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ON MORE RAPID CONVERGENCE TO A DENSITY

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Abstract. Let the set $A \subset \mathbb{N}$ have positive asymptotic density d and the set $|A(n) - nd|$ be not bounded above. Then for any $d' \in (0, d)$ there exists a $B \subset A$, such that the asymptotic density of B is d' and for infinitely many n we have $|B(n)n^{-1} - d'|$ tends to zero more rapidly than $|A(n)n^{-1} - d|$. This solves an open question of Rita Giuliano at al. [1].

1. Introduction

Denote by $\mathbb{N}$ the set of all positive integers. For $A \subset \mathbb{N}$ and a real number x let $A(x)$ denote the counting function of the set A. The asymptotic density of A is defined as

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if the limit exists. Note that $A = \{a_1 < a_2 < \cdots\}$ has asymptotic density d if and only if

$$\lim_{n\to\infty} \frac{n}{a_n} = d.$$

The paper by R. Giuliano, G. Grekos and L. Mišík [1] is a collection of open problems on densities. The aim of this note is to solve the Open Problem 12 in [1] which reads as follows:

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Suppose that the set $A \subset \mathbb{N}$ has asymptotic density $d > 0$. Let $f(n) = |A(n)n^{-1} - d|$ which tends to zero as n tends to ∞. Suppose, that

$$\limsup_{n\to\infty} nf(n) = \infty.$$

Is there $d' \in (0, d]$ and $B \subset A$ such that for $g(n) = |B(n)n^{-1} - d'|$ we have

1) *$g(n)$ tends to 0, as n tends to ∞; and*
2) $\liminf\limits_{n\to\infty} \dfrac{g(n)}{f(n)} = 0$*?*

2. Results

The following lemma will be useful.

Lemma 2.1. *Suppose that the set $A \subset \mathbb{N}$ has asymptotic density d. Then for an arbitrary $d' \in (0, d)$ there exists $D \subset A$ such that $d(D) = d'$.*

Proof. Let $A = \{a_1 < a_2 < \cdots\}$ and $\alpha = \frac{d}{d'}$. Define

$$D = \{a_{\lfloor n\alpha \rfloor} : n \in \mathbb{N}\},$$

where $\lfloor x \rfloor$ denotes the integer part of x. Clearly, $(n+1)\alpha - n\alpha > 1$ and therefore the numbers $\lfloor n\alpha \rfloor$ are all different. We have

$$d(D) = \lim_{n\to\infty} \frac{n}{a_{\lfloor n\alpha \rfloor}} = \lim_{n\to\infty} \frac{n}{\lfloor n\alpha \rfloor} \cdot \frac{\lfloor n\alpha \rfloor}{a_{\lfloor n\alpha \rfloor}} = \frac{1}{\alpha} d. \qquad \blacksquare$$

Theorem 2.1. *Let the set $A \subset \mathbb{N}$ have positive density d and*

$$\limsup_{n\to\infty} |A(n) - nd| = \infty.$$

Then for any $d' \in (0, d)$ there exists a $B \subset A$ such that $d(B) = d'$ and

$$\liminf_{n\to\infty} \frac{|B(n)n^{-1} - d'|}{|A(n)n^{-1} - d|} = 0.$$

Proof. Let $\varphi(n) = A(n) - nd$. There are two cases to consider.

Case I: $\varphi(n)$ is not bounded above.
In this case there exists a subsequence (n_i) of positive integers such that

$\lim_{n\to\infty} \varphi(n_i) = \infty$. There is no loss of generality in assuming that $n_i < n_{i+1}$, e.g. $n_{i+1} > 2^{n_i}$, $i = 1, 2, \dots$. Using the previous lemma, we get a set $D \subset A$ which has asymptotic density d'.

The basic idea is to eliminate $\lfloor \frac{\varphi(n_i)}{\alpha} \rfloor$ elements from the set D which are less than n_i and adding the same number of elements from $A \setminus D$ to the set D after the position of n_i.

For each i let m_i, n_i be such integers, that the cardinalities of both sets $(m_i, n_i) \cap D$ and $(n_i, k_i) \cap (A \setminus D)$ are equal to $\lfloor \frac{\varphi(n_i)}{\alpha} \rfloor$. Set

$$B = \bigcup_{i=1}^{\infty} \big(D \setminus (m_i, n_i)\big) \cup \big((n_i, k_i) \cap (A \setminus D)\big).$$

From the density of the set A it follows that

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = 0.$$

Therefore $d(B) = d(D) = d'$. We have

$$|B(n_i) - n_i d'| = \left| \left(D(n_i) - \left\lfloor \frac{\varphi(n_i)}{\alpha} \right\rfloor \right) - \frac{n_i d}{\alpha} \right| = \left| \left\lfloor \frac{A(n_i)}{\alpha} \right\rfloor - \left\lfloor \frac{\varphi(n_i)}{\alpha} \right\rfloor - \frac{n_i d}{\alpha} \right| < 2.$$

Using this fact we immediately have

$$\lim_{i\to\infty} \frac{|B(n_i)n_i^{-1} - d'|}{|A(n_i)n_i^{-1} - d|} = \lim_{i\to\infty} \frac{|B(n_i) - n_i d'|}{|A(n_i) - n_i d|} \le \lim_{i\to\infty} \frac{2}{\varphi(n_i)} = 0.$$

Case II: $\varphi(n)$ is not bounded below.
This case can be handled analogously. In this case there exists a subsequence (n_i) of positive integers such that $\lim_{n\to\infty} \varphi(n_i) = -\infty$. Now the numbers m_i, k_i have the property that the sets $(m_i, n_i) \cap (A \setminus D)$ and $(n_i, k_i) \cap D$ are of the same cardinality $\lfloor \frac{-\varphi(n_i)}{\alpha} \rfloor$. Define

$$B = \bigcup_{i=1}^{\infty} \big(D \setminus (n_i, k_i)\big) \cup \big((m_i, n_i) \cap (A \setminus D)\big).$$

The rest of the proof is similar to Case I, so we leave it to the reader. ∎

Remark 2.1. We now turn to the case $d = d'$. In general, the above theorem does not hold in this case. To see this, let us consider the set $A \subset \mathbb{N}$ with positive asymptotic density and with the property that for any positive integer n

$$A(n) \le nd$$

holds. Then for arbitrary $B \subset A$ we have

$$\frac{|B(n)n^{-1} - d|}{|A(n)n^{-1} - d|} = \frac{nd - B(n)}{nd - A(n)} \geq 1.$$

References

[1] **Giuliano, R., G. Grekos and L. Mišík,** Open problems on densities II, in: *Diophantine analysis and related fields 2010*, DARF–2010. Proceedings of the conference (ed: Komatsu, Takao), Musashino, Tokyo, Japan, March 4–5, 2010. Melville, NY: American Institute of Physics (AIP). AIP Conference Proceedings 1264, 2010, 114-128.

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