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A RANDOM GRAPH MODEL BASED ON 3-INTERACTIONS

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Abstract. We consider a random graph model evolving in discrete time-steps that is based on 3-interactions among vertices. Triangles, edges and vertices have different weights; objects with larger weight are more likely to participate in future interactions. We prove the scale free property of the model by exploring the asymptotic behaviour of the weight distribution. We also find the asympotics of the weight of a fixed vertex.

1. Introduction

Random graphs evolving by some "preferential attachment" rule are inevitable in modelling real-world networks. There is a vast number of publications inventing and studying different models of that kind, but in most of them the dynamics is only driven by vertex-vertex interactions. However, one can easily find networks (that is, objects equipped with links) in economy or other areas where simultaneous interactions can take place among three or even more vertices, and those interactions determine the evolution of the process.

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We consider a random graph model evolving in discrete time-steps. The most important feature of the model is the presence of 3-interactions among vertices. In our graph process vertices, edges and triangles get nonnegative, integer valued random weights which grow with time. The dynamics is driven by these weights.

The model we are going to deal with resembles those in [1], [3], and [4], but there are essential differences. In [3] there is no interaction between more than two vertices, and the weight of a vertex is simply equal to its degree. In [4] and [1] interactions among groups of vertices do appear, but with completely different dynamics. In all three papers the existence of a power law asymptotic degree distribution is proved, and this is what we are also interested in.

Our goal is to prove that the ratio of vertices of weight w tends to some positive constant x_w almost surely, as the number of steps goes to infinity. We will give a recursion for x_w, from where it will be easy to see the polynomial decay of x_w as $w \to \infty$. This is the so-called scale free property [2]. We will also determine the asymptotics of the weight of any fixed vertex. In the proofs martingale methods from [4] are used.

2. The model

We start with a single triangle. This has initial weight 1, and all its three edges have weight 1. Later on, we will add vertices and edges to the graph randomly. Vertices, edges, and triangles will have nonnegative integer-valued weights, which increase according to the random evolution of the graph.

The graph evolves in discrete time-steps. The sum of the weights of triangles will be increased by 1 at each step, while the total weight of edges will be increased by 3 step by step.

At each step either a new vertex is added, which then interacts with two already existing vertices, or 3 old vertices interact. This has to be decided at the beginning of the step, independently of the past. The probability that a new vertex is born is p at every step; this is a parameter of the model. We will need $0 < p \leq 1$.

Assume that in the nth step a new vertex is added to the graph. We choose two of the old vertices randomly; they will interact with the new vertex. With probability r, independently of the past, the choice is done according to the "preferential attachment" rule, and with probability $1-r$ it is done "uniformly". r is a fixed parameter of the model. More precisely, in the case of "preferential attachment" we choose the endpoints of an already existing edge having weight

w with probability $\frac{w}{3n}$. Note that the sum of the edge weights is equal to $3n$ at this moment. In the case of "uniform" selection two vertices are chosen with each pair having equal probability to be selected; that is, we perform sampling without replacement. This allows us to generate edges between old vertices that were not connected before.

Then the new vertex interacts with the two selected vertices. This means that the triangle they form comes to existence with initial weight 1. The two new edges connecting the new vertex to the other two get weight 1 each. We connect the old vertices if they are not connected yet with an edge of weight 1. This may only happen with uniform selection. If the two old vertices are already connected, then we increase the weight of that edge by 1. To put it in another way, we increase the weights of all three edges of the 3-interaction by 1. This is the end of the step where a new vertex is generated.

With probability $1-p$, 3 of the old vertices will interact. With probability q they will be chosen according to the "preferential attachment" rule, and with probability $1-q$ they will be chosen "uniformly". The choice is also independent of the past. This q is the third parameter of the model.

In the "preferential attachment" case we choose an already existing triangle of weight w with probability proportional to its weight, that is, with probability $\frac{w}{n}$. Note that there may exist triangles with zero weight in the graph. A triangle has positive weight if and only if it has already appeared in a 3-interaction before.

On the other hand, in the "uniform" case three distinct vertices are chosen such that every triplet has the same probability to be selected. This is again sampling without replacement from all the existing vertices.

In both cases, having selected the three vertices to interact, we draw the edges of the triangle that are not present yet. Then the weight of the triangle is increased by 1, as well as the weights of the three sides of the triangle. That is, the initial weight of a newly generated edge is 1, while the old ones' weights are increased by 1.

Now we define the weights of vertices. The weight of a vertex is the sum of the weights of the triangles that contain it. Note that this is just the half of the sum of weights of edges from it, because whenever a vertex takes part in a 3-interaction, the first sum is increased by 1, and the latter one is increased by 2.

Similarly, the weight of an edge is simply the sum of the weights of the triangles that contain it. In fact, it would not really be necessary to introduce edge weights, for the only occurrence of edge weights is the case of a new vertex combined with "preferential attachment" selection, where the following rule would give the same result. Select a triangle with weight-proportional probability, then choose one of its sides at random.

Denote by $\mathcal{F}_n$ the σ-field generated by the first n steps, and by V_n the number of vertices after the nth step. Thus $V_0 = 3$. Since we decide independently at each step whether a new vertex is born, by the strong law of large numbers we obtain that

$$V_n = pn + o\left(n^{1/2+\varepsilon}\right) \text{ a.s.} \tag{2.1}$$

for all $\varepsilon > 0$.

Throughout this paper, for two sequences $(a_n), (b_n)$ of nonnegative numbers, $a_n \sim b_n$ means that $b_n > 0$ except finitely many terms, and $a_n/b_n \to 1$ as $n \to \infty$.

3. Asymptotic weight distribution

We are interested in the distribution of weights of vertices. As we mentioned before, this is the half of the degree of a vertex counted with multiplicity.

Scale free property often emerges in models where preferential attachment rules are applied. Therefore, throughout the paper we suppose that the parameters do not exclude preferential attachment; that is, either $r > 0$, or $q > 0$ and $p < 1$.

Let $X[n, w]$ denote the number of vertices of weight w after n steps. Our goal is to examine the asymptotic behaviour of $\frac{X[n,w]}{V_n}$; more precisely, to prove that the ratio of vertices of weight w is convergent almost surely. The limits are deterministic constants, which form a polynomially decaying sequence as $w \to \infty$. We may refer to this fact as the scale free property of the model, following the terminology of Albert and Barabási [2]. We also compute the characteristic exponent.

Theorem 3.1. *For $w = 1, 2, \ldots$ we have*

$$\frac{X[n, w]}{V_n} \to x_w$$

almost surely, as $n \to \infty$. The limits x_w are positive constants satisfying the following recursion

$$x_1 = \frac{1}{\alpha + \beta + 1}, \quad x_w = \frac{\alpha(w-1) + \beta}{\alpha w + \beta + 1} x_{w-1}, \quad w \geq 2,$$

where

$$\alpha = \frac{2}{3} pr + (1-p) q > 0, \quad \beta = \frac{1}{p} \big[2p(1-r) + 3(1-p)(1-q)\big].$$

Moreover,

$$x_w \sim C w^{-\left(1+\frac{1}{\alpha}\right)},$$

as $w \to \infty$, with some positive constant C.

Finally we remark that (x_w) is a probability distribution, as its sum is equal to 1.

Proof. First we compute the probability that a given vertex of actual weight w takes part in the 3-interaction at the nth step.

If at the nth step a new vertex is generated and we follow the "preferential attachment" rule, then this probability is equal to $\frac{2w}{3n}$, since the total weight of edges is $3n$, and the sum of weights of edges from the given vertex is just the double of its weight w.

On the other hand, there are $\binom{V_{n-1}}{2}$ pairs of vertices, and every vertex is contained in $V_{n-1}-1$ of them. Hence the probability of being chosen at uniform selection is $\frac{2}{V_{n-1}}$.

Now let us examine the case when old vertices interact. A vertex of weight w is contained in triangles of total weight w by definition, while the total sum of triangles is equal to n after $n-1$ steps. Hence the probability of being chosen is $\frac{w}{n}$ by the "preferential attachment" rule. With "uniform selection" it is clearly $\binom{V_{n-1}-1}{2}/\binom{V_{n-1}}{3} = \frac{3}{V_{n-1}}$.

Putting these together we get that the probability that a vertex of weight w takes part in the interaction of step n is given by

$$p\left[r\,\frac{2w}{3n} + (1-r)\,\frac{2}{V_{n-1}}\right] + (1-p)\left[q\,\frac{w}{n} + (1-q)\,\frac{3}{V_{n-1}}\right] = \frac{\alpha w}{n} + \frac{\beta p}{V_{n-1}}\,. \tag{3.1}$$

Now we determine the conditional expectation of $X[n,w]$ with respect to $\mathcal{F}_{n-1}$. The weights can change at most by 1. After $n-1$ steps we have $X\left[n-1,w\right]$ vertices of weight w. Each of them increases its weight with the probability given above; while vertices of weight $w-1$ will count if they take part in the 3-interaction at step n. Using the additive property of expectation we obtain that for $n \ge 1$, $w \ge 1$ the following holds.

$$\begin{aligned}
\mathbb{E}(X[n,w]\,|\,\mathcal{F}_{n-1}) &= X[n-1,w] - X[n-1,w]\left[\frac{\alpha w}{n} + \frac{\beta p}{V_{n-1}}\right] + \\
&\quad + X[n-1,w-1]\left[\frac{\alpha(w-1)}{n} + \frac{\beta p}{V_{n-1}}\right] + p\delta_{w,1} = \\
&= X[n-1,w]\left[1 - \frac{\alpha w}{n} - \frac{\beta p}{V_{n-1}}\right] + \\
&\quad + X[n-1,w-1]\left[\frac{\alpha(w-1)}{n} + \frac{\beta p}{V_{n-1}}\right] + p\delta_{w,1},
\end{aligned} \tag{3.2}$$

where $X[n-1,0]$ is meant to be zero. The last term is only present for $w=1$, because the weight of the new vertex is 1.

We define the following normalizing constants.

$$c[n,w]=\prod_{i=1}^{n-1}\left(1-\frac{\alpha w}{i}-\frac{\beta p}{V_{i-1}}\right)^{-1},\quad n\ge 1,\ w\ge 1.$$

By equation (2.1), with any positive ε less than $\frac{1}{2}$ we have

$$\log c[n,w]=\sum_{i=1}^{n-1}-\log\left(1-\frac{\alpha w}{i}-\frac{\beta}{i+o\left(i^{1/2+\varepsilon}\right)}\right)=$$
$$=\sum_{i=1}^{n-1}\left(\frac{\alpha w}{i}+\frac{\beta}{i}+o\big(i^{-3/2+\varepsilon}\big)\right)=(\alpha w+\beta)\sum_{i=1}^{n-1}\frac{1}{i}+O(1)$$

almost surely, where the error term converges as $n\to\infty$. This implies

$$c[n,w]\sim a_w n^{\alpha w+\beta}\quad \text{a.s.} \tag{3.3}$$

as $n\to\infty$, where a_w is a positive random variable.

Introduce $Z[n,w]=c[n,w]X[n,w]$, $n\ge 1$, $w\ge 1$. From equation (3.2) it is clear that $\big(Z[n,w],\,\mathcal{F}_n\big)$ is a nonnegative submartingale for every positive integer w. Consider the Doob–Meyer decomposition $Z[n,w]=M[n,w]+A[n,w]$, where $M[n,w]$ is a martingale and $A[n,w]$ is a predictable increasing process. Based on equation (3.2) we have

$$\begin{aligned}A[n,w]&=\mathbb{E}Z[1,w]+\sum_{i=2}^{n}\big(\mathbb{E}(Z[i,w]\,|\,\mathcal{F}_{i-1})-Z[i-1,w]\big)=\\&=\mathbb{E}Z[1,w]+\sum_{i=2}^{n}c[i,w]\left(X[i-1,w-1]\left(\frac{\alpha(w-1)}{i}+\frac{\beta p}{V_{i-1}}\right)+p\delta_{w,1}\right).\end{aligned}\tag{3.4}$$

We will also need a bound on the variation of the martingale part. By using equation (3.3) we obtain

$$\begin{aligned}B[n,w]&=\sum_{i=2}^{n}\operatorname{Var}(Z[i,w]\,|\,\mathcal{F}_{i-1})=\sum_{i=2}^{n}c[i,w]^2\operatorname{Var}(X[i,w]\,|\,\mathcal{F}_{i-1})=\\&=\sum_{i=2}^{n}c[i,w]^2\operatorname{Var}\big(X[i,w]-X[i-1,w]\;\big|\;\mathcal{F}_{i-1}\big)\le\\&\le\sum_{i=2}^{n}c[i,w]^2\,\mathbb{E}\left(\big(X[i,w]-X[i-1,w]\big)^2\Big|\,\mathcal{F}_{i-1}\right)\le\\&\le 9\sum_{i=2}^{n}c\,[i,w]^2=O\left(n^{2(\alpha w+\beta)+1}\right).\end{aligned}\tag{3.5}$$

First we used the facts that $c[i,w]$ is measurable with respect to $\mathcal{F}_{i-1}$, and, since there is exactly one 3-interaction at each step, the change of X is less than or equal to 3. Note that $B[n,w]$ is just the increasing process in the Doob–Meyer decomposition of $M[n,w]^2$.

The proof continues by induction on w. For $w=1$ we obtain that

$$A[n,1] \sim p \sum_{i=2}^{n} c[i,1] \sim p \sum_{i=2}^{n} a_1 i^{\alpha+\beta} \sim p \cdot \frac{a_1}{\alpha+\beta+1} \cdot n^{\alpha+\beta+1} \tag{3.6}$$

almost surely, as $n \to \infty$.

On the other hand, $B[n,1] = O\left(n^{2(\alpha+\beta)+1}\right)$, hence, by applying Proposition VII-2-4 of Neveu [5] to $M[n,w]$ we get

$$M[n,w] = o\left(B[n,1]^{1/2} \log B[n,1]\right) = o\big(A[n,1]\big)$$

(see Section 6 of [4] for more details of this argument). Finally we obtain that

$$Z[n,1] \sim A[n,1] \quad \text{a.s.}$$

as $n \to \infty$. Using the asymptotics of $c\,[n,1]$ and $A\,[n,1]$, that is, equations (3.3) and (3.6), then equation (2.1) and the definition of $Z\,[n,1]$, we get that

$$\frac{X[n,1]}{V_n} = \frac{Z[n,1]}{c[n,1]V_n} \sim \frac{\dfrac{a_1}{\alpha+\beta+1}\, pn^{\alpha+\beta+1}}{a_1 n^{\alpha+\beta}\, pn} \to \frac{1}{\alpha+\beta+1}$$

almost surely, as $n \to \infty$.

Thus the theorem holds for $w=1$ with $x_1 = \frac{1}{\alpha+\beta+1}$.

Suppose that the statement of Theorem 3.1 holds for $w-1$ for some fixed $w \geq 2$; that is, the ratio of vertices of weight $w-1$ converges to some constant x_{w-1}. By using this fact we can compute the asymptotics of $A[n,w]$. From (3.4) we have

$$\begin{aligned}
A[n,w] &\sim \sum_{i=2}^{n} c[i,w] X[i-1,w-1] \left(\frac{\alpha(w-1)}{i} + \frac{\beta p}{V_{i-1}}\right) \sim \\
&\sim \sum_{i=2}^{n} a_w\, i^{\alpha w+\beta}\, x_{w-1}\, V_{i-1} \left(\frac{\alpha(w-1)}{i} + \frac{\beta p}{V_{i-1}}\right) \sim \\
&\sim a_w x_{w-1} \sum_{i=2}^{n} p\,(\alpha(w-1)+\beta)\, i^{\alpha w+\beta} \sim \\
&\sim \frac{a_w x_{w-1} p\,(\alpha(w-1)+\beta)}{\alpha w+\beta+1}\, n^{\alpha w+\beta+1}
\end{aligned}$$

almost surely, as $n \to \infty$. Here we also used that α and β are both nonnegative, which is clear from their definition.

From inequality (3.5) we know that $B[n,w] = O\left(n^{2(\alpha w+\beta)+1}\right)$, thus Proposition VII-2-4 of [5] can be applied again to conclude that

$$M[n,w] = o\left(B[n,w]^{1/2}\log B[n,w]\right) = o\big(A[n,w]\big).$$

We end up with

$$X[n,w] \sim \frac{\dfrac{a_w x_{w-1} p(\alpha(w-1)+\beta)}{\alpha w+\beta+1}\, n^{\alpha w+\beta+1}}{a_w\, n^{\alpha w+\beta}} = x_{w-1}\,\frac{\alpha(w-1)+\beta}{\alpha w+\beta+1}\,np$$

almost surely, as $n \to \infty$. Hence

$$\lim_{n\to\infty}\frac{X[n,w]}{V_n} = x_{w-1}\,\frac{\alpha(w-1)+\beta}{\alpha w+\beta+1} \qquad \text{a.s.}$$

Thus the induction step is complete: x_w exists, and it is positive and finite.

Furthermore, we have a recursion for x_w, from where

$$(3.7) \qquad x_w = x_1\prod_{j=2}^{w}\frac{\alpha(j-1)+\beta}{\alpha j+\beta+1} = \frac{1}{\alpha w+\beta+1}\prod_{j=1}^{w-1}\frac{j+\frac{\beta}{\alpha}}{j+\frac{\beta+1}{\alpha}}$$
$$= \frac{\Gamma\left(1+\frac{\beta+1}{\alpha}\right)\Gamma\left(w+\frac{\beta}{\alpha}\right)}{\alpha\,\Gamma\left(1+\frac{\beta}{\alpha}\right)\Gamma\left(w+\frac{\beta+1}{\alpha}+1\right)} \sim C w^{-\left(1+\frac{1}{\alpha}\right)}$$

with some positive constant C, as w tends to infinity. The proof of the theorem is complete. ■

Remark. From (3.7) it follows that

$$x_w = \frac{\prod_{j=1}^{w-1}(\alpha j+\beta)}{\prod_{j=1}^{w}(\alpha j+\beta+1)} = y_w - y_{w-1},$$

where

$$y_w = \prod_{j=1}^{w}\frac{\alpha j+\beta}{\alpha j+\beta+1} \to 0,$$

hence

$$\sum_{w=1}^{\infty} x_w = y_0 = 1.$$

4. The weight of a fixed vertex

In this section our goal is to determine the asymptotics of the weight of a fixed vertex. Since the weight of a vertex is just the half of its degree when edges are counted with multiplicity, we could reformulate our result to obtain the asymptotics of the degree.

It is clear that the weights of the vertices of the starting triangle are interchangeable, therefore it is not necessary to deal with all the three. Let only one of them be labelled by 0, the other two will remain unlabelled. Moreover, let the further vertices get labels 1, 2, etc, in the order they are added to the graph. Let $W[n,j]$ be the weight of vertex j after step n, provided it exists. Otherwise let $W[n,j]$ be equal to zero. Obviously, vertex j cannot exist before step j. Let $I[n,j]$ denote the indicator of the event $\{W[n,j]>1\}$.

We introduce the sequences

$$b_n=\prod_{i=1}^{n}\left(1+\frac{\alpha}{i}\right)^{-1},\quad d_n=\beta p\sum_{i=1}^{n}\frac{b_i}{V_{i-1}}\,,$$

with α, β defined in Theorem 3.1. Note that b_n is deterministic, while d_n is random, but $\mathcal{F}_{n-1}$-measurable.

Lemma 4.1. *Let j and k be fixed integers, $0\le j\le k$, and let $Z[n,j]=$ $=b_nW[n,j]-d_n$. Then $\big(Z[n,j]I[k,j],\,\mathcal{F}_n\big)$ is a martingale for $n\ge k$.*

Proof. According to equation (3.1), the probability that vertex j gets new edges at step $n+1$ is equal to $\dfrac{\alpha W[n,j]}{n+1}+\dfrac{\beta p}{V_n}$, provided that it already exists, which surely holds if the indicator $I[k,j]$ differs from 0. This implies that

$$\mathbb{E}\big(I[k,j]W[n+1,j]\mid\mathcal{F}_n\big)=I[k,j]W[n,j]+I[k,j]\left(\frac{\alpha W[n,j]}{n+1}+\frac{\beta p}{V_n}\right)=$$
$$=I[k,j]W[n,j]\left(1+\frac{\alpha}{n+1}\right)+I[k,j]\frac{\beta p}{V_n}\,.$$

Multiplying both sides by b_{n+1}, we get by definition that

$$\mathbb{E}\big(b_{n+1}W[n+1,j]I[k,j]\mid\mathcal{F}_n\big)=I[k,j]\left(b_nW[n,j]+b_{n+1}\frac{\beta p}{V_n}\right)=$$
$$=I[k,j]\big(b_nW[n,j]-d_n+d_{n+1}\big),$$

which completes the proof of the lemma, since d_{n+1} is $\mathcal{F}_n$-measurable. ∎

Theorem 4.1. *Fix $j \geq 0$. Then $W[n,j] \sim \zeta_j n^\alpha$ almost surely as $n \to \infty$, where ζ_j is a positive random variable.*

Proof. First we show that this holds with a nonnegative ζ_j.

Almost surely on the event that vertex j exists after step n we have

$$\mathbb{P}\big(W[n+1,j] = W[n,j]+1 \mid \mathcal{F}_n\big) \geq \frac{\alpha}{n+1}\,.$$

Using the Lévy-type generalization of Borel–Cantelli-lemma [5, VII-2-6] we get that $W[n,j] \to \infty$ with probability 1.

From the definition of b_n it easily follows that

$$b_n = \frac{\Gamma(n+1)\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)} \sim \Gamma(1+\alpha) n^{-\alpha}, \tag{4.1}$$

as $n \to \infty$. Hence, by using equation (2.1) and the positivity of α, we get

$$d_n = \beta p \sum_{i=1}^{n} \frac{b_i}{V_{i-1}} = \beta\,\Gamma(1+\alpha) \sum_{i=1}^{n-1} i^{-\alpha-1}\big(1+o(1)\big).$$

Thus d_n is almost surely convergent as $n \to \infty$, hence the martingale of Lemma 4.1 is bounded from below. This martingale has bounded differences, for

$$Z[n+1,j] - Z[n,j] \leq b_n\big(W[n+1,j] - W[n,j]\big) \leq b_n \leq 1,$$

and

$$Z[n,j] - Z[n+1,j] \leq \big(b_n - b_{n+1}\big) W[n,j] + \big(d_{n+1} - d_n\big) \leq$$
$$\leq b_{n+1}\alpha + b_{n+1}\,\frac{\beta p}{3} \leq \alpha + \frac{\beta p}{3}\,.$$

By Proposition VII-3-9 of [5] such a martingale either converges or oscillates between $-\infty$ and $+\infty$, but now the latter is excluded, hence it must converge almost surely.

Going further, we get that $b_n W[n,j]$ is convergent almost everywhere on the event that the weight of vertex j is greater than 1 after step k. Since that weight tends to infinity, the limit as $k \to \infty$ of this increasing sequence of events has probability 1. Thus $b_n W[n,j]$ is almost surely convergent, and by equation (4.1) we get that the statement of the theorem holds with a nonnegative ζ_j.

Now we only have to prove that ζ_j is positive with probability 1.

In what follows, if the indicator in the numerator is zero, let us define the fractions to be zero. Similarly to the previous lemma, for $n \geq k$ we can write

$$\mathbb{E}\left(\frac{I[k,j]}{W[n+1,j]-1}\,\middle|\,\mathcal{F}_n\right) = \left(\frac{\alpha W[n,j]}{n+1}+\frac{\beta p}{V_n}\right)\frac{I[k,j]}{W[n,j]}+ \\ +\left[1-\left(\frac{\alpha W[n,j]}{n+1}+\frac{\beta p}{V_n}\right)\right]\frac{I[k,j]}{W[n,j]-1}.$$

It is clear that

$$\left(\frac{\alpha W[n,j]}{n+1}+\frac{\beta p}{V_n}\right)\left(\frac{I[k,j]}{W[n,j]}-\frac{I[k,j]}{W[n,j]-1}\right) \leq -\frac{\alpha I[k,j]}{(n+1)\,(W[n,j]-1)}.$$

Hence we get

$$\mathbb{E}\left(\frac{I[k,j]}{W[n+1,j]-1}\,\middle|\,\mathcal{F}_n\right) \leq \frac{I[k,j]}{W[n,j]-1}\left(1-\frac{\alpha}{n+1}\right).$$

From this it follows that

$$\left(\frac{e_n I[k,j]}{W[n,j]-1}\,,\ \mathcal{F}_n\right)$$

is a supermartingale for $n \geq j$, where

$$e_n = \prod_{i=1}^{n}\left(1-\frac{\alpha}{i}\right)^{-1} = \frac{\Gamma(1-\alpha)\Gamma(n+1)}{\Gamma(n+1-\alpha)} \sim \Gamma(1-\alpha)n^{\alpha}.$$

This supermartingale is nonnegative, hence it converges almost surely. Since $\lim_{k\to\infty} I[k,j] = 1$ holds a.s., we obtain that $\frac{e_n}{W[n,j]-1}$ is also convergent almost surely as $n \to \infty$. This implies that $\zeta_j > 0$ with probability 1, as stated. ■

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