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RESTRICTED SUMMABILITY OF MULTI-DIMENSIONAL VILENKIN–FOURIER SERIES

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. It is proved that the maximal operator of the (C,α) $(\alpha = (\alpha_1,\ldots,\alpha_d))$ and Riesz means of a multi-dimensional Vilenkin–Fourier series is bounded from H_p to L_p $(1/(\alpha_k+1) < p < \infty)$ and is of weak type $(1,1)$, provided that the supremum in the maximal operator is taken over a cone-like set. As a consequence we obtain the a.e. convergence of the summability means of a function $f \in L_1$ to f.

1. Introduction

It can be found in Zygmund [16] (Vol. I, p.94) that the trigonometric Cesàro or (C,α) means $\sigma_n^\alpha f$ $(\alpha > 0)$ of a one-dimensional function $f \in L_1(\mathbb{T})$ converge a.e. to f as $n \to \infty$. Moreover, it is known (see Zygmund [16, Vol. I, pp.

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154-156]) that the maximal operator of the (C, α) means $\sigma_*^\alpha := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha|$ is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

For two-dimensional trigonometric Fourier series Marcinkiewicz and Zygmund [6] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbb{T}^2)$ converge a.e. to f as $n \to \infty$ in the restricted sense. This means that n must be in a positive cone, i.e., $2^{-\tau} \le n_i/n_j \le 2^\tau$ for every $i, j = 1, 2$ and for some $\tau \ge 0$. The author [13] extended this result to the (C, α) and Riesz means of the trigonometric Fourier series for higher dimensions, too. We proved also that the restricted maximal operator

$$\sigma_*^\alpha := \sup_{\substack{2^{-\tau}\le n_i/n_j \le 2^\tau \\ i,j=1,\ldots,d}} |\sigma_n^\alpha|$$

is bounded from H_p to L_p for $\max\{1/(\alpha_j+1)\} < p < \infty$ where $\alpha = (\alpha_1, \ldots, \alpha_d)$. By interpolation we obtained the weak $(1,1)$ inequality for σ_*^α which guarantees the preceding convergence results. Recently Gát [4] introduced more general sets than cones, the so called cone-like sets, and proved the preceding convergence theorem for two-dimensional Fejér means. The author [15] extended this result to higher dimensions, to Cesàro and Riesz means and proved also the above maximal inequality.

For one-dimensional Walsh–Fourier series the convergence result is due to Fine [2] and the weak $(1,1)$ inequality for $\alpha = 1$ to Schipp [7]. Fujii [3] proved that σ_*^1 is bounded from H_1 to L_1 (see also Schipp, Simon [8]). For Vilenkin–Fourier series the results are due to Simon [10]. The author [12, 14] proved the convergence theorem and the maximal inequality mentioned above for multi-dimensional Cesàro and Riesz means of Vilenkin–Fourier series, provided that the n is in a cone.

More recently Gát and Nagy [5] extended the convergence for cone-like sets and for two-dimensional Fejér means of Walsh-Fourier series. In this paper we generalize the preceding results and prove the convergence and maximal inequality for cone-like sets and for Cesàro and Riesz means of more-dimensional Vilenkin–Fourier series.

2. Martingale Hardy spaces and cone-like sets

For a set $\mathbb{X} \ne \emptyset$ let $\mathbb{X}^d$ be its Cartesian product $\mathbb{X} \times \ldots \times \mathbb{X}$ taken with itself d-times. To define the d-dimensional Vilenkin systems we need a sequence

$p := (p_n, n \in \mathbb{N})$ of natural numbers whose terms are at least 2. We suppose always that this sequence is bounded. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k, \qquad (n \in \mathbb{N}).$$

By a *Vilenkin interval* we mean one of the form $[k/P_n, (k+1)/P_n)$ for some $k, n \in \mathbb{N}$, $0 \leq k < P_n$. Given $n \in \mathbb{N}$ and $x \in [0,1)$ let $I_n(x)$ denote the Vilenkin interval of length $1/P_n$ which contains x. Clearly, the Vilenkin rectangle of area $1/P_{n_1} \times \ldots \times 1/P_{n_d}$ containing $x \in [0,1)^d$ is given by $I_n(x) := I_{n_1}(x_1) \times \ldots \times I_{n_d}(x_d)$. For $n := (n_1, \ldots, n_d) \in \mathbb{N}^d$ the σ-algebra generated by the Vilenkin rectangles $\{I_n(x), x \in [0,1)^d\}$ will be denoted by $\mathcal{F}_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. We briefly write L_p instead of the $L_p([0,1)^d, \lambda)$ space. The Lebesgue measure is denoted by λ in any dimension. We denote the Lebesgue measure of a set H also by $|H|$.

Suppose that for all $j = 2, \ldots, d$, $\gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that $\lim_\infty \gamma_j = \infty$. Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

$$c_{j,1}\gamma_j(x) \leq \gamma_j(\xi x) \leq c_{j,2}\gamma_j(x) \qquad (x > 0). \tag{1}$$

Let $c_{j,1} = \xi^{\tau_{j,1}}$ and $c_{j,2} = \xi^{\tau_{j,2}}$ $(j = 2, \ldots, d)$. For convenience we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$, $c_{1,1} = c_{1,2} = \xi$ and $\tau_{1,1} = \tau_{1,2} = 1$. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\delta = (\delta_1, \ldots, \delta_d)$ with $\delta_1 = 1$ and fixed $\delta_j \geq 1$ $(j = 2, \ldots, d)$. We will investigate the maximal operator of the summability means and the convergence over a *cone-like set* (with respect to the first dimension)

$$L := \{n \in \mathbb{N}^d : \delta_j^{-1}\gamma_j(n_1) \leq n_j \leq \delta_j\gamma_j(n_1), j = 2, \ldots, d\}. \tag{2}$$

Cone-like sets were introduced and investigated first by Gát [4]. The condition on γ_j seems to be natural, because he [4] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if (1) holds.

To consider summability means over a cone-like set we need to define new martingale Hardy spaces depending on γ. Given $n_1 \in \mathbb{N}$ we define $n_2, \ldots, n_d$ by $\gamma_j^0(P_{n_1}) := P_{n_j}$, where $P_{n_j} \leq \gamma_j(P_{n_1}) < P_{n_j+1}$ $(j = 2, \ldots, d)$. Let $\overline{n}_1 := (n_1, n_2, \ldots, n_d)$. Since the functions γ_j are increasing, the sequence $(\overline{n}_1, n_1 \in \mathbb{N})$ is increasing, too. We investigate the class of (*one-parameter*) *martingales* $f = (f_{\overline{n}_1}, n_1 \in \mathbb{N})$ with respect to $(\mathcal{F}_{\overline{n}_1}, n_1 \in \mathbb{N})$.

For $0 < p \le \infty$ the *martingale Hardy space* $H_p^\gamma([0,1)^d) = H_p^\gamma$ consists of all martingales for which

$$\|f\|_{H_p^\gamma} := \| \sup_{n_1 \in \mathbb{N}} |f_{\overline{n}_1}| \|_p < \infty.$$

It is known (see e.g. Weisz [13]) that $H_p^\gamma \sim L_p$ for $1 < p \le \infty$ where $\sim$ denotes the equivalence of the norms and spaces.

3. Cesàro and Riesz summability of Vilenkin–Fourier series

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \qquad 0 \le x_k < p_k, \ x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called *generalized Rademacher functions*, where $\imath = \sqrt{-1}$. The functions corresponding to the sequence $(2, 2, \ldots)$ are called Rademacher functions.

The product system generated by the generalized Rademacher functions is the *one-dimensional Vilenkin system*:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$. The product system corresponding to the Rademacher functions is called *Walsh system* (see Vilenkin [11] or Schipp, Wade, Simon and Pál [9]).

The Kronecker product $(w_n; n \in \mathbb{N}^d)$ of d Vilenkin systems is said to be the *d-dimensional Vilenkin system*. Thus

$$w_n(x) := w_{n_1}(x_1) \cdots w_{n_d}(x_d)$$

where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in [0,1)^d$. If we consider in each coordinate a different sequence $(p_n^{(j)}, n \in \mathbb{N})$ and a different Vilenkin system

$(w_n^{(j)}; n \in \mathbb{N}^d)$ $(j = 1, \ldots, d)$, then the same results hold. For simplicity we suppose that each Vilenkin system is the same.

If $f \in L_1$ then the number $\hat{f}(n) := \int_{[0,1)^d} f w_n \, d\lambda$ $(n \in \mathbb{N}^d)$ is said to be the nth *Vilenkin–Fourier coefficients* of f. We can extend this definition to martingales in the usual way (see Weisz [13]).

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $0 < \alpha_k \le 1$ $(k = 1, \ldots, d)$ and let

$$A_j^\beta := \binom{j+\beta}{j} = \frac{(\beta+1)(\beta+2)\ldots(\beta+j)}{j!} \qquad (j \in \mathbb{N}; \beta \ne -1, -2, \ldots).$$

It is known that $A_j^\beta \sim O(j^\beta)$ $(j \in \mathbb{N})$ (see Zygmund [16]). The (C,α) *or Cesàro means* and the *Riesz means* of a martingale f are defined by

$$\sigma_n^\alpha f := \frac{1}{\prod_{i=1}^d A_{n_i-1}^{\alpha_i}} \sum_{j=1}^{d} \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d A_{n_i-m_i-1}^{\alpha_i} \Big) \hat{f}(m) w_m$$

and

$$\sigma_n^{\alpha,\beta} f := \frac{1}{\prod_{i=1}^d {n_i}^{\alpha_i \beta_i}} \sum_{j=1}^{d} \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d (n_i^{\beta_i} - m_i^{\beta_i})^{\alpha_i} \Big) \hat{f}(m) w_m,$$

where $\beta = (\beta_1, \ldots, \beta_d)$ and $0 < \alpha_k \le 1 \le \beta_k$ $(k = 1, \ldots, d)$. The functions

$$K_n^\alpha := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-k-1}^\alpha w_k, \quad \text{and} \quad K_n^{\alpha,\beta} := \frac{1}{n^{\alpha\beta}} \sum_{k=0}^{n-1} (n^\beta - k^\beta)^\alpha w_k$$

are the one-dimensional *Cesàro* and *Riesz kernels*. If $\alpha = 1$ or $\alpha = \beta = 1$ then we obtain the *Fejér means*

$$\sigma_n^1 f := \sum_{j=1}^{d} \sum_{m_j=0}^{n_j-1} \Big(\prod_{i=1}^d (1 - \frac{m_i}{n_i}) \Big) \hat{f}(m) w_m = \frac{1}{\prod_{i=1}^d n_i} \sum_{j=1}^{d} \sum_{m_j=0}^{n_j-1} s_m f.$$

Since the results of this paper are independent of β, both the (C,α) and Riesz kernels will be denoted by K_n^α and the corresponding summability means by σ_n^α. It is simple to show that

$$\sigma_n^\alpha f(x) = \int_{[0,1)^d} f(t) (K_{n_1}^{\alpha_1}(x_1 \dot{-} t_1) \cdots K_{n_d}^{\alpha_d}(x_d \dot{-} t_d)) \, dt \qquad (n \in \mathbb{N}^d)$$

if $f \in L_1$. Note that the group operations $\dot{+}$ and $\dot{-}$ were defined in Vilenkin [11] or in Schipp, Wade, Simon, Pál [9].

For a given γ, δ satisfying the above conditions the *restricted maximal operator* is defined by

$$\sigma_\gamma^\alpha f := \sup_{n \in L} |\sigma_n^\alpha f|,$$

where the cone-like set L is defined in (2). If $\gamma_j = \mathcal{I}$ for all $j = 2, \ldots, d$ then we get a cone.

4. Estimations of the (C, α) and Riesz kernels

Recall (see Fine [1] and Vilenkin [11]) that the *Vilenkin-Dirichlet kernels* $D_k := \sum_{j=0}^{k-1} w_j$ satisfy

$$D_{P_k}(x) = \begin{cases} P_k, & \text{if } x \in [0, P_k^{-1}) \\ 0, & \text{if } x \in [P_k^{-1}, 1) \end{cases} \qquad (k \in \mathbb{N}). \tag{3}$$

If we write n in the form $n = r_1 P_{n_1} + r_2 P_{n_2} + \ldots + r_v P_{n_v}$ with $n_1 > n_2 > \, > \ldots > n_v \geq 0$ and $0 < r_i < p_i$ $(i = 1, \ldots, v)$, then let $n^{(0)} := n$ and $n^{(i)} := n^{(i-1)} - r_i P_{n_i}$. We have estimated the (C, α) and Riesz kernels in [14].

Theorem 1 ([14]) *For $0 < \alpha \leq 1 \leq \beta$ we have*

$$|K_n^\alpha(x)| \leq C n^{-\alpha} \sum_{k=1}^{v} \sum_{j=0}^{n_k} \sum_{i=j}^{n_k} \sum_{h=0}^{p_j - 1} P_i^{\alpha-1} P_j D_{P_i}(x \dot{+} h P_{j+1}^{-1}), \qquad (n \in \mathbb{N}). \tag{4}$$

The uniform boundedness of the integrals of the kernel functions follows easily from this (see [14]): for $0 < \alpha \leq 1 \leq \beta$ we have

$$\int_0^1 |K_n^\alpha| \, d\lambda \leq C, \qquad (n \in \mathbb{N}). \tag{5}$$

Lemma 1. *If $1 \leq s \leq K$, $0 < \alpha \leq 1 \leq \beta$ and $1/(\alpha + 1) < p \leq 1$ then*

$$\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} \Big(\int_0^{P_K^{-1}} |K_n^\alpha(x \dot{+} t)| \, dt \Big)^p \, dx \leq C_p P_K^{-1},$$

where C_p is depending on s, p and α.

Proof. If $j \geq K-s$ and $x \notin [0, P_{K-s}^{-1})$ then $x\dot{+}hP_{j+1}^{-1} \notin [0, P_{K-s}^{-1})$. Thus

$$\int_0^{P_K^{-1}} D_{P_i}(x\dot{+}hP_{j+1}^{-1}\dot{+}t)\,dt = 0$$

for $x \notin [0, P_{K-s}^{-1})$, $i \geq j \geq K-s$ and $h = 0, \ldots, p_j - 1$. Applying (4) we conclude

$$\begin{aligned}
&\int_0^{P_K^{-1}} |K_n^\alpha(x\dot{+}t)|\,dt \leq \\
&\leq \quad Cn^{-\alpha} \sum_{\substack{k=1\\ n_k<K-s}}^{v} \sum_{j=0}^{n_k}\sum_{i=j}^{n_k}\sum_{h=0}^{p_j-1} P_i^{\alpha-1}P_j \int_0^{P_K^{-1}} D_{P_i}(x\dot{+}hP_{j+1}^{-1}\dot{+}t)\,dt + \\
&\qquad + Cn^{-\alpha} \sum_{\substack{k=1\\ n_k\geq K-s}}^{v} \sum_{j=0}^{K-s-1}\sum_{i=j}^{K-1}\sum_{h=0}^{p_j-1} P_i^{\alpha-1}P_j \int_0^{P_K^{-1}} D_{P_i}(x\dot{+}hP_{j+1}^{-1}\dot{+}t)\,dt + \\
&\qquad + Cn^{-\alpha} \sum_{\substack{k=1\\ n_k\geq K-s}}^{v} \sum_{j=0}^{K-s-1}\sum_{i=K}^{n_k}\sum_{h=0}^{p_j-1} P_i^{\alpha-1}P_j \int_0^{P_K^{-1}} D_{P_i}(x\dot{+}hP_{j+1}^{-1}\dot{+}t)\,dt = \\
&= \quad (A_n) + (B_n) + (C_n).
\end{aligned}$$

It is easy to see, that equality (3) implies

$$\int_0^{P_K^{-1}} D_{P_i}(x\dot{+}hP_{j+1}^{-1}\dot{+}t)\,dt = P_iP_K^{-1}1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1}\dot{+}P_i^{-1})}(x)$$

for $j \leq i \leq K-1$. Thus

$$(A_n) \leq CP_{K-s}^{-\alpha} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l}\sum_{i=j}^{K-1}\sum_{h=0}^{p_j-1} P_i^{\alpha-1}P_jP_iP_K^{-1}1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1}\dot{+}P_i^{-1})}(x).$$

Consequently, if $p > 1/(\alpha+1)$ and $\alpha p \neq 1$ then

$$\begin{aligned}
\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} (A_n)^p \, d\lambda &\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} \sum_{i=j}^{K-1} P_i^{\alpha p - 1} P_j^p \leq \\
&\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} P_j^{\alpha p + p - 1} \leq \\
&\leq C_p P_K^{-\alpha p - p} \sum_{l=1}^{K-s-1} P_l^{\alpha p + p - 1} \leq \\
&\leq C_p P_K^{-1}.
\end{aligned}$$

Recall that the sequence (p_j) is bounded. If $\alpha p = 1$, in other words, if $\alpha = p = 1$ then

$$\begin{aligned}
\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} (A_n)^p \, d\lambda \leq C_p P_K^{-\alpha p - p} & \sum_{l=1}^{K-s-1} \sum_{j=0}^{l} (K-j) P_j^p \leq \\
&\leq C_1 P_K^{-1} \sum_{j=1}^{K-s-1} (K-j)^2 P_j P_K^{-1} \leq \\
&\leq C_1 P_K^{-1} \sum_{j=1}^{K-s-1} (K-j)^2 2^{j-K} \leq \\
&\leq C_1 P_K^{-1}.
\end{aligned}$$

Since $P_{n_1}^{-\alpha} P_{K-s-1}^{\alpha} (n_1 - K + s + 1) \leq 2^{-\alpha(n_1 - K + s + 1)} (n_1 - K + s + 1)$, which is bounded, we obtain

$$(B_n) \leq$$

$$\leq C P_{n_1}^{-\alpha} (n_1 - K + s + 1) \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j - 1} P_i^{\alpha - 1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} h P_{j+1}^{-1} \dot{+} t) \, dt \leq$$

$$\leq C P_{K-s-1}^{-\alpha} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} \sum_{h=0}^{p_j - 1} P_i^{\alpha - 1} P_j P_i P_K^{-1} 1_{[h P_{j+1}^{-1}, h P_{j+1}^{-1} \dot{+} P_i^{-1})}(x).$$

Hence

$$\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} (B_n)^p \, d\lambda \leq C_p P_K^{-\alpha p - p} \sum_{j=0}^{K-s-1} \sum_{i=j}^{K-1} P_i^{\alpha p - 1} P_j^p \leq C_p P_K^{-1}$$

as before. The case $\alpha = p = 1$ can be handled similarly.

If $i \geq K$ then (3) implies

$$\int_0^{P_K^{-1}} D_{P_i}(x \dot{+} hP_{j+1}^{-1} \dot{+} t)\, dt = 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x).$$

Similarly as above we can see that

$$(C_n) \leq$$

$$\leq Cn^{-\alpha/3} \sum_{\substack{k=1 \\ n_k \geq K-s}}^{v} \sum_{j=0}^{K-s-1} \sum_{i=K}^{n_k} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j \int_0^{P_K^{-1}} D_{P_i}(x \dot{+} hP_{j+1}^{-1} \dot{+} t)\, dt \leq$$

$$\leq CP_{n_1}^{-\alpha/3}(n_1 - K + s + 1) \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x) \leq$$

$$\leq CP_{K-s-1}^{-\alpha/3} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} \sum_{h=0}^{p_j-1} P_i^{\alpha/3-1} P_j 1_{[hP_{j+1}^{-1}, hP_{j+1}^{-1} \dot{+} P_K^{-1})}(x).$$

Consequently,

$$\int_{P_{K-s}^{-1}}^{1} \sup_{n \geq P_{K-s}} (C_n)^p\, d\lambda \leq C_p P_K^{-\alpha p/3} \sum_{j=0}^{K-s-1} \sum_{i=K}^{\infty} P_i^{(\alpha/3-1)p} P_j^p P_K^{-1} \leq C_p P_K^{-1},$$

which shows the lemma. ∎

5. The boundedness of the maximal operators on Hardy spaces

A bounded measurable function a is a *p-atom* if there exists a Vilenkin rectangle $I \in \mathcal{F}_{\overline{n}_1}$ such that

(i) $\operatorname{supp} a \subset I$,

(ii) $\|a\|_\infty \leq |I|^{-1/p}$,

(iii) $\int_I a\, d\lambda = 0$.

Theorem 2. *Suppose that*

$$\max\{1/(\alpha_j+1), j=1,\dots,d\} =: p_0 < p < \infty$$

and $0<\alpha_j \le 1 \le \beta_j$ $(j=1,\dots,d)$. *Then*

$$\|\sigma_\gamma^\alpha f\|_p \le C_p \|f\|_{H_p} \qquad (f\in H_p). \tag{6}$$

In particular, if $f\in L_1$ *then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\gamma^\alpha f > \rho) \le C\|f\|_1. \tag{7}$$

Proof. We have to show that the operator σ_γ^α is bounded from L_∞ to L_∞ and

$$\int\limits_{[0,1)^d} |\sigma_\gamma^\alpha a|^p \, d\lambda \le C_p \tag{8}$$

for every p-atom a (see Weisz [13]).

The boundedness follows from (5). Let a be an arbitrary p-atom with support $I=I_1\times\ldots\times I_d$ and $|I_1|=P_K^{-1}$, $|I_j|=\gamma_j^0(P_K)^{-1}$ $(j=2,\dots,d;$ $K\in\mathbb{N})$. Recall that $\gamma_1^0=\mathcal{I}$ and $\gamma_j^0(P_K):=P_{K_j}$, if $P_{K_j}\le\gamma_j(P_K)<P_{K_j+1}$ $(j=2,\dots,d; K,K_j\in\mathbb{N})$. We can assume that $I_j=[0,P_{K_j}^{-1})$ $(j=1,\dots,d)$. It is easy to see that $\hat{a}(n)=0$ if $n_j<\gamma_j^0(P_K)$ for all $j=1,\dots,d$. In this case $\sigma_n^\alpha a=0$.

Suppose that $n_1<P_{K-r}$ for some $r\in\mathbb{N}$. Let $\delta_j=\xi^{\mu_j}$ and $a_j\tau_{j,1}\le\mu_j<$ $<(a_j+1)\tau_{j,1}$ for some $a_j\in\mathbb{N}$. By the definition of the cone-like set and by (1) we have

$$n_j \le \xi^{\mu_j}\gamma_j(n_1) \le \xi^{(a_j+1)\tau_{j,1}}\gamma_j(P_{K-r}) \le \gamma_j(\xi^{a_j+1}P_{K-r}).$$

Choose $a,b_j\in\mathbb{N}$ such that $\xi\le 2^a$ and $m=\sup_{j\in\mathbb{N}} p_j \le \xi^{\tau_{j,1}b_j}$. Then

$$\begin{aligned} n_j &\le \xi^{-\tau_{j,1}b_j}\gamma_j(\xi^{a_j+1+b_j}P_{K-r}) \le \frac{1}{m}\gamma_j(2^{a(a_j+1+b_j)}P_{K-r}) \le \\ &\le \frac{1}{m}\gamma_j(2^r P_{K-r}) \le \frac{1}{m}\gamma_j(P_K) \le \gamma_j^0(P_K) \end{aligned}$$

for all $j=2,\dots,d$, where let $r:=\max_{j=2,\dots,d}\{a(a_j+1+b_j)\}$. In this case $\sigma_n^\alpha a=0$.

Thus we can suppose that $n_1\ge P_{K-r}$. By the right hand side of (1),

$$\begin{aligned} n_j &\ge \xi^{-(a_j+1)\tau_{j,1}}\gamma_j(P_{K-r}) \ge \xi^{-(a_j+1)\tau_{j,1}}\xi^{-\tau_{j,2}br}\gamma_j(P_{K-r}\xi^{br}) \ge \\ &\ge \xi^{-(a_j+1)\tau_{j,1}-\tau_{j,2}br}\gamma_j(P_{K-r}m^r) \ge 2^{-a((a_j+1)\tau_{j,1}+\tau_{j,2}br)}\gamma_j(P_K) \ge \\ &\ge 2^{-s}P_{K_j} \ge P_{K_j-s}, \end{aligned}$$

where $b, s \in \mathbb{N}$ are chosen such that $m \le \xi^b$ and

$$\max_{j=2,\ldots,d} \{a((a_j+1)\tau_{j,1} + \tau_{j,2}br)\} \le s.$$

We can suppose that $s \ge r$. Therefore

$$\sigma_\gamma^\alpha a \le \sup_{n_j \ge P_{K_j - s}, j=1,\ldots,d} |\sigma_n^\alpha a|.$$

By the L_∞ boundedness of σ_γ^α we conclude

$$\int\limits_{\prod_{j=1}^d [0, P_{K_j-s}^{-1})} |\sigma_\gamma^\alpha a|^p \, d\lambda \le C_p \|a\|_\infty^p \prod_{j=1}^d P_{K_j-s}^{-1} \le C_p \prod_{j=1}^d P_{K_j} \prod_{j=1}^d P_{K_j-s}^{-1} \le C_p.$$

To compute the integral over $[0,1)^d \setminus \prod_{j=1}^d [0, P_{K_j-s}^{-1})$ it is enough to integrate over

$$H_k := [0,1) \setminus [0, P_{K_1-s}^{-1}) \times \ldots \times [0,1) \setminus [0, P_{K_k-s}^{-1}) \times [0, P_{K_{k+1}-s}^{-1}) \times \ldots \times [0, P_{K_d-s}^{-1})$$

for $k = 1, \ldots, d$. Using (5) and the definition of the atom we can see that

$$\begin{aligned} |\sigma_n^\alpha a(x)| &\le \int\limits_{\prod_{j=1}^d [0, P_{K_j}^{-1})} |a(t)| (|K_{n_1}^{\alpha_1}(x_1 \dot{+} t_1)| \times \cdots \times |K_{n_d}^{\alpha_d}(x_d \dot{+} t_d)|) \, dt \le \\ &\le C \Big(\prod_{j=1}^d P_{K_j}^{1/p} \Big) \prod_{j=1}^k \int\limits_{[0, P_{K_j}^{-1})} |K_{n_j}^{\alpha_j}(x_j \dot{+} t_j)| \, dt_j. \end{aligned}$$

Lemma 1 implies that

$$\int\limits_{H_k} |\sigma_\gamma^\alpha a(x)|^p \, dx \le C_p \prod_{j=1}^d P_{K_j} \prod_{j=1}^k P_{K_j}^{-1} \prod_{j=k+1}^d P_{K_j-s}^{-1} = C_p$$

which verifies (8) as well as (6) for each $p_0 < p \le 1$. The weak type $(1,1)$ inequality in (7) follows by interpolation. ∎

This theorem was proved by the author in [12, 14] for cones, i.e. if each $\gamma_j = \mathcal{I}$, and in [15] for trigonometric Fourier series.

Observe that the set of the Vilenkin polynomials is dense in L_1. The weak type $(1,1)$ inequality in Theorem 2 and the usual density argument of Marcinkievicz and Zygmund [6] imply

Corollary 1. *If* $0 < \alpha_j \leq 1 \leq \beta_j$ $(j = 1, \dots, d)$ *and* $f \in L_1$ *then*

$$\lim_{n\to\infty, n\in L} \sigma_n^\alpha f = f \qquad a.e.$$

The a.e. convergence of $\sigma_n^\alpha f$ was proved by Gát and Nagy [5] for two-dimensional Fejér means.

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