From this it can be seen, that $\mathcal{I}_{n,k}(x)$ has the same value for every $x \in I_{n,\ell}$. By changing indexes $i = n - j - 1$ in the product we get:

$$\mathcal{I}_{n,k}(x) = 2^{-n} \prod_{i=0}^{n-1} L(x^{n-i}_{[\ell 2^{-i}]}, x^{n-i}_{[k 2^{-i}]}), \quad (x \in I_{n,\ell}, 0 \le k < 2^n, 0 \le n \le N). \tag{3.6}$$

From the condition (2.8) follows

$$L(x', x'') = 2\delta_{x',x''}, \quad (A(x') = A(x'') = x).$$

In case of $k \neq \ell$ $(0 \leq k, \ell < 2^n)$, there exists an index i $(0 < i < n)$ such that $x' := [\ell 2^{-i}] \neq x'' = [k2^{-i}]$ and $A(x') := [\ell 2^{-i-1}] = A(x'') = [k2^{-i-1}]$.

This implies that in the product (3.6) there is at least one zero factor (the factor that belongs to index i).

If $k = \ell$ then all factors of the mentioned product are equal to 2, so the product equals to 1.

The scaling equation (3.5) follows from the equation (3.4) by using the condition (2.8). ∎

Similarly to the original properties ((1.2), (1.3)) we can introduce the Haar-like functions

$$\mathcal{H}_{n,k} := \mathcal{I}_{n+1,2k} - \mathcal{I}_{n+1,2k+1} \quad (0 \leq k < 2^n, n = 0, 1, \cdots, N-1).$$

Theorem 3.2. *The system $\mathcal{H}_{n,k}$ $(0 \leq k < 2^n, n = 0, 1, \cdots, N-1)$ is a discrete orthogonal Haar-like system with respect to the scalar product*

$$\langle f, g\rangle := 2^{-N} \sum_{x \in X_N} f(x)\overline{g}(x). \tag{3.7}$$

More precisely

$$\langle \mathcal{H}_{n,k}, \mathcal{H}_{m,\ell}\rangle = 2^{-n} \cdot \delta_{n,m} \cdot \delta_{k,\ell},$$
$$(0 \leq k < 2^n,\ 0 \leq n < N,\ 0 \leq \ell < 2^m,\ 0 \leq m < N).$$

Proof. If $I_{n,k} = I_{m,\ell}$, then

$$\langle \mathcal{H}_{n,k}, \mathcal{H}_{m,\ell}\rangle = 2^{-N} \cdot \sum_{x \in I_{n,k}} 1 = 2^{-N} \cdot 2^{N-n} = 2^{-n}.$$

If $I_{n,k} \neq I_{m,\ell}$ and $m \leq n$, then eiter $I_{n,k} \cap I_{m,\ell} = \emptyset$ or there exist $p \in \mathbb{N}^*$ and $s \in \{0, 1, \ldots, 2^p - 1\}$ such that $I_{n,k} = I_{m+p,2^p\ell+s}$.

In the first case $\mathcal{H}_{n,k} \cdot \mathcal{H}_{m,\ell} = 0$ on X_n.

In the second case $\mathcal{H}_{n,k} \cdot \mathcal{H}_{m,\ell} = \pm\mathcal{H}_{n,k}$. Consequently,

$$\langle \mathcal{H}_{n,k}, \mathcal{H}_{m,\ell}\rangle = \pm 2^{-N} \textstyle\sum_{x \in I_{n,k}} \mathcal{H}_{n,k} = 0.$$ ∎

Remark 3.1. Orthonormed Haar-like system can be constructed by the formula

$$H_{n,k} := 2^{n/2} \cdot \mathcal{H}_{n,k},\ (0 \leq k < 2^n, n = 0, 1, \ldots, N-1).$$

4. Examples and special cases

The Walsh–Chebyshev system was presented in [5]. This system can be generated by this construction too.

Let us start from the interval $X = (-1, 1)$. The function $A(x) = 2x^2 - 1$, $x \in X$ is a two-fold map on X. Indeed, if $A(x') = A(x'')$ (and $x' \neq x''$) then $x' = -x''$. Iterating the 2^n-fold map A^n we get $A^n(x) = T_{2^n}(x)$, i.e. the Chebyshev-polynomial of index 2^n.

In this case the points of discrete set X_n are the Chebyshev-abscissas.

The Rademacher-like system $\{\varphi_k^0, \varphi_k^1\}$, $(k \in \mathbb{N})$ and $\{\psi_k^0, \psi_k^1\}$ $(k \in \mathbb{N})$ can be generated by

$$\{\varphi^0, \varphi^1\} = \{1, x\}, \quad \{\psi^0(x) = 1, \psi^1(x) = 2x/(A(x) + 1)\}.$$

In this case the biorthogonality relation (2.8) is satisfied and the generated Rademacher-like functions are

$$\varphi_k^1 = T_{2^k}(x), \quad \psi_k^1(x) = 2T_{2^k}(x)/(1 + T_{2^{k+1}}(x)) \quad (k \in \mathbb{N}).$$

The functions Φ_m of the product system are polynomials of degree m and the functions of the biorthogonal system Ψ_m are rational functions. The system $\Phi_m (m \in \mathbb{N})$ is called Walsh–Chebyshev system.

Using the mixed Dirichlet-kernel of these systems the Haar-Chebyshev scaling functions and Haar-Chebyshev functions can be derived.

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