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ON REAL VALUED ADDITIVE FUNCTIONS MODULO 1

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Abstract. We determine class of five completely additive real valued functions satisfying particular relations.

1. Introduction

1.1. Notations

Let $\mathbb{G}$ be an additive commutative semigroup with identity element 0. Let $\mathcal{A}_{\mathbb{G}}$ and $\mathcal{A}^*_{\mathbb{G}}$ denote the set of $\mathbb{G}$ valued additive and completely additive functions respectively.

In case $\mathbb{G} = \mathbb{R}$, then we simply write $\mathcal{A}$ (respectively $\mathcal{A}^*$) and when $\mathbb{H} = \mathbb{C}$, then we write $\mathcal{M}$ (respectively $\mathcal{M}^*$). The domain of $f \in \mathcal{A}_{\mathbb{G}}$ ($\mathcal{A}^*_{\mathbb{G}}$) can be extended to $\mathbb{Z}$ by defining $f(-1) = f(0) = 0$. Then $f(n) = f(|n|)$, and $f(nm) = f(n) + f(m)$ remain valid in $n, m \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. Similarly, for $g \in \mathcal{M}_{\mathbb{H}}$, defining $g(-1) = g(0) = 1$ and $g(-n) = g(n)$, we can extend g over $\mathbb{Z}$ by $g(n) = g(|n|)$. Then $g(nm) = g(n)g(m)$ holds, if $(n, m) = 1$ and $m, n \in \mathbb{Z}^*$.

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1.2. Regular behaviour of additive and multiplicative functions

P. Erdős [2] proved that if $f \in \mathcal{A}$ be such that $f(n+1) - f(n) \to 0$ as $n \to \infty$, then $f(n)$ is a constant multiple of $\log n$. Since then this beautiful and simple assertion saw a plenty of generalizations.

It is natural to determine all $g \in \mathcal{M}$ for which $g(n+1) - g(n) \to 0$ as $n \to \infty$. It clearly holds if $g(n) \to 0$ $(n \to \infty)$, or if $g(n) = n^s$ $(n \in \mathbb{N})$ and $\Re s < 1$. In 1984, celebrating P. Erdős's 70th anniversary in a conference, I. Kátai conjectured that no more solution exists. E. Wirsing proved this assertion and the proof was sent in a letter to I. Kátai. More than ten years later Y. Tang and S. Pintsung proved the same assertion. Finally, they wrote a joint paper together with E. Wirsing [11].

The result of Wirsing–Tang–Pintsung would imply that:

If $f \in \mathcal{A}$ and

$$f(n+1) - f(n) \to 0 \pmod 1 \tag{1.1}$$

then $f(n) \equiv c \log n \pmod 1$ holds for some $c \in \mathbb{R}$.

Let $g(n) = e^{2\pi i f(n)}$. From (1.1) we have $g(n+1)\overline{g(n)} \to 1$ $(n \to \infty)$, whence

$$|g(n+1) - g(n)|^2 = 2 - 2\mathrm{Re}(g(n+1)\overline{g(n)}) \to 0$$

and so, from $|g(n)| = 1$ we have that $g(n) = n^{i\tau}$. Thus,

$$f(n) - \frac{\tau}{2\pi} \log n \equiv 0 \pmod 1.$$

It is not hard to show that:

If $f, g \in \mathcal{M}$ and $g(n+1) - f(n) \to 0$ $(n \to \infty)$, then either $f(n) \to 0$ and $g(n) \to 0$, or $f(n) = g(n) = n^{i\tau}$ holds for all $n \in \mathbb{N}$.

Thus, if $f, g \in \mathcal{A}$ and $g(n+1) - f(n) \to 0 \pmod 1$, then

$$g(n) \equiv f(n) \equiv \tau \log n \pmod 1.$$

1.3. Conjectures of I. Kátai

In these directions the following conjectures are due to I. Kátai.

Conjecture 1. *If* $f_0,\ f_1,\ \dots, f_k \in \mathcal{A}^*$ *and*

$$f_0(n)+f_1(n+1)+\ \dots + f_k(n+k) \pmod 1 \ \to 0,$$

as $n\to\infty$, *then there are* $\tau_0,\dots,\tau_k\in\mathbb{R}$ *such that*

$$\tau_0+\dots+\tau_k=0$$

and

$$f_0(n)\equiv\tau_0\log n \pmod 1,\dots,f_k(n)\equiv\tau_k\log n \pmod 1$$

for all $n\in\mathbb{N}$.

Conjecture 2. *Let* $f_0,\ f_1,\ \dots, f_k\in\mathcal{A}^*$ *and,*

$$L_n=f_0(n)+f_1(n+1)+\dots+f_k(n+k). \tag{1.2}$$

If $L_n\equiv 0 \pmod 1$ $(n\in\mathbb{N})$, *then*

$$f_0(n)\equiv f_1(n)\equiv\dots\equiv f_k(n)\equiv 0 \pmod 1. \tag{1.3}$$

This conjecture is known for $k=2,3$ (see [4] and [5]). In this paper we prove this conjecture and its variants for the case $k=4$ by assuming that the relation $L_n\equiv 0 \pmod 1$ holds for all $n\in\mathbb{Z}$. R. Styer [10] determined all those $f_0,f_1,f_2\in\mathcal{A}$ so that,

$$f_0(n)+f_1(n+1)+f_2(n+2)\equiv 0 \pmod 1 \ (n\in\mathbb{N}).$$

In [6] it was proved that for arbitrary $a,b\in\mathbb{N}$, all solutions $f_1,f_2,f_3\in\mathcal{A}^*$ of

$$f_1(n-a)+f_2(n)+f_3(n+b)\equiv 0 \pmod 1 \ (n\in\mathbb{N},\ n\ge a+1)$$

form a finite dimensional space. If $f_j(q)\equiv 0 \pmod 1$ $(i=1,2,3)$ holds for all primes $q\le\max(3,a+b)$, then $f_j(n)\equiv 0 \pmod 1$ $(j=1,2,3)$ and for all $n\in\mathbb{N}$.

Let $g_0,\dots,g_k$ be complex valued completely additive functions on $\mathbb{Z}[i]$ (the ring of Gaussian integers). Assume that $g_j(0)=0$ and $g_j(\epsilon)=0$ for $\epsilon=\pm1,\ \pm i$ and that $g_j(\alpha\beta)=g_j(\alpha)+g_j(\beta)$ holds for every $\alpha,\ \beta\in\mathbb{Z}[i]$. Let

$$S_k(\alpha)=\sum_{j=0}^{k} g_j(\alpha+j).$$

Assume that

$$S_k(\alpha)\in\mathbb{Z}[i] \ \ (\alpha\in\mathbb{Z}[i]). \tag{1.4}$$

It is expected that (1.4) would imply $g_j(\alpha)\in\mathbb{Z}[i]$ $(j=0,1,\dots,k)$. This has been proved in [9] for $k=3$ and in [7] for $k=5$.

I. Kátai in [3] stated a weaker conjecture:

Conjecture 3. *If* $P(x) = 1 + A_1x + A_2x^2 + \ldots + A_kx^k \in \mathbb{R}[x] \setminus \mathbb{Q}[x]$ *and* $f \in \mathcal{A}^*$ *satisfy*

$$f(n) + A_1f(n+1) + A_2f(n+2) + \ldots + A_kf(n+k) \equiv 0 \pmod 1.$$

Then $f(n) = 0$ *for all* $n \in \mathbb{N}$.

This is true for $k = 2$ and for $k = 3$ (see [3, 4, 5]). It is clear that conjecture 2 implies conjecture 3. In [8] A. Kovács and B. M. Phong proved Conjecture 3 for $k = 4$.

1.4. Our aim

Let $A_0(n), A_1(n), \ldots, A_k(n) \in \mathbb{Q}$ for all $n \in \mathbb{N}$. We are interested to determine all those $f_0, f_1, \ldots, f_k \in \mathcal{A}^*$ for which

$$f_0(A_0(n)) + f_1(A_1(n)) + \ldots + f_k(A_k(n)) \equiv 0 \pmod 1 \tag{1.5}$$

holds.

The domain of f can be extended to $\mathbb{Q}_+$ (the group of positive rationals) by defining $f(\frac{n}{m}) = f(n) - f(m)$. Let $\mathbb{Q}_+^{k+1}$ be the $(k+1)$-fold direct product of $\mathbb{Q}_+$. Let $\mathcal{B}$ be the subgroup of $\mathbb{Q}_+^{k+1}$ generated by the elements $(A_0(n), A_1(n), \ldots, A_k(n))$. Clearly, if $(\alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathcal{B}$, then

$$f_0(\alpha_0) + f_1(\alpha_1) + \ldots + f_k(\alpha_k) \equiv 0 \pmod 1.$$

If $\mathcal{B} = \mathbb{Q}_+^{k+1}$, then $f_0(\beta_0) + f_1(\beta_1) + \ldots + f_k(\beta_k) \equiv 0 \pmod 1$ holds for $\beta_\ell = n$ and $\beta_\nu = 1$ for all $\nu \neq \ell$, and so $f_\ell(n) \equiv 0 \pmod 1$ holds for all $\ell = 0, \ldots, k$.

If $\mathcal{B} \neq \mathbb{Q}_+^{k+1}$, then it may occur that there exists such a solution of (1.5) for which $f_j(n) \equiv 0 \pmod 1$, $j = 0, \ldots, k$ does not hold identically.

Let c be a fixed constant and

$$\mathcal{D} = \{(\beta_0, \ldots, \beta_k) \mid \beta_j = p \leq c, \beta_\nu = 1 \quad \text{if} \quad \nu \neq j,\ p \in \mathcal{P}\}.$$

Let us assume that $\mathcal{DB} = \mathbb{Q}_+^{k+1}$. Then one has:

If $(f_0^{(h)}, \ldots, f_k^{(h)})(h = 1, 2)$ *are such solutions of* (1.5) *for which* $f_\nu^{(1)}(p) \equiv f_\nu^{(2)}(p) \pmod 1$ *for* $\nu = 0, 1, \ldots, k$, $p \leq K$, *then*

$$f_\nu^{(1)}(n) \equiv f_\nu^{(2)}(n) \pmod 1 \quad (n \in \mathbb{N};\ \nu = 0, 1, \ldots, k).$$

This is obvious, since for $f_\nu(n) = f_\nu^{(1)}(n) - f_\nu^{(2)}(n)$ the relation

$$\sum_{j=0}^{k} f_j(\alpha_j) \equiv 0 \pmod 1$$

holds for every $(\alpha_0, \ldots, \alpha_k) \in \mathbb{Q}_+^{k+1}$.

Let $\xi_n = (A_0(n), \cdots, A_k(n))$ and assume that the group $\mathcal{B} = \mathbb{Q}_+^{k+1}$. Then, for any given $(r_0, \cdots, r_k) \in \mathbb{Q}_+^{k+1}$ there exist suitable $n_1, \cdots, n_t \in \mathbb{N}$ for which

$$(r_0, \cdots, r_k) = \prod_{j=1}^{t} \xi_{n_j}^{\epsilon_j},$$

$(\epsilon_j \in \{-1, 1\})$ i.e. that,

$$r_\ell = \prod_{j=1}^{t} A_\ell(n_j)^{\epsilon_j} \quad (\ell = 0, 1, \cdots, k).$$

Thus one has,

Theorem 1. *Let $\mathcal{B}$ be the group generated by ξ_n $(n = 1, 2, \cdots)$ and $\mathcal{B} = \mathbb{Q}_+^{k+1}$. Let $\mathbb{G}$ be an Abelian group, $\mathbb{G}_0$ be an arbitrary subgroup of $\mathbb{G}$. Let $f_j \in \mathcal{A}_{\mathbb{G}}^*$, and assume that*

$$t_n = \sum_{j=0}^{k} f_j(A_j(n)) \in \mathbb{G}_0 \quad (n = 1, 2 \cdots).$$

Then $f_j(n) \in \mathbb{G}_0$ for all $n \in \mathbb{N}$ and $j = 0, \cdots, k$.

We recommend [1] for further study.

1.5. Statement of the results

We shall prove the following three theorems.

Theorem 2. *Let $f_0, f_1, f_2, f_3, f_4 \in \mathcal{A}^*$. Assume that*

$$\mathcal{A}_f(n) = f_0(n) + f_1(n+1) + f_2(n+2) + f_3(n+3) + f_4(n+4) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. Then

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$.

Theorem 3. *Let $f_0, f_1, f_2, f_3, f_4 \in \mathcal{A}^*$. Assume that*

$$\mathcal{B}_f(n) = f_0(n) + f_1(n+2) + f_2(n+3) + f_3(n+4) + f_4(n+6) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. Then

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$.

Theorem 4. *Let* $f_0,\ f_1,\ f_2,\ f_3,\ f_4 \in \mathcal{A}^*$. *Assume that*

$$\mathcal{C}_f(n) = f_0(n) + f_1(n+1) + f_2(n+3) + f_3(n+5) + f_4(n+6) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. *Then*

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$.

2. Proof of Theorem 2

Firstly we prove a few lemmas.

Lemma 1. *Let* $\mathcal{T}_0,\ \mathcal{T}_1, \mathcal{T}_2 \in \mathcal{A}^*$. *Assume that*

$$\mathcal{T}_0(n) + \mathcal{T}_1(n+1) + \mathcal{T}_2(n+2) - \mathcal{T}_2(n+4) - \mathcal{T}_1(n+5) - \mathcal{T}_0(n+6) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{N}$. *Then*

$$\mathcal{T}_0(n) \equiv \mathcal{T}_1(n) \equiv \mathcal{T}_2(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{N}$.

Proof. This is Theorem 1 in [7]. ■

Lemma 2. *Let* $a_0,\ a_1,\ a_2 \in \mathcal{A}^*$. *Assume that*

$$\text{(2.1)}\quad \mathcal{H}(n) = a_0(n) + a_1(n+1) + a_2(n+2) + a_1(n+3) + a_0(n+4) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{N}$. *If*

$$\text{(2.2)}\qquad a_0(n) \equiv a_1(n) \equiv a_2(n) \equiv 0 \pmod 1 \quad \textit{for} \quad n \le 12.$$

Then,

$$\text{(2.3)}\qquad a_0(n) \equiv a_1(n) \equiv a_2(n) \equiv 0 \pmod 1 \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. Assume that the conditions (2.1) and (2.2) are satisfied and (2.3) is not true. Then there is a minimal positive integer n_0 with $n_0 > 12$ for which $a_i(n_0) \not\equiv 0 \pmod 1$. Then n_0 should be a prime $p \ge 13$. Let $a_2(p) \equiv \xi \not\equiv 0$

(mod 1). Using $\mathcal{H}(p-2)\equiv 0 \pmod 1$ we have that $a_0(p+2)\equiv -\xi \pmod 1$ and $p+2\in\mathcal{P}$. Thus

$$p\equiv 2 \pmod 3. \tag{2.4}$$

Using (2.4) and $p\ge 13$, we have $2|p+3,\ 3|p+4,\ 2|p+5$, consequently we infer from $\mathcal{H}(p+2)\equiv 0 \pmod 1$ that $a_0(p+6)\equiv -\xi \pmod 1$ and $p+6\in\mathcal{P}$. Since

$$\mathcal{H}(p+6)=a_0(p+6)+a_1(p+7)+a_2(p+8)+a_1(p+9)$$
$$+a_0(p+10)\equiv 0 \pmod 1$$

and $2|p+7,\ 2|p+9,\ 3|p+10$, therefore $a_2(p+8)\equiv -\xi \pmod 1$ and $p+8\in\mathcal{P}$. Thus we have proved that $p,p+2,p+6,p+8\in\mathcal{P}$, which implies that

$$p\equiv 1 \pmod 5. \tag{2.5}$$

Next, we prove the following assertion:

$$\textit{if}\quad p\in\mathcal{P},\ q<2p-3,\ \textit{then}\quad a_1(q)\equiv 0 \pmod 1. \tag{2.6}$$

This clearly holds if $q<p$. Let $p\le q<2p-3$. Then either $3|q-2$ or $3|q+2$. Since

$$\mathcal{H}(q-1)=a_0(q-1)+a_1(q)+a_2(q+1)+a_1(q+2)+a_0(q+3)\equiv 0 \pmod 1$$

and

$$\mathcal{H}(q-3)=a_0(q-3)+a_1(q-2)+a_2(q-1)+a_1(q)+a_0(q+1)\equiv 0 \pmod 1$$

and $2|q+\ell,\ \frac{q+\ell}{2}<p$ if $\ell=-3,-1,1,3$. Thus

$$a_1(q)+a_1(q+2)\equiv 0 \pmod 1 \quad \text{and} \quad a_1(q-2)+a_1(q)\equiv 0 \pmod 1.$$

Since either $a_1(q-2)\equiv 0 \pmod 1$ or $a_1(q+2)\equiv 0 \pmod 1$, consequently $a_1(q)\equiv 0 \pmod 1$. Hence (2.6) is proved.

From

$$\mathcal{H}(2p+1)=a_0(2p+1)+a_1(2p+2)+a_2(2p+3)+a_1(2p+4)$$
$$+a_0(2p+5)\equiv 0 \pmod 1,$$

observing from (2.4), (2.5) and (2.6) that $4|2p+2,\ 5|2p+3,\ 3|2p+5$, and that $a_1(2p+4)\equiv a_1(p+2)\equiv 0 \pmod 1$, we deduce that $a_0(2p+1)\equiv 0 \pmod 1$. Therefore, $\mathcal{H}(4p-2)\equiv 0 \pmod 1$ implies that

$$a_0(4p-2)+a_1(4p-1)+a_2(4p)+a_1(4p+1)+a_0(4p+2)\equiv 0 \pmod 1.$$

Since $6|4p-2,\ 5|4p+1$ and $a_0(2p+1)\equiv 0 \pmod 1$, therefore

$$a_2(p)+a_1(4p-1)\equiv 0 \pmod 1.$$

Thus,

$$a_1(4p-1)\equiv -\xi \pmod 1,\ \text{and}\ 4p-1\in\mathcal{P}. \tag{2.7}$$

Since $\mathcal{H}(2p-3)\equiv 0 \pmod 1$, $4|2p-2,\ 3|2p-1$ and $a_0(2p+1)\equiv a_1(p)\equiv \equiv 0 \pmod 1$, therefore $a_0(2p-3)\equiv 0 \pmod 1$.

From $\mathcal{H}(4p-6)\equiv 0 \pmod 1$ we deduce that

$$a_0(4p-6)+a_1(4p-5)+a_2(4p-4)+a_1(4p-3)+a_0(4p-2)\equiv 0 \pmod 1.$$

It is obvious that $6|4p-2$ implies $a_0(4p-2)\equiv 0 \pmod 1$, $3|4p-5$. Thus either $4p-5=3q,\ q\in\mathcal{P},\ q<2p-3$, or $\frac{4p-5}{5}$ is not a prime. In both cases we deduce from (2.6) that $a_1(4p-5)\equiv 0 \pmod 1$. Thus we derive,

$$a_0(2p-3)+a_1(4p-3)\equiv 0 \pmod 1.$$

Consequently,

$$a_1(4p-3)\equiv 0 \pmod 1. \tag{2.8}$$

Finally, from $\mathcal{H}(4p-4)\equiv 0 \pmod 1$ we have

$$a_0(4p-4)+a_1(4p-3)+a_2(4p-2)+a_1(4p-1)+a_0(4p)\equiv 0 \pmod 1.$$

Since $a_0(p)\equiv 0 \pmod 1$, $8|4p-4,\ 6|4p-2$, we get from (2.8) that $a_1(4p-1)\equiv \equiv 0 \pmod 1$. This contradicts (2.7). ■

Lemma 3. *Let* $a_0,\ a_1,\ a_2\in\mathcal{A}^*$ *and*

$$\mathcal{H}(n)=a_0(n)+a_1(n+1)+a_2(n+2)+a_1(n+3)+a_0(n+4).$$

If

$$\mathcal{H}(n)\equiv 0 \pmod 1 \quad \textit{for all} \quad n\in\mathbb{N}, \tag{2.9}$$

then (2.2) *is true, i.e.*

$$a_0(n)\equiv a_1(n)\equiv a_2(n)\equiv 0 \pmod 1 \quad \textit{for} \quad n\le 12.$$

Proof. Let $\mathcal{B}$ be the subgroup of $\mathbb{Q}_+^3$ generated by the sequences

$$L_n=\Big(n(n+4),\ (n+1)(n+3),\ n+2\Big) \quad (n\in\mathbb{N}).$$

It is easy to see (by (2.9)) that,

$$a_0(a) + a_1(b) + a_2(c) \equiv 0 \pmod 1 \quad \text{for all} \quad (a,b,c) \in \mathcal{B}. \tag{2.10}$$

We use the following notations for a prime p:

$$a_p = (p,1,1),\ b_p = (1,p,1) \quad \text{and} \quad c_p = (1,1,p).$$

We show that $a_p, b_p,$ and $c_p \in \mathcal{B}$ for all primes $p \le 11$. This assertion along with (2.10) would imply (2.2).

Using a simple Maple program, for,

$$n \in \{1,2,3,4,5,8,12,7,11,14,48,9,13,16,22,23,19,15,25,26,28,31\},$$

we can give $a_p,\ b_q,\ c_r$ for primes $p,q \le 31,$ and $r \le 17$ in terms of L_n and $a_2,\ a_3,\ b_2,\ b_3,\ c_2,\ c_3$ and c_5.

Table 1

n	L_n	$a_p,\ b_q,\ c_r$
1	$(5, 2^3, 3)$	$a_5 = \frac{L_1}{b_2^3 c_3}$
2	$(2^2.3, 3.5, 2^2)$	$b_5 = \frac{L_2}{a_2^2 a_3 b_3 c_2^2},$
3	$(3.7, 2^3.3, 5)$	$a_7 = \frac{L_3}{a_3 b_2^3 b_3 c_5}$
4	$(2^5, 5.7, 2.3)$	$b_7 = \frac{L_4}{a_2^5 b_5 c_2 c_3} = \frac{L_4 a_3 b_3 c_2}{L_2 a_2^3 c_3}$
5	$(3^2.5, 2^4.3, 7)$	$c_7 = \frac{L_5}{a_3^2 a_5 b_2^4 b_3} = \frac{L_5 c_3}{L_1 a_3^2 b_2 b_3}$
8	$(2^5.3, 3^2.11, 2.5)$	$b_{11} = \frac{L_8}{a_2^5 a_3 b_3^2 c_2 c_5} = \frac{L_8}{a_2^5 a_3 b_3^2 c_2 c_5}$
12	$(2^6.3, 3.5.13, 2.7)$	$b_{13} = \frac{L_{12}}{a_2^6 a_3 b_3 b_5 c_2 c_7} = \frac{L_1 L_{12} a_3^2 b_3 c_2 b_2}{L_2 L_5 a_2^4 c_3}$
7	$(7.11, 2^4.5, 3^2)$	$a_{11} = \frac{L_7}{a_7 b_2^4 b_5 c_3^2} = \frac{L_7 a_3^2 b_3^2 c_5 a_2^2 c_2^2}{L_3 b_2 L_2 c_3^2}$
11	$(3.5.11, 2^3.3.7, 13)$	$c_{13} = \frac{L_{11}}{a_3 a_5 a_{11} b_2^3 b_3 b_7} = \frac{L_2^2 L_3 L_{11} b_2 c_3^4 a_2}{L_1 L_4 L_7 a_3^4 b_3^4 c_5 c_2^3}$
14	$(2^2.3^2.7, 3.5.17, 2^4)$	$b_{17} = \frac{L_{14}}{a_2^2 a_3^2 a_7 b_3 b_5 c_2^4} = \frac{L_{14} b_2^3 b_3 c_5}{L_2 L_3 c_2^2}$
48	$(2^6.3.13, 3.7^2.17, 2.5^2)$	$a_{13} = \frac{L_{48}}{a_2^6 a_3 b_3 b_7^2 b_{17} c_2 c_5^2} = \frac{L_2^3 L_3 L_{48} c_3^2}{L_4^2 L_{14} a_3^3 b_2^3 b3^4 c_2 c_5^3}$
9	$(3^2.13, 2^3.3.5, 11)$	$c_{11} = \frac{L_9}{a_3^2 a_{13} b_2^3 b_3 b_5} = \frac{L_4^2 L_9 L_{14} a_3^2 b_3^4 c_2^3 c_5^3 a_2^2}{L_3 L_2^4 L_{48} c_3^2}$
13	$(13.17, 2^5.7, 3.5)$	$a_{17} = \frac{L_{13}}{a_{13} b_2^5 b_7 c_3 c_5} = \frac{L_4 L_{13} L_{14} a_3^2 b_3^3 c_5^2 a_2^3}{L_2^2 L_3 L_{48} c_3^2 b_2^2}$
16	$(2^6.5, 17.19, 2.3^2)$	$b_{19} = \frac{L_{16}}{a_2^6 a_5 b_{17} c_2 c_3^2} = \frac{L_2 L_3 L_{16} c_2}{L_1 L_{14} a_2^6 c_3 b_3 c_5}$
22	$(2^2.11.13, 5^2.23, 2^3.3)$	$b_{23} = \frac{L_{22}}{a_2^2 a_{11} a_{13} b_5^2 c_2^3 c_3} = \frac{L_4^2 L_{14} L_{22} a_3^3 b_2^4 b_3^4 c_5^2}{L_2^4 L_7 L_{48} c_3}$
23	$(3^3.23, 2^4.3.13, 5^2)$	$a_{23} = \frac{L_{23}}{a_3^3 b_2^4 b_3 b_{13} c_5^2} = \frac{L_2 L_5 L_{23} a_2^4 c_3}{L_1 L_{12} a_3^5 b_2^5 b_3^2 c_2 c_5^2}$
19	$(19.23, 2^3.5.11, 3.7)$	$a_{19} = \frac{L_{19}}{a_{23} b_2^3 b_5 b_{11} c_3 c_7} = \frac{L_1^2 L_{12} L_{19} a_3^9 b_2^3 b_3^6 a_2^3 c_2^4 c_5^3}{L_2^2 L_5^2 L_8 L_{23} c_3^3}$
15	$(3.5.19, 2^5.3^2, 17)$	$c_{17} = \frac{L_{15}}{a_3 a_5 a_{19} b_2^5 b_3^2} = \frac{L_2^2 L_5^2 L_8 L_{15} L_{23} c_3^4}{L1^3 L_{12} L_{19} a_2^3 a_3^{10} b_2^5 b_3^8 c_2^4 c_5^3}$

n	L_n	$a_p,\ b_q,\ c_r$
25	$(5^2.29, 2^3.7.13, 3^3)$	$a_{29} = \frac{L_{25}}{a_5^2 b_2^3 b_7 b_{13} c_3^3} = \frac{L_2^2 L_5 L_{25} b_2^2 c_3 a_2^7}{L_1^3 L_4 L_{12} a_3^3 b_3^2 c_2^2}$
26	$(2^2.3.5.13, 3^3.29, 2^2.7)$	$b_{29} = \frac{L_{26}}{a_2^2 a_3 a_5 a_{13} b_3^3 c_2^2 c_7} = \frac{L_4^2 L_{14} L_{26} a_3^4 b_2^7 b_3^2 c_5^3}{L_2^3 L_3 L_5 L_{48} a_2^2 c_2 c_3^2}$
28	$(2^7.7, 29.31, 2.3.5)$	$b_{31} = \frac{L_{28}}{a_2^7 a_7 b_{29} c_2 c_3 c_5} = \frac{L_2^3 L_5 L_{28} L_{48} c_3}{L_4^2 L_{14} L26 a_2^5 a_3^3 b_2^4 b_3 c_5^3}$
31	$(5.7.31, 2^6.17, 3.11)$	$a_{31} = \frac{L_{31}}{a_5 a_7 b_2^6 b_{17} c_3 c_{11}} = \frac{L_2^5 L_3 L_{31} L_{48} c_3^2}{L_1 L_4^2 L_9 L_{14}^2 b_2^3 a_3 b_3^4 c_5^3 c_2 a_2^2}$

Now, by using the above relations for $n = 6,\ 10,\ 18,\ 24,\ 32,\ 30,\ 54$ and $n = 62$, we will get the following 8 equations.

(2.11) $$E_1 := \frac{L_2 L_6}{L_1 L_4} = \frac{a_3^2 b_3^3 c_2^4}{a_2 b_2^3 c_3^2} \in \mathcal{B},$$

(2.12) $$E_2 := \frac{L_2 L_5 L_{10}}{L_1^2 L_3 L_8 L_{12}} = \frac{c_2^2}{a_2^7 b_2^5 b_3^2 c_3 c_5^2} \in \mathcal{B},$$

(2.13) $$E_3 := \frac{L_1 L_2 L_{14} L_{18}}{L_4 L_7 L_{16}} = \frac{a_3^5 b_3^3 c_2^6 c_5}{a_2^5 b_2 c_3^4} \in \mathcal{B},$$

(2.14) $$E_4 := \frac{L_1 L_4 L_7 L_{24}}{L_2^4 L_3^2 L_{11}} = \frac{a_2^2 c_3^4}{a_3^6 b_2^2 b_3^4 c_2^6 c_5^2} \in \mathcal{B},$$

(2.15) $$E_5 := \frac{L_1^3 L_{12} L_{19} L_{32}}{L_2^2 L_4 L_5^2 L_8^2 L_{15} L_{23}} = \frac{c_3^3}{a_2^6 a_3^9 b_2^5 b_3^9 c_2^5 c_5^4} \in \mathcal{B},$$

(2.16) $$E_6 := \frac{L_3 L_4 L_{26} L_{30}}{L_1 L_2 L_5 L_8 L_{13} L_{28}} = \frac{b_3 c_2^4}{a_2^5 a_3 b_2^9 c_3^2 c_5^2} \in \mathcal{B},$$

(2.17) $$E_7 := \frac{L_1^5 L_4 L_{12} L_{14} L_{54}}{L_2^4 L_3 L_5^2 L_8 L_{16} L_{25}} = \frac{b_2 c_3}{a_2^4 a_3^4 b_3^6 c_2 c_5^2} \in \mathcal{B}$$

and

(2.18) $$E_8 := \frac{L_4 L_5 L_9 L_{14}^2 L_{62}}{L_2^3 L_7 L_{12} L_{31} L_{48}} = \frac{a_3^4 b_3 c_2^7}{a_2^7 b_2^3 c_3^2 c_5^2} \in \mathcal{B}.$$

This system has solutions in $a_2, a_3, b_2, b_3, c_2, c_3, c_5$, which are given in terms of $E_1, \cdots, E_8$. Thus $a_2, a_3, b_2, b_3, c_2, c_3, c_5$ are elements of $\mathcal{B}$.

The solutions of the above equations (2.11)–(2.18) can be obtained now as follows: we express a_2 from the expression of E_1 in (2.11), similarly c_5 from (2.13). After taking these expressions of a_2 and c_5 into equations (2.11)–(2.18), we get c_3, b_3, c_2 and a_3 from the expressions of $\frac{E_2}{E_4^2}$, $\frac{E_6}{E_7}$, $\frac{E_4}{E_5}$ and $\frac{E_4^{17}}{E_7^{11}}$, respectively. Finally the solution b_2 can be gotten from (2.14) and (2.18) in the expression of $\frac{E_8^{15}}{E_4^{32}}$. The solutions are:

$$a_2 = \frac{E_1^{905} E_2^{151} E_3^{72} E_4^{109} E_5^{228}}{E_6^{528} E_7^{77} E_8^{75}}, \qquad a_3 = \frac{E_1^{11} E_2^{2} E_5^{6}}{E_3^{4} E_4^{5} E_6^{7} E_7^{4}}$$

$$b_2 = \frac{E_1^{135} E_2^{25} E_3^{26} E_4^{38} E_5^{21}}{E_6^{77} E_8^{15}}, \qquad b_3 = \frac{E_3^{18} E_4^{25} E_6^{43} E_7^{23}}{E_1^{69} E_2^{8} E_5^{37}},$$

$$c_2 = \frac{E_6^{266} E_7^{20} E_8^{45}}{E_1^{461} E_2^{82} E_3^{62} E_4^{92} E_5^{94}}, \qquad c_3 = \frac{E_6^{969} E_7^{109} E_8^{150}}{E_1^{1670} E_2^{287} E_3^{176} E_4^{263} E_5^{383}},$$

and

$$c_5 = \frac{E_1^{898} E_2^{138} E_3^{21} E_4^{33} E_5^{274}}{E_6^{531} E_7^{118} E_8^{60}}.$$

Finally, it is obvious from Table 1 that

$$a_5,\ b_5,\ a_7,\ b_7,\ a_{11},\ b_{11} \quad \text{and} \quad c_{11}$$

are elements of $\mathcal{B}$. This completes the proof of Lemma 3. ■

Proof of Theorem 2. Assume that $f_0,\ f_1,\ f_2,\ f_3,\ f_4 \in \mathcal{A}^*$ satisfy the condition

$$\mathcal{A}_f(n) := f_0(n) + f_1(n+1) + f_2(n+2) + f_3(n+3) + f_4(n+4) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. Then

$$\mathcal{A}_f(-n-4) = f_4(n) + f_3(n+1) + f_2(n+2) + f_1(n+3) + f_0(n+4) \equiv 0 \pmod 1.$$

Let

$$\varphi_0(n) = f_0(n) - f_4(n) \quad \text{and} \quad \varphi_1(n) = f_1(n) - f_3(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Thus, we deduce from the above relations that,

$$\varphi_0(n) + \varphi_1(n+1) - \varphi_1(n+3) - \varphi_0(n+4) \equiv 0 \pmod 1 \quad \text{for all} \quad n \in \mathbb{Z}.$$

From Lemma 1, we have that $\varphi_0(n) \equiv \varphi_1(n) \equiv 0 \pmod 1$, consequently $f_0(n) \equiv f_4(n) \pmod 1$ and $f_1(n) \equiv f_3(n) \pmod 1$ for all $n \in \mathbb{Z}$. Hence

$$\mathcal{A}_f(n) \equiv f_0(n) + f_1(n+1) + f_2(n+2) + f_1(n+3) + f_0(n+4) \equiv 0 \pmod 1$$

is true for all $n \in \mathbb{Z}$. The conditions of Lemma 2 and Lemma 3 are satisfied by taking $a_j(n) = f_j(n)\ \ (j = 0, 1, 2)$ and

$$\mathcal{H}(n) = f_0(n) + f_1(n+1) + f_2(n+2) + f_1(n+3) + f_0(n+4).$$

Thus

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$. This completes the proof. ■

We can deduce an interesting result from Lemma 3.

Theorem 5. *If $\mathcal{B}$ denotes the subgroup of $\mathbb{Q}_+^3$ generated by the sequences*

$$L_n = \Big(n(n+4),\ (n+1)(n+3),\ n+2\Big) \quad (n \in \mathbb{N}),$$

then we have

$$\mathcal{B} = \mathbb{Q}_+^3.$$

3. Proof of Theorem 3

We follow similar strategy as in the case of Theorem 2 and prove a couple of lemmas before completing the proof of the theorem.

Lemma 4. *Let $b_0,\ b_1,\ b_2 \in \mathcal{A}^*$. Assume that*

$$\mathcal{S}(n) := b_0(n) + b_1(n+2) + b_2(n+3) + b_1(n+4) + b_0(n+6) \equiv 0 \pmod 1$$

for all $n \in \mathbb{N}$. If

$$b_0(n) \equiv b_1(n) \equiv b_2(n) \equiv 0 \pmod 1 \quad \textit{for} \quad n \leq 10,$$

then

$$b_0(n) \equiv b_1(n) \equiv b_2(n) \equiv 0 \pmod 1 \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. Let n_0 be the minimal positive integer for which $b_j(n_0) \not\equiv \not\equiv 0 \pmod 1$ holds for some $j \in \{0, 1, 2\}$. It is clear that n_0 should be a prime P, $P \geq 11$. $\mathcal{S}(P-6) \equiv \pmod 1$ implies that $b_0(P) \equiv 0 \pmod 1$,

$\mathcal{S}(P-3) \equiv 0 \pmod 1$ similarly that $b_2(P) \equiv 0 \pmod 1$. It remains to consider the case when $b_1(P) \equiv \xi \pmod 1$. Then $S(P-4) \equiv 0 \pmod 1$ implies that $b_0(P+2) \equiv -\xi \pmod 1$, $P+2$ is a prime, thus $P \equiv 2 \pmod 3$. From $S(P-2) \equiv 0 \pmod 1$ we obtain that

$$b_0(P-2) + b_1(P) + b_2(P+1) + b_1(P+2) + b_0(P+4) \equiv 0 \pmod 1,$$

which, by $3|P+4,\ 2|P+1$ implies that

$$b_1(P) + b_1(P+2) \equiv 0 \pmod 1, \quad \text{i.e} \quad b_1(P+2) \equiv -\xi \pmod 1.$$

Finally, we infer from $4|2P+2,\ 3|2P+5,\ 4|2P+6,\ 6|2P+8$ and $S(2P+2) \equiv \equiv 0 \pmod 1$ that

$$b_0(2P+2) + b_1(2P+4) + b_2(2P+5) + b_1(2P+6) + b_0(2P+8) \equiv 0 \pmod 1,$$

and so

$$b_1(P+2) \equiv 0 \pmod 1.$$

This contradicts to the fact that $b_1(P+2) \equiv -\xi \pmod 1$ and consequently the Lemma 4 is proved. ■

Lemma 5. *Let $b_0,\ b_1,\ b_2 \in \mathcal{A}^*$. If*

$$b_0(n) + b_1(n+2) + b_2(n+3) + b_1(n+4) + b_0(n+6) \equiv 0 \pmod 1,$$

for all $n \in \mathbb{N}$, then

$$b_0(n) \equiv b_1(n) \equiv b_2(n) \equiv 0 \pmod 1 \quad \textit{for} \quad n \leq 10.$$

Proof. The proof is similar to the proof of Lemma 3. Let $\mathcal{D}$ be the subgroup of $\mathbb{Q}_+^3$ generated by the sequences

$$D_n := \big(n(n+6),\ (n+2)(n+4),\ n+3\big) \quad (n \in \mathbb{N}).$$

From (3.4) we obtain that,

$$b_0(a) + b_1(b) + b_2(c) \equiv 0 \pmod 1 \quad \text{for all} \quad (a,b,c) \in \mathcal{D}.$$

We shall use the following notations (p is prime):

$$A_p := (p,1,1) \in \mathcal{D},\ B_p := (1,p,1 \in \mathcal{D}) \quad \text{and} \quad C_p := (1,1,p) \in \mathcal{D}.$$

We shall prove that $A_p, B_p \in \mathcal{D}$ and $C_p \in \mathcal{B}$ for all primes $p \leq 7$. This will prove Lemma 5.

First, by using a simple Maple program, we shall give $A_p,\ B_q,\ C_\pi$ for primes $p \le 23,\ q \le 23,\ \pi \le 23$ in terms of L_n and $A_2,\ A_3,\ B_2,\ B_3,\ C_2,\ C_3$ and A_5.

n	D_n	$A_p,\ B_q,\ C_\pi$
2	$(2^4,\ 2^3.3,\ 5)$	$C_5 = \frac{D_2}{A_2^4 B_2^3 B_3}$
4	$(2^3.5, 2^4.3, 7)$	$C_7 = \frac{D_4}{A_2^3 A_5 B_2^4 B_3}$
6	$(2^3.3^2, 2^4.5, 3^2)$	$B_5 = \frac{D_6}{A_2^3 A_3^2 B_2^4 C_3^2}$
1	$(7,\ 3.5,\ 2^2)$	$A_7 = \frac{D_1}{B_3 B_5 C_2^2} = \frac{D_1 A_2^3 A_3^2 B_2^4 C_3^2}{D_6 B_3 C_2^2}$
3	$(3^3,\ 5.7,\ 2.3)$	$B_7 = \frac{D_3}{A_3^3 B_5 C_2 C_3} = \frac{D_3 A_2^3 B_2^4 C_3}{A_3 D_6 C_2}$
18	$(2^4.3^3,\ 2^3.5.11,\ 3.7)$	$B_{11} = \frac{D_{18}}{A_2^4 A_3^3 B_2^3 B_5 C_3 C_7} = \frac{D_{18} A_2^2 A_5 B_2^5 B_3 C_3}{D_4 D_6 A_3}$
5	$(5.11,\ 3^2.7,\ 2^3)$	$A_{11} = \frac{D_5}{A_5 B_3^2 B_7 C_2^3} = \frac{D_5 D_6 A_3}{D_3 A_2^3 A_5 B_2^4 B_3^2 C_2^2 C_3}$
7	$(7.13,\ 3^2.11,\ 2.5)$	$A_{13} = \frac{D_7}{A_7 B_3^2 B_{11} C_2 C_5} = \frac{D_4 D_6^2 D_7 C_2}{D_1 D_2 D_{18} A_2 A_3 A_5 B_2^6 B_3 C_3^3}$
8	$(2^4.7,\ 2^3.3.5,\ 11)$	$C_{11} = \frac{D_8}{A_2^4 A_7 B_2^3 B_3 B_5} = \frac{D_8 C_2^2}{D_1 A_2^4 B_2^3}$
9	$(3^3.5,\ 11.13,\ 2^2.3)$	$B_{13} = \frac{D_9}{A_3^3 A_5 B_{11} C_2^2 C_3} = \frac{D_4 D_6 D_9}{D_{18} A_2^2 A_3^2 A_5^2 B_2^5 B_3 C_2^2 C_3^2}$
10	$(2^5.5,\ 2^3.3.7,\ 13)$	$C_{13} = \frac{D_{10}}{A_2^5 A_5 B_2^3 B_3 B_7} = \frac{D_6 D_{10} A_3 C_2}{D_3 A_2^8 A_5 B_2^7 B_3 C_3}$
14	$(2^3.5.7,\ 2^5.3^2,\ 17)$	$C_{17} = \frac{D_{14}}{A_2^3 A_5 A_7 B_2^5 B_3^2} = \frac{D_6 D_{14} C_2^2}{D_1 A_2^6 A_3^2 A_5 B_3 B_2^9 C_3^2}$
11	$(11.17, 3.5.13, 2.7)$	$A_{17} = \frac{D_{11}}{A_{11} B_3 B_5 B_{13} C_2 C_7} =$ $= \frac{D_3 D_{11} D_{18} A_2^{11} A_3^3 A_5^4 B_2^{17} B_3^3 C_2^3 C_3^5}{D_4^2 D_5 D_6^3 D_9}$
21	$(3^4.7, 5^2.23, 2^3.3)$	$B_{23} = \frac{D_{21}}{A_3^4 A_7 B_5^2 C_2^3 C_3} = \frac{D_{21} A_2^3 B_2^4 B_3 C_3}{D_1 D_6 A_3^2 C_2}$
16	$(2^5.11,\ 2^3.3^2.5,\ 19)$	$C_{19} = \frac{D_{16}}{A_2^5 A_{11} B_2^3 B_3^2 B_5} = \frac{D_3 D_{16} A_2 A_3 A_5 B_2^5 C_2^2 C_3^3}{D_5 D_6^2}$
20	$(2^3.5.13, 2^4.3.11, 23)$	$C_{23} = \frac{D_{20}}{A_2^3 A_5 A_{13} B_2^4 B_3 B_{11}} = \frac{D_1 D_2 D_{20} A_3^2 C_3^2}{D_6 D_7 A_2^4 A_5 B_2^3 B_3 C_2}$
30	$(2^3.3^3.5,\ 2^6.17,\ 3.11)$	$B_{17} = \frac{D_{30}}{A_2^3 A_3^3 A_5 B_2^6 C_3 C_{11}} = \frac{D_1 D_{30} A_2}{D_8 A_3^3 A_5 B_2^3 C_2^2 C_3}$
13	$(13.19,\ 3.5.17,\ 2^4)$	$A_{19} = \frac{D_{13}}{A_{13} B_3 B_5 B_{17} C_2^4} = \frac{D_2 D_8 D_{13} D_{18} A_2^3 A_3^6 A_5^2 B_2^{13} C_3^6}{D_4 D_6^3 D_7 D_{30} C_2^3}$
15	$(3^2.5.7,\ 17.19,\ 2.3^2)$	$B_{19} = \frac{D_{15}}{A_3^2 A_5 A_7 B_{17} C_2 C_3^2} = \frac{D_6 D_8 D_{15} B_3 C_2^3}{D_1^2 D_{30} A_2^4 A_3 B_2 C_3^3}$
17	$(17.23, 3.7.19, 2^2.5)$	$A_{23} = \frac{D_{17}}{A_{17} B_3 B_7 B_{19} C_2^2 C_5} =$ $= \frac{D_1^2 D_4^2 D_5 D_6^3 D_9 D_{17} D_{30}}{D_2 D_3^2 D_8 D_{11} D_{18} D_{15} A_2^6 A_3 A_5^4 B_2^{17} B_3^4 C_2^7 C_3^3}$

Table 2

Now, by using the above relations for $n = 12,\ 19,\ 22,\ 24,\ 32,\ 42,\ 46$ and $n = 48$, we will get 8 equations.

For $n = 12$, we have $D_{12} = A_2^3 A_3^3 B_2^5 B_7 C_3 C_5 = \frac{D_2 D_3 A_2^2 A_3^2 B_2^6 C_3^2}{D_6 C_2 B_3}$. Consequently

$$F_1 := \frac{D_6 D_{12}}{D_2 D_3} = \frac{A_2^2 A_3^2 B_2^6 C_3^2}{B_3 C_2} \in \mathcal{D}. \tag{3.1}$$

For $n = 19$, we infer from $A_{19},\ B_7,\ B_{23},\ C_{11}$ and

$$D_{19} = A_5^2 A_{19} B_3 B_7 B_{23} C_2 C_{11} = \frac{D_2 D_3 D_8^2 D_{13} D_{18} D_{2}1 A_2^5 A_3^3 A_5^4 B_2^{18} B_3^2 C_3^8}{D_1^2 D_4 D_6^5 D_7 D_{30} C_2^2}$$

that

$$F_2 := \frac{D_{19} D_1^2 D_4 D_6^5 D_7 D_{30}}{D_2 D_3 D_8^2 D_{13} D_{18} D_{21}} = \frac{A_2^5 A_3^3 A_5^4 B_2^{18} B_3^2 C_3^8}{C_2^2} \in \mathcal{D}. \tag{3.2}$$

As,

$$D_{22} = A_2^3 A_7 A_{11} B_2^4 B_3 B_{13} C_5^2 = \frac{D_1 D_2^2 D_4 D_5 D_6 D_9 A_3}{D_3 D_{18} A_2^7 A_5^3 B_2^7 B_3^5 C_3 C_2^6},$$

we have,

$$F_3 := \frac{D_3 D_{18} D_{22}}{D_1 D_2^2 D_4 D_5 D_6 D_9} = \frac{A_3}{A_2^7 A_5^3 B_2^7 B_3^5 C_2^6 C_3} \in \mathcal{D}. \tag{3.3}$$

Similarly, we get from B_7 and B_{13} that

$$D_{24} = A_2^4 A_3^2 A_5 B_2^3 B_7 B_{13} C_3^3 = \frac{D_3 D_4 D_9 A_2^5 B_2^2 C_3^2}{D_{18} A_3 A_5 B_3 C_2^3}.$$

This implies that,

$$F_4 := \frac{D_{18} D_{24}}{D_3 D_4 D_9} = \frac{A_2^5 B_2^2 C_3^2}{A_3 A_5 B_3 C_2^3} \in \mathcal{D}. \tag{3.4}$$

For $n = 32$, we get from $A_{19},\ B_{17},\ C_5,\ C_7$ that

$$D_{32} = A_2^6 A_{19} B_2^3 B_3^2 B_{17} C_5 C_7 = \frac{D_1 D_2^2 D_{13} D_{18} A_2^3 A_3^3 B_2^6 C_3^5}{D_6^3 D_7 C_2^5},$$

which gives

$$F_5 := \frac{D_6^3 D_7 D_{32}}{D_{13} D_{18} D_2^2 D_1} = \frac{A_2^3 A_3^3 B_2^6 C_3^5}{C_2^5} \in \mathcal{D}. \tag{3.5}$$

For $n = 42,\ 46,$ and 48, we get the following equations:

$$F_6 := \frac{D_4 D_6^3 D_{42}}{D_2 D_{18} D_{21}} = \frac{A_2^9 A_3 A_5 B_2^{13} C_3^6}{C_2^3} \in \mathcal{D}, \tag{3.6}$$

$$F_7 := \frac{D_2^2 D_{11} D_3^2 D_{15} D_8 D_{18}^2 D_{46}}{D_7 D_1 D_6^7 D_4^5 D_{17} D_5 D_9 D_{30}} = \frac{1}{A_2^{16} A_3^6 A_5^7 B_2^{34} B_3^6 C_2^6 C_3^{10}} \in \mathcal{D}, \tag{3.7}$$

$$F_8 := \frac{D_1 D_{18} D_{48}}{D_4 D_6^4 D_9 D_{14}} = \frac{1}{A_2^9 A_3^4 A_5^3 B_2^{19} B_3^2 C_3^7} \in \mathcal{D}. \tag{3.8}$$

The solutions of the above equations (3.1)-(3.8) can be obtained now as follows: we express B_3 from (3.1), similarly A_5 from (3.4), A_3 from (3.6). Therefore, we get C_2 from (3.3) and (3.6), C_3 from (3.3) and (3.5), B_2 from (3.2) and (3.5). Finally the solution A_2 can be gotten from (3.3) and (3.7). The solutions are:

$$A_2 = \frac{F_1^{11} F_2^7 F_4^{17} F_7^{19}}{F_3^{10} F_5^{11} F_6^{25} F_8^{39}}, A_3 = \frac{F_1^{838} F_2^{503} F_4^{1174} F_7^{1178}}{F_3^{672} F_5^{673} F_6^{1679} F_8^{2357}}, A_5 = \frac{F_3^{1816} F_5^{1812} F_6^{4536} F_8^{6342}}{F_1^{2274} F_2^{1362} F_4^{3176} F_7^{3173}},$$

$$B_2 = \frac{F_3^{125} F_5^{127} F_6^{282} F_8^{452}}{F_1^{142} F_2^{83} F_4^{185} F_7^{228}}, \quad B_3 = \frac{F_1^{1526} F_2^{919} F_4^{2158} F_7^{2090}}{F_3^{1203} F_5^{1199} F_6^{3055} F_8^{4186}},$$

$$C_2 = \frac{F_1^{181} F_2^{103} F_4^{222} F_7^{304}}{F_3^{165} F_5^{167} F_6^{351} F_8^{598}}, \quad C_3 = \frac{F_1^{431} F_2^{250} F_4^{554} F_7^{684}}{F_3^{377} F_5^{380} F_6^{845} F_8^{1352}}.$$

They are elements of $\mathcal{D}$ and so Lemma 5 is proved. ■

Proof of Theorem 3. Let $f_0,\ f_1,\ f_2,\ f_3,\ f_4 \in \mathcal{A}^*$ and,

$$\mathcal{B}_f(n) = f_0(n) + f_1(n+2) + f_2(n+3) + f_3(n+4) + f_4(n+6) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. Then

$$\mathcal{B}_f(-n-6) = f_4(n) + f_3(n+2) + f_2(n+3) + f_1(n+4) + f_0(n+6) \equiv 0 \pmod 1.$$

Let $\psi_0(n) := f_0(n) - f_4(n)$ and $\psi_1(n) := f_1(n) - f_3(n)$ for all $n \in \mathbb{Z}$. Thus, we have,

$$\psi_0(n) + \psi_1(n+2) - \psi_1(n+4) - \psi_0(n+6) \equiv 0 \pmod 1 \quad \text{for all} \quad n \in \mathbb{Z}.$$

From Lemma 1 we have that $\psi_0(n) \equiv \psi_1(n) \equiv 0 \pmod 1$, consequently $f_0(n) \equiv \equiv f_4(n) \pmod 1$ and $f_1(n) \equiv f_3(n) \pmod 1$ for all $n \in \mathbb{Z}$. Hence,

$$\mathcal{B}_f(n) \equiv f_0(n) + f_1(n+2) + f_2(n+3) + f_1(n+4) + f_0(n+6) \equiv 0 \pmod 1$$

is true for all $n \in \mathbb{Z}$. The conditions of Lemma 4 and Lemma 5 are satisfied by taking $b_j(n) = f_j(n)$ $(j = 0, 1, 2)$ and

$$\mathcal{S}(n) = b_0(n) + b_1(n+2) + b_2(n+3) + b_1(n+4) + b_0(n+6).$$

Thus

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$ and this completes the proof. ■

From Lemma 4 and Lemma 5 we obtain

Theorem 6. *If $\mathcal{D}$ denotes the subgroup of $\mathbb{Q}_+^3$ generated by the sequences*

$$D_n = \Big(n(n+6),\ (n+2)(n+4),\ n+3\Big) \quad (n \in \mathbb{N}),$$

then we have

$$\mathcal{D} = \mathbb{Q}_+^3.$$

4. Proof of Theorem 4

Lemma 6. *Let $c_0,\ c_1,\ c_2 \in \mathcal{A}^*$. If*

$$c_0(n) + c_1(n+1) + c_2(n+3) + c_1(n+5) + c_0(n+6) \equiv 0 \pmod 1 \tag{4.1}$$

for all $n \in \mathbb{N}$, then

$$c_0(n) \equiv c_1(n) \equiv c_2(n) \equiv 0 \pmod 1 \quad \text{for} \quad n \in \mathbb{N}. \tag{4.2}$$

Proof. In order to prove Lemma 6, we shall use the following fact:

$$\text{If } (4.1) \text{ holds for all } n \in \mathbb{N}, \text{then } (4.2) \text{ holds for } n \le 11. \tag{4.3}$$

This can be shown in the same way as we proved Lemma 3 and lemma 5. Let,

$$T(n) = c_0(n) + c_1(n+1) + c_2(n+3) + c_1(n+5) + c_0(n+6) \equiv 0 \pmod 1.$$

Let n_0 be the smallest positive integer n for which $c_j(n) \not\equiv 0 \pmod 1$ for at least one j. Then n_0 is a prime p and $p > 11$. It is easily seen that $T(p-5) \equiv 0 \pmod 1$ and $T(p-6) \equiv 0 \pmod 1$ imply that $c_0(p) \equiv c_1(p) \equiv$ $\equiv 0 \pmod 1$. Let $c_2(p) \equiv \nu \not\equiv 0 \pmod 1$, then $T(p-3) \equiv 0 \pmod 1$ implies that $c_1(p+2) \equiv -\nu \pmod 1$.

From $T(p+1) \equiv 0 \pmod 1$ we have that $c_1(p+6) \equiv \nu \pmod 1$ and from $T(p+5) \equiv 0 \pmod 1$ we have $c_2(p+8) \equiv -\nu \pmod 1$. As $p \equiv 2 \pmod 3$, and so $3|p+10$, $2|p+11$ and $\frac{p+11}{2} < p$. It is obvious from $p, p+2, p+6, p+8 \in \mathcal{P}$ that $p \equiv 1 \pmod 5$. We have $0 \equiv T(2p-3) \equiv c_0(2p-3) + c_2(p) \pmod 1$, thus $c_0(2p-3) \equiv -\nu \pmod 1$.

Let us consider now

$$0 \equiv T(2p-6j-3) \equiv 0 \pmod 1$$

for $j = 1,2,3,4,5$. Since $2|2p-6j-2$, $2|2p-6j$, $2|2p-6j+2$, we have

$$c_1(2p-6j-2)+c_2(2p-6j)+c_1(2p-6j+2)+c_0(2p-6j+3) \equiv 0 \pmod 1,$$

and so

$$c_1(2p-6j-3)+c_1(2p-6j-3) \equiv 0 \pmod 1 \quad (j=1,2,3,4,5).$$

Hence $c_0(2p-9) \equiv \nu \pmod 1$, $c_0(2p-15) \equiv -\nu \pmod 1$, $c_0(2p-21) \equiv$ $\equiv -\nu \pmod 1$, $c_0(2p-27) \equiv \nu \pmod 1$, which with $5|2p-27$ implies that $\nu = 0$. ■

Proof of Theorem 4. Let $f_0, f_1, f_2, f_3, f_4 \in \mathcal{A}^*$ and,

$$\mathcal{C}_f(n) = f_0(n)+f_1(n+1)+f_2(n+3)+f_3(n+5)+f_4(n+6) \equiv 0 \pmod 1$$

for all $n \in \mathbb{Z}$. Then

$$\mathcal{B}_f(-n-6) = f_4(n)+f_3(n+1)+f_2(n+3)+f_1(n+5)+f_0(n+6) \equiv 0 \pmod 1.$$

Let

$$\kappa_0(n) = f_0(n)-f_4(n) \quad \text{and} \quad \kappa_1(n) := f_1(n)-f_3(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Thus, we deduce from the above relations that

$$\kappa_0(n)+\kappa_1(n+1)-\kappa_1(n+5)-\kappa_0(n+6) \equiv 0 \pmod 1 \quad \text{for all} \quad n \in \mathbb{Z}.$$

From Lemma 1 we have $\kappa_0(n) \equiv \kappa_1(n) \equiv 0 \pmod 1$, and so $f_0(n) \equiv f_4(n) \pmod 1$ and $f_1(n) \equiv f_3(n) \pmod 1$ for all $n \in \mathbb{Z}$. Hence,

$$\mathcal{C}_f(n) \equiv f_0(n)+f_1(n+1)+f_2(n+3)+f_1(n+5)+f_0(n+6) \equiv 0 \pmod 1$$

is true for all $n \in \mathbb{Z}$. Thus the conditions of Lemma 6 are satisfied by taking $c_j(n) = f_j(n)$ $(j=0,1,2)$ and

$$T(n) = c_0(n)+c_1(n+1)+c_2(n+3)+c_1(n+5)+c_0(n+6).$$

Thus

$$f_0(n) \equiv f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod 1$$

holds for all $n \in \mathbb{Z}$. ■

Thus we obtain (from the last lemma),

Theorem 7. *If $\mathcal{T}$ denotes the subgroup of $\mathbb{Q}_+^3$ generated by the sequences*

$$T_n = \Big(n(n+6),\ (n+1)(n+5),\ n+3\Big) \quad (n \in \mathbb{N}),$$

then we have

$$\mathcal{T} = \mathbb{Q}_+^3.$$

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