Annales Univ. Sci. Budapest., Sect. Comp. 35 (2011) 285–304

ON FEJÉR TYPE SUMMABILITY OF WEIGHTED LAGRANGE INTERPOLATION ON THE LAGUERRE ROOTS

L. Szili (Budapest, Hungary)

Dedicated to Professor Antal Járai on his 60th and Professor Péter Vértesi on his 70th birthdays

Abstract. The sequence of certain arithmetic means of the Lagrange interpolation on the roots of Laguerre polynomials is shown to be uniformly convergent in suitable weighted function spaces.

1. Introduction

Let $w_\alpha(x) := x^\alpha e^{-x}$ $(x \in \mathbb{R}^+ := (0, +\infty),\ \alpha > -1)$ be a Laguerre weight and denote by $U_n(w_\alpha)$ $(n \in \mathbb{N} := \{1, 2, \ldots\})$ the root system of $p_n(w_\alpha)$ $(n \in \in \mathbb{N}_0 := \{0, 1, \ldots\})$ (orthonormal polynomials with respect to the weight w_α). We shall consider a Fejér type summation of Lagrange interpolation on $U_n(w_\alpha)$ $(n \in \mathbb{N})$. The corresponding polynomials will be denoted by $\sigma_n\big(f, U_n(w_\alpha), \cdot\big)$ (see (2.8)).

The goal of this paper is to give conditions for the parameters $\alpha > -1, \gamma \geq 0$ ensuring

$$\lim_{n\to+\infty} \big\| \big(f - \sigma_n\big(f, U_n(w_\alpha), \cdot\big)\big)\sqrt{w_\gamma}\big\|_\infty = 0$$

for all $f \in C_{\sqrt{w_\gamma}}$ (see Section 2.1), where $\|\cdot\|_\infty$ denotes the maximum norm.

2010 Mathematics Subject Classification: 40C05, 40D05, 40G05, 41A05, 41A10.

Key words and phrases: Weighted interpolation, weighted approximation, summation methods, uniform convergence, Laguerre weights.

The Project is supported by the European Union and co-financed by the European Social Fund (grant agreement no. TAMOP 4.2.1/B-09/1/KMR-2010-0003).

https://doi.org/10.71352/ac.35.285

2. Notations and preliminaries

We are going to summarize definitions and statements on function spaces, weighted approximation, weighted Lagrange interpolation, which we shall need in the following sections.

2.1. Some weighted uniform spaces. Setting

$$w_\gamma(x) := x^\gamma e^{-x} \qquad (x \in \mathbb{R}_0^+ := [0, +\infty),\ \gamma \geq 0),$$

we define the weighted functional space $C_{\sqrt{w_\gamma}}$ as follows:

i) for $\gamma > 0$, $f \in C_{\sqrt{w_\gamma}}$ iff f is a continuous function in any segment $[a, b] \subset \mathbb{R}^+$ and

$$\lim_{x \to 0+0} f(x)\sqrt{w_\gamma(x)} = 0 = \lim_{x \to +\infty} f(x)\sqrt{w_\gamma(x)};$$

ii) for $\gamma = 0$, $f \in C_{\sqrt{w_0}}$ iff f is continuous in $[0, +\infty)$ and

$$\lim_{x \to +\infty} f(x)\sqrt{w_0(x)} = 0.$$

In other words, when $\gamma > 0$, the function f in $C_{\sqrt{w_\gamma}}$ could take very large values, with polynomial growth, as x approaches zero from the right, and could have an exponential growth as $x \to +\infty$.

If we introduce the norm

$$\|f\|_{\sqrt{w_\gamma}} := \left\|f\sqrt{w_\gamma}\right\|_\infty := \max_{x \in \mathbb{R}_0^+} \left|f(x)\right|\sqrt{w_\gamma(x)},$$

in $C_{\sqrt{w_\gamma}}$, $\gamma \geq 0$, then we get the Banach space $\left(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\right)$.

2.2. Weighted polynomial approximation. We recall two fundamental results with respect to the polynomial approximation in the function space $\left(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\right)$.

The first fact is that *the set of polynomials are dense in the function space* $\left(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\right)$. More precisely, if we denote by $\mathcal{P}_n$ the linear space of all polynomials of degree at most n and by

$$E_n(f, \sqrt{w_\gamma}) := \inf_{P \in \mathcal{P}_n} \|(f - P)\sqrt{w_\gamma}\|_\infty = \inf_{P \in \mathcal{P}_n} \|f - P\|_{w_\gamma}$$

the best polynomial approximation of the function $f \in C_{\sqrt{w_\gamma}}$, then we have

$$\lim_{n\to+\infty} E_n(f, \sqrt{w_\gamma}) = 0$$

(see for example [9, p. 11] and [1, p. 186]).

The second fact is associated with *the Mhaskar–Rahmanov–Saff number*: For every $\gamma \geq 0$ and $n \in \mathbb{N}$ there are positive real numbers $a_n := a_n(\gamma)$ and $b_n := b_n(\gamma)$ such that for any polynomial $P \in \mathcal{P}_n$ we get

$$\|P\|_{\sqrt{w_\gamma}} = \|P\sqrt{w_\gamma}\|_\infty = \max_{x\in\mathbb{R}_0^+} \big|P(x)\big|\sqrt{w_\gamma(x)} = \max_{a_n\leq x\leq b_n} \big|P(x)\big|\sqrt{w_\gamma(x)} \tag{2.1}$$

and

$$\|P\sqrt{w_\gamma}\|_\infty > \big|P(x)\big|\sqrt{w_\gamma(x)} \qquad \text{for all } 0 \leq x < a_n \text{ and } b_n < x.$$

Moreover, for every $\gamma \geq 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} a_n &: = a_n(\gamma) = (2n+\gamma)\left(1 - \sqrt{1 - \frac{\gamma^2}{(\gamma+2n)^2}}\right) > \frac{\gamma^2}{4n+2\gamma}, \\ b_n &: = b_n(\gamma) = (2n+\gamma)\left(1 + \sqrt{1 - \frac{\gamma^2}{(\gamma+2n)^2}}\right) = 4n + 2\gamma + \frac{C}{n} \end{aligned} \tag{2.2}$$

with a constant $C > 0$ independent of n (see for example [6, (2.1)]).

2.3. Weighted Lagrange interpolation. Let

$$p_n(w_\alpha, x) \qquad (x \in \mathbb{R}_0^+,\ n \in \mathbb{N}_0,\ \alpha > -1)$$

be the sequence of *orthonormal* Laguerre polynomials with positive leading coefficients. Let us denote by

$$U_n(w_\alpha) := \{y_{k,n} := y_{k,n}(w_\alpha) \mid k = 1, 2, \ldots, n\} \quad (n \in \mathbb{N}) \tag{2.3}$$

the n different roots of $p_n(w_\alpha, \cdot)$. We index them as

$$0 < y_{1,n}(w_\alpha) < y_{2,n}(w_\alpha) < \cdots < y_{n-1,n}(w_\alpha) < y_{n,n}(w_\alpha) < \infty.$$

For a given function $f : \mathbb{R}_0^+ \to \mathbb{R}$ we denote by $L_n(f, U_n(w_\alpha), \cdot)$ *the Lagrange interpolatory polynomial* of degree $\leq n-1$ at the zeros of $p_n(w_\alpha)$, i.e.

$$L_n\left(f, U_n(w_\alpha), y_{k,n}\right) = f(y_{k,n}) \qquad (k = 1, 2, \ldots, n).$$

We have

$$L_n\left(f,U_n(w_\alpha),x\right)=\sum_{k=1}^{n} f(y_{k,n})\ell_{k,n}\left(U_n(w_\alpha),x\right)$$
$$(x\in\mathbb{R}_0^+,\quad n\in\mathbb{N}),$$

where

$$\ell_{k,n}\left(U_n(w_\alpha),x\right)=\frac{p_n(w_\alpha,x)}{p_n'(w_\alpha,y_{k,n})(x-y_{k,n})}$$
$$(x\in\mathbb{R}_0^+;\ k=1,2,\dots,n;\ n\in\mathbb{N})$$

are *the fundamental polynomials* associated with the nodes $U_n(w_\alpha)$.

Consider the (uniform) convergence of the sequence $L_n\big(f,U_n(w_\alpha),\cdot\big)$ $(n\in\mathbb{N})$ in the Banach space $\big(C_{\sqrt{w_\gamma}},\|\cdot\|_{\sqrt{w_\gamma}}\big)$. In other words, for a function $f\in C_{\sqrt{w_\gamma}}$ we have to investigate the real sequence

$$\varrho_n(f):=\left\|\Big(f-L_n\big(f,U_n(w_\alpha),\cdot\big)\Big)\sqrt{w_\gamma}\right\|_\infty\quad(n\in\mathbb{N}).$$

In other words, we approximate the function $f\sqrt{w_\gamma}$ by *the weighted Lagrange interpolatory polynomials*

$$L_n\big(f,U_n(w_\alpha),x\big)\sqrt{w_\gamma(x)}\quad(x\in\mathbb{R}_0^+,\ n\in\mathbb{N}).\tag{2.4}$$

The main question is: is it true that $\varrho_n(f)\to 0$ $(n\to+\infty)$ for all $f\in C_{\sqrt{w_\gamma}}$ or not?

The classical Lebesgue estimate for the weighted Lagrange interpolation is the following: take the best uniform approximation $P_{n-1}(f)$ to $f\in C_{w_\gamma}$ (the existence of such a $P_{n-1}(f)$ is obvious), and consider

$$\begin{aligned}&\Big|f(x)-L_n\big(f,U_n(w_\alpha),x\big)\Big|\sqrt{w_\gamma(x)}\le\\ &\le\Big|f(x)-P_{n-1}(f,x)\Big|\sqrt{w_\gamma(x)}+\Big|L_n\Big(f-P_{n-1}(f),U_n(w_\alpha),x\Big)\Big|\sqrt{w_\gamma(x)}\le\\ &\le E_{n-1}\big(f,\sqrt{w_\gamma}\big)\left(1+\sum_{k=1}^{n}\Big|\ell_{k,n}\big(U_n(w_\alpha,x)\big)\Big|\frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}}\right).\end{aligned}$$

This estimate shows that the pointwise/uniform convergence of the sequence (2.4) depends on the orders of *the weighted Lebesgue functions*:

$$\lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma},x\big):=\sum_{k=1}^{n}\Big|\ell_{k,n}\big(U_n(w_\alpha,x)\big)\Big|\frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}}$$
$$(x\in\mathbb{R}_0^+,\ n\in\mathbb{N}),$$

and on the orders of *the weighted Lebesgue constants*:

$$\Lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma}\big) := \sup_{x\in\mathbb{R}_0^+} \lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma},x\big) \quad (n\in\mathbb{N}).$$

It is clear that for all $\gamma\ge 0$, $\alpha>-1$ and $n\in\mathbb{N}$

$$\begin{aligned} \mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma}) \ &: \ \big(C_{\sqrt{w_\gamma}},\|\cdot\|_{\sqrt{w_\gamma}}\big)\to\mathcal{P}_{n-1}\subset\big(C_{\sqrt{w_\gamma}},\|\cdot\|_{\sqrt{w_\gamma}}\big) \\ \mathcal{L}_n(f,U_n(w_\alpha),\sqrt{w_\gamma}) &:= L_n\big(f,U_n(w_\alpha),\cdot\big) \end{aligned} \tag{2.5}$$

is a bounded linear operator and its norm is

$$\begin{aligned} \|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\| :&= \sup_{0\not\equiv f\in C_{\sqrt{w_\gamma}}} \frac{\|\mathcal{L}_n(f,U_n(w_\alpha),\sqrt{w_\gamma})\|_{\sqrt{w_\gamma}}}{\|f\|_{\sqrt{w_\gamma}}} = \\ &= \sup_{0\not\equiv f\in C_{w_\gamma}} \frac{\|L_n\big(f,U_n(w_\alpha),\cdot\big)\sqrt{w_\gamma}\|_\infty}{\|f\sqrt{w_\gamma}\|_\infty}. \end{aligned}$$

Since

$$\begin{aligned} L_n\big(f,U_n(w_\alpha),x\big) &= \sum_{k=1}^n f(y_{k,n})\ell_{k,n}\big(U_n(w_\alpha),x\big) = \\ &= \sum_{k=1}^n f(y_{k,n})\sqrt{w_\gamma(y_{k,n})}\cdot\ell_{k,n}\big(U_n(w_\alpha),x\big)\cdot\frac{1}{\sqrt{w_\gamma(y_{k,n})}}, \end{aligned}$$

thus by a usual argument we have that the norm of the operator (2.5) equals to the n-th Lebesgue constant, i.e.

$$\big\|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\big\| = \Lambda_n\big(U_n(w_\alpha),\sqrt{w_\gamma}\big) \quad (n\in\mathbb{N}).$$

The pointwise/uniform convergence of $L_n(f,U_n(w_\alpha),\cdot)$ $(n\in\mathbb{N})$ in different function spaces were investigated by several authors (see [3], [8], [6]). For example in 2001, G. Mastroianni and D. Occorsio showed that for arbitrary $\gamma\ge 0$ and $\alpha>-1$ the order of the norm of the operator $\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})$ is $n^{1/6}$ (see [6, Theorem 3.3]), i.e.

$$\|\mathcal{L}_n(\cdot,U_n(w_\alpha),\sqrt{w_\gamma})\| \sim n^{1/6} \quad (n\in\mathbb{N}).$$

Here and in the sequel, if A and B are two expressions depending on certain indices and variables, then we write

$$A\sim B, \qquad \text{if and only if} \qquad 0<C_1\le\left|\frac{A}{B}\right|\le C_2$$

uniformly for the indices and variables considered.

From results of P. Vértesi it follows that for *any* interpolatory matrix $X_n \subset \subset \mathbb{R}_0^+$ $(n \in \mathbb{N})$ the order of the corresponding weighted Lebesgue constants is at least $\log n$, i.e. if $\gamma \geq 0$ and $X_n \subset \mathbb{R}^+$ $(n \in \mathbb{N})$ is an arbitrary interpolatory matrix then there exists a constant $C > 0$ independent of n such that

$$\Lambda_n(X_n, \sqrt{w_\gamma}) = \|\mathcal{L}_n(\cdot, X_n, \sqrt{w_\gamma})\| \geq C \log n \quad (n \in \mathbb{N}).$$

See [16, Theorem 7.2], [14] and [15]. Thus using the Banach–Steinhaus theorem we obtain the following Faber type result:

Theorem A. *If $\gamma \geq 0$ and $X_n \subset \mathbb{R}^+$ $(n \in \mathbb{N})$ is an arbitrary interpolatory matrix then there exists a function $f \in C_{\sqrt{w_\gamma}}$ for which the relation*

$$\left\|(f - L_n(f, X_n, \cdot))\sqrt{w_\gamma}\right\|_\infty \to 0 \quad \textit{as} \quad n \to +\infty$$

does not hold.

In [6] G. Mastroianni and D. Occorsio also proved that there exist point systems for which the optimal Lebesgue constants can be attained. We recall only the following result:

Theorem B (see [6, Theorem 3.4]). *If $\mathcal{V}_{n+1} := U_n(w_\alpha) \cup \{4n\}$, then*

$$\|\mathcal{L}_{n+1}\big(\cdot, \mathcal{V}_{n+1}, \sqrt{w_\gamma}\big)\| = \Lambda_{n+1}\big(\mathcal{V}_{n+1}, \sqrt{w_\gamma}\big) \sim \log n \quad (n \in \mathbb{N})$$

if and only if the parameters $\alpha > -1$ and $\gamma \geq 0$ satisfy the additional conditions:

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}.$$

2.4. Fejér type sums. Using the Christoffel–Darboux formula [12, Theorem 3.2.2] we write the Lagrange interpolatory polynomials as

$$L_n\big(f, U_n(w_\alpha), x\big) = \sum_{l=0}^{n-1} c_{l,n}(f) p_l(w_\alpha, x) \quad (x \in \mathbb{R}_0^+,\ n \in \mathbb{N}), \tag{2.6}$$

where

$$c_{l,n}(f) := \sum_{k=1}^{n} f(y_{k,n}) p_l\left(w_\alpha, y_{k,n}\right) \lambda_{k,n} \quad (l = 0, 1, \dots, n-1,\ n \in \mathbb{N}). \tag{2.7}$$

Here and in the sequel $\lambda_{k,n} := \lambda_{k,n}(w_\alpha)$ $(k = 1, 2, \dots, n,\ n \in \mathbb{N})$ denote the Christoffel numbers with respect to the weight w_α (cf. [12, (15.3.5)]).

Using (2.6) and (2.7) we have

$$L_n(f,x) := L_n\big(U_n(w_\alpha), f, x\big) = \sum_{l=0}^{n-1} c_{l,n}(f) p_l(w_\alpha, x) =$$
$$= \sum_{k=1}^{n} f\left(y_{k,n}\right) K_{n-1}\left(x, y_{k,n}\right) \lambda_{k,n},$$

where

$$K_{n-1}(x,y) := \sum_{l=0}^{n-1} p_l\left(w_\alpha, x\right) p_l\left(w_\alpha, y\right)$$
$$\big(x, y \in \mathbb{R}_0^+,\ n \in \mathbb{N}\big).$$

Let

$$L_{n,m}(f,x) := \sum_{l=0}^{m} c_{l,n}(f) p_l(w_\alpha, x) = \sum_{k=1}^{n} f\left(y_{k,n}\right) K_m\left(x, y_{k,n}\right) \lambda_{k,n}$$
$$\big(x \in \mathbb{R}_0^+,\ m = 0, 1, \ldots, n-1,\ n \in \mathbb{N}\big).$$

The *Fejér means of the Lagrange interpolation* of the function $f : \mathbb{R}_0^+ \to \mathbb{R}$ are defined as the arithmetic means of the sums $L_{n,0}$, $L_{n,1}$, …, $L_{n,n-1}$, i.e.

$$\sigma_n(f,x) := \sigma_n\big(f, U_n(w_\alpha), x\big) :=$$
$$:= \frac{L_{n,0}(f,x) + L_{n,1}(f,x) + \cdots + L_{n,n-1}(f,x)}{n} \tag{2.8}$$
$$(x \in \mathbb{R}_0^+,\ n \in \mathbb{N}).$$

From the above formulas we have

$$\sigma_n(f,x) = \sum_{l=0}^{n-1} \left(1 - \frac{l}{n}\right) c_{l,n}(f) p_l(w_\alpha, x) =$$
$$= \sum_{k=1}^{n} f(y_{k,n}) \Big\{ \frac{1}{n} \sum_{m=0}^{n-1} K_m(x, y_{k,n}) \Big\} \lambda_{k,n} = \tag{2.9}$$
$$= \sum_{k=1}^{n} f(y_{k,n}) K_n^{(1)}(x, y_{k,n}) \lambda_{k,n},$$

where

$$K_n^{(1)}(x,y) := \frac{1}{n} \sum_{m=0}^{n-1} K_m(x,y) = \sum_{l=0}^{n-1} \left(1 - \frac{l}{n}\right) p_l(w_\alpha, x) p_l(w_\alpha, y) \tag{2.10}$$
$$\big(x, y \in \mathbb{R}_0^+,\ n \in \mathbb{N}\big).$$

Remark 1. It is important to observe that we defined the Fejér means of Lagrange interpolation by considering the means (2.8) and *not* by the means

$$\frac{L_0(f,x)+L_1(f,x)+\cdots+L_{n-1}(f,x)}{n} \quad (x\in\mathbb{R}_0^+,\ n\in\mathbb{N}). \tag{2.11}$$

Several earlier results suggest that the two methods (2.8) and (2.11) have different behavior with respect to uniform convergence.

For example in the trigonometric case J. Marcinkiewicz [4] proved that the method corresponding to (2.8) is uniformly convergent in $C_{2\pi}$ (the Banach space of 2π periodic continuous functions defined on $\mathbb{R}$ endowed with the maximum norm), moreover there exists a function $f\in C_{2\pi}$ such that the sequence corresponding to (2.11) diverges at a point. In other words we have an analogue of the classical theorem of L. Fejér about the uniform convergence of the $(C,1)$ means of the partial sums of the trigonometric Fourier series *only for suitable* arithmetic means of the Lagrange interpolation.

The situation is similar if we consider the Lagrange interpolation on the roots of the Chebyshev polynomials of the first kind. In [13] A.K. Varma and T.M. Mills showed that the (2.8) type means of the Lagrange interpolation uniformly convergent for every $f\in C[-1,1]$. Moreover in [2] P. Erdős and G. Halász proved that there exists a continuous function for which the (2.11) type means are almost everywhere divergent on the interval $[-1,1]$.

3. Uniform convergence of suitable arithmetic means

The main goal of this paper is to show that the (2.8) type arithmetic means of the Lagrange interpolation on the roots of Laguerre polynomials is uniformly convergent in suitable weighted function spaces.

Theorem. *Let* $\alpha>-1$ *and* $0\le\gamma=:\alpha+2r$, *i.e.* $\sqrt{w_\gamma(x)}=\sqrt{w_\alpha(x)}x^r$ $(x\in\mathbb{R}^+)$. *If*

$$-\min\left(\frac{\alpha}{2},\frac{1}{4}\right)<r\le\frac{7}{6}, \tag{3.1}$$

then

$$\lim_{n\to+\infty}\left\|\Big(f-\sigma_n(f,U_n(w_\alpha),\cdot)\Big)\sqrt{w_\gamma}\right\|_\infty=0 \tag{3.2}$$

holds for all $f\in C_{\sqrt{w_\gamma}}$.

Remark 2. We intend to investigate the convergence of the method (2.11) in a subsequent paper.

Remark 3. The formulas (2.6) and (2.7) show that the Lagrange interpolation polynomials on the roots of Laguerre polynomials can be considered as a discrete version of partial sums of the Fourier series with respect to the system of Laguerre polynomials. In [9] E.L. Poiani proved (among other things) that the sequence of the $(C, 1)$ means of the Laguerre series of an arbitrary function $f \in C_{w_\gamma}$ $(\gamma = 2r + \alpha,\ \alpha > -1)$ converges to f in the space $\left(C_{w_\gamma, \|\cdot\|_{w_\gamma}}\right)$, if

$$-\min\left(\frac{\alpha}{2}, \frac{1}{2}\right) < r < 1 + \min\left(\frac{\alpha}{2}, \frac{1}{4}\right) \quad \text{and} \quad -\frac{1}{2} \le r \le \frac{7}{6}.$$

4. Proof of the Theorem

4.1. Laguerre polynomials. We mention some relations with respect to the Laguerre polynomials which will be used later. Let $\{p_n(w_\alpha)\}$, $\alpha > -1$, be the sequence of *orthonormal* Laguerre polynomials with positive leading coefficients. The zeros $y_{k,n} := y_{k,n}(w_\alpha)$ of $p_n(w_\alpha)$, $n \ge 1$ satisfy

$$\frac{C_1}{n} < y_{1,n} < y_{2,n} < \ldots < y_{n,n} = 4n + 2\alpha + 2 - C_2 \sqrt[3]{4n}, \tag{4.1}$$

$$y_{k,n} \sim \frac{k^2}{n} \qquad (k = 1, 2, \ldots, n,\ n \in \mathbb{N}) \tag{4.2}$$

(see [12, Section 6.32] and [5, Section 2.3.5]).

Here and what follows $C, C_1, \ldots$ will always denote positive constants (not necessarily the same at each occurrence) being independent of parameters k and n.

It is known that

$$\triangle y_{k,n} := y_{k+1,n} - y_{k,n} \sim \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \tag{4.3}$$
$$(k = 1, 2, \ldots, n-1,\ n \in \mathbb{N}),$$

and for $y_{k,n} \le x \le y_{k+1,n}$ $(k = 1, 2, \ldots, n-1)$ we have

$$\sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \sim \sqrt{\frac{x}{4n - x}} \sim \sqrt{\frac{y_{k+1,n}}{4n - y_{k+1,n}}}$$

uniformly in k and n (see [6, (2.4) and (2.5)]). From (4.2) and (4.3) it follows that

$$|y_{j,n} - y_{k,n}| \geq C\,\frac{|j^2 - k^2|}{n} \qquad (j,k = 1,2,\dots,n). \tag{4.4}$$

For the Christoffel numbers we have

$$\lambda_{k,n} := \lambda_{k,n}(w_\alpha) \sim w_\alpha(y_{k,n})\sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \sim w_\alpha(y_{k,n})\triangle y_{k,n} \tag{4.5}$$

uniformly in $k = 1,2,\dots,n$ and $n \in \mathbb{N}$ (see [6, (2.7)]).

In an article of B. Muckenhoupt and D.W. Webb [7] there is a pointwise upper estimate for the kernel of (C,δ) $(\delta > 0)$ Cesàro means of Laguerre–Fourier series (see also [17]). We shall use this result only with respect to $(C,1)$ means, that is for the kernel function $K_n^{(1)}(x,y)$ (see (2.10)): Let $\alpha > -1$. Then we have

$$\begin{gathered} \left|K_n^{(1)}(x,y)\right| \leq \frac{C}{\sqrt{w_\alpha(x)}\,\sqrt{w_\alpha(y)}}\,G_n(x,y) \\ (0 < x, y < \nu(n) + \sqrt[3]{\nu(n)},\ n \in \mathbb{N}), \end{gathered} \tag{4.6}$$

where $\nu := \nu(n) := 4n + 2\alpha + 2$,

$$\begin{aligned} &G_n(x,y) := \\ &:= \frac{1}{\nu}\,\mathcal{M}_n(x)\mathcal{M}_n(y)\,\frac{(x+y)\Big[\nu^{1/3} + |x-\nu| + |y-\nu|\Big]^2}{(x+y) + (x-y)^2\Big[\nu^{1/3} + |x-\nu| + |y-\nu|\Big]} \end{aligned} \tag{4.7}$$

and

$$\mathcal{M}_n(x) := \frac{x^{\alpha/2}\big(x + \frac{1}{\nu}\big)^{-\alpha/2 - 1/4}}{\sqrt[4]{\nu^{1/3} + |x-\nu|}} \tag{4.8}$$

(see [7, p. 1124]).

Denote by $y_{j,n}$ one of the closest root(s) to x (shortly $x \approx y_{j,n}, j = j(n)$). Using the above relations we obtain that

$$\mathcal{M}_n(x) \sim \mathcal{M}_n(y_{j,n}) \sim \begin{cases} \dfrac{1}{\sqrt{j}}, & \text{if } \dfrac{c}{n} \leq x \leq \dfrac{\nu}{2} \\[2ex] \dfrac{1}{\sqrt[4]{n|y_{j,n} - \nu|}}, & \text{if } \dfrac{\nu}{2} \leq x \leq \nu - \sqrt[3]{\nu} \\[2ex] \dfrac{1}{\sqrt[3]{n}}, & \text{if } \nu - \sqrt[3]{\nu} \leq x \leq \nu + \sqrt[3]{\nu} \end{cases} \tag{4.9}$$

for $x \in [c/n, \nu + \sqrt[3]{\nu}]$.

4.2. Uniform boundedness. Let us consider for every $n \in \mathbb{N}$ the bounded linear operator

$$\mathcal{F}_n : \left(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\right) \to \mathcal{P}_n \subset \left(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\right)$$

$$\mathcal{F}_n f := \sigma_n(f, U_n(w_\alpha), \cdot).$$

For the norm of the operator $\mathcal{F}_n$ we obtain that (see (2.9))

$$\|\mathcal{F}_n\| := \sup_{0 \not\equiv f \in C_{\sqrt{w_\gamma}}} \frac{\|\mathcal{F}_n f\|_{\sqrt{w_\gamma}}}{\|f\|_{\sqrt{w_\gamma}}} = \sup_{0 \not\equiv f \in C_{\sqrt{w_\gamma}}} \frac{\left\|\sigma_n\left(f, U_n(w_\alpha), \cdot\right)\sqrt{w_\gamma}\right\|_\infty}{\left\|f\sqrt{w_\gamma}\right\|_\infty} =$$

$$= \max_{x \in \mathbb{R}_0^+} \sum_{k=1}^{n} \left|K_n^{(1)}(x, y_{k,n})\right| \frac{\sqrt{w_\gamma(x)}}{\sqrt{w_\gamma(y_{k,n})}} \lambda_{k,n}.$$

The core of the proof of the Theorem is contained in the following lemma, which states the uniform boundedness of the operator sequence $(\mathcal{F}_n)$.

Lemma 4.1. *Let $\alpha > -1$ and r satisfy the inequality* (3.1). *Then there exists a constant $C > 0$ independent of $n \in \mathbb{N}$ such that*

$$\|\mathcal{F}_n\| = \max_{x \in \mathbb{R}_0^+} \sum_{k=1}^{n} \left|K_n^{(1)}(x, y_{k,n})\right| \frac{\sqrt{w_\alpha(x)}}{\sqrt{w_\alpha(y_{k,n})}} \left(\frac{x}{y_{k,n}}\right)^r \lambda_{k,n} \le C. \tag{4.10}$$

Proof. We shall use the following important equality (see [11, Lemma 1]): If $\gamma \ge 0$, $m \le n \in \mathbb{N}$ and $q_k \in \mathcal{P}_n$ $(k = 1, 2, \ldots, m)$ are arbitrary polynomials then

$$\max_{x \in \mathbb{R}_0^+} \left[\sqrt{w_\gamma}(x) \sum_{k=1}^{m} |q_k(x)|\right] = \max_{a_n \le x \le b_n} \left[\sqrt{w_\gamma}(x) \sum_{k=1}^{m} |q_k(x)|\right].$$

Therefore by (4.5)–(4.7) it is enough to show that

$$\max_{a_n \le x \le b_n} \sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \le C, \tag{4.11}$$

where

$$\frac{c}{n} \le a_n = a_n(\gamma) \le x \le b_n = b_n(\gamma) < \nu + \sqrt[3]{\nu}.$$

In order to prove (4.11), we distinguish several cases.

Case 1: Let $x \in [a_n, \frac{\nu}{2}]$ and

$$\begin{aligned}(4.12)\qquad &\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} = \\ &\sum_{y_{k,n} \le \frac{\nu}{2}} \dots + \sum_{y_{k,n} > \frac{\nu}{2}} \dots =: A_n^{(1)}(x) + A_n^{(2)}(x).\end{aligned}$$

Since $\nu^{1/3} + |x - \nu| + |y_{k,n} - \nu| \sim n$ $(k = 1, 2, \dots, n,\ n \in \mathbb{N})$ thus by (4.2), (4.4), (4.7) and (4.9) we have

$$A_n^{(1)}(x) \le C_1 \sum_{y_{k,n} \le \frac{\nu}{2}} n \frac{j^2 + k^2}{j^2 + k^2 + |j^2 - k^2|^2} \frac{1}{\sqrt{kj}} \left(\frac{j}{k}\right)^{2r} \frac{k}{n} \le$$

$$\le C_2 \left\{ \sum_{k \le \frac{j}{2}} \frac{j^{2r-5/2}}{k^{2r-1/2}} + \sum_{\frac{j}{2} \le k \le 2j} \frac{1}{1 + (k-j)^2} + \sum_{k \ge 2j} \frac{j^{2r-1/2}}{k^{2r+3/2}} \right\}.$$

The second sum is bounded. For the first sum we obtain that

$$\sum_{k \le j/2} \frac{j^{2r-5/2}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log j}{j}, & \text{if } r = \dfrac{3}{4} \\ j^{2r-5/2}, & \text{if } r > \dfrac{3}{4} \\ \dfrac{1}{j}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

and these expressions are bounded (independently of j and n), if $r \le 5/4$. Moreover by

$$\sum_{k=j}^{n} \frac{1}{k^s} \sim \begin{cases} \log \frac{n}{j}, & \text{if } s = 1 \\ \left| n^{-s+1} - j^{-s+1} \right|, & \text{if } s \ne 1 \end{cases}$$

we have

$$\sum_{k \ge 2j} \frac{j^{2r-1/2}}{k^{2r+3/2}} \sim \begin{cases} \dfrac{\log \frac{n}{j}}{j}, & \text{if } r = -\dfrac{1}{4} \\ \dfrac{1}{j} \left| \left(\dfrac{j}{n}\right)^{2r+1/2} - 1 \right|, & \text{if } r \ne -\dfrac{1}{4} \end{cases}$$

whence the third sum is bounded (independently of j and n), if $r > -\frac{1}{4}$. Therefore

$$(4.13)\qquad A_n^{(1)}(x) \le C \quad \left(x \in [a_n, \tfrac{\nu}{2}],\ n \in \mathbb{N}\right), \quad \text{if } -\tfrac{1}{4} < r \le \tfrac{5}{4}.$$

Let us consider $A_n^{(2)}(x)$. Since $y_{k,n} \ge \frac{\nu}{2}$ thus by (4.2), (4.4), (4.7) and (4.9) we have

$$\begin{aligned} A_n^{(2)}(x) &\le \\ \le C_1 \sum_{y_{k,n}\ge\frac{\nu}{2}} \frac{n}{1+|y_{j,n}-y_{k,n}|^2} \frac{1}{\sqrt{j}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \left(\frac{j}{k}\right)^{2r} \sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} &\le \\ \le C_2 \left\{ \sum_{\frac{\nu}{2}\le y_{k,n}\le\frac{x+y_{n,n}}{2}} \cdots + \sum_{\frac{x+y_{n,n}}{2}<y_{k,n}} \cdots \right\} &=: \\ =: A_n^{(21)}(x) + A_n^{(22)}(x). \end{aligned} \tag{4.14}$$

If $\frac{\nu}{2} \le y_{k,n} \le \frac{x+y_{n,n}}{2}$ then $|y_{k,n}-\nu| \sim n$ thus by (4.2) and (4.4) we obtain that

$$A_n^{(21)}(x) \le C_1 \left(\frac{j}{n}\right)^{2r-1/2} \sum_{\frac{\nu}{2}\le y_{k,n}\le\frac{x+y_{n,n}}{2}} \frac{1}{1+|k-j|^2}.$$

If $x \approx y_{j,n} \le \frac{\nu}{4}$ and $y_{k,n} \ge \frac{\nu}{2}$ then $|k-j| \ge cn$ therefore in this case

$$A_n^{(21)}(x) \le C\frac{1}{j}\left(\frac{j}{n}\right)^{2r+1/2},$$

which is bounded (independently of j and n), if $r \ge -\frac{1}{4}$. Moreover, if $x \approx y_{j,n} \ge \ge \frac{\nu}{4}$ then $j \sim n$ hence $A_n^{(21)}$ is bounded for all r.

If $y_{k,n} \ge (x+y_{n,n})/2$ then $|y_{j,n}-y_{k,n}| \sim n$ thus by (4.3) and (4.14) we obtain that

$$\begin{aligned} A_n^{(22)}(x) &\le C_1 \frac{j^{2r-1/2}}{n^{2r+5/4}} \sum_{\frac{x+y_{n,n}}{2}\le y_{k,n}} \frac{\triangle y_{k,n}}{\sqrt[4]{|y_{k,n}-\nu|}} \le \\ &\le C_2 \frac{j^{2r-1/2}}{n^{2r+5/4}} \int_{\nu/2}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu-t}} dt \le \\ &\le C_3 \frac{j^{2r-1/2}}{n^{2r+5/4}} n^{3/4} = C_3 \left(\frac{j}{n}\right)^{2r+1/2} \frac{1}{j}, \end{aligned}$$

and this is bounded, if $r \ge -\frac{1}{4}$. Consequently

$$A_n^{(2)}(x) \le C \quad \left(x \in \left[a_n, \tfrac{\nu}{2}\right],\ n \in \mathbb{N}\right), \quad \text{if } -\tfrac{1}{4} \le r. \tag{4.15}$$

By (4.13)–(4.15) we get: there exists a constant $C>0$ independent of x and n such that

$$A_n^{(1)}(x)+A_n^{(2)}(x)\le C \quad \left(x\in\left[a_n,\tfrac{\nu}{2}\right],\ n\in\mathbb{N}\right) \quad \text{if } -\tfrac{1}{4}<r\le\tfrac{5}{4}. \tag{4.16}$$

Case 2: Let $x\in[\frac{1}{2}\nu,\frac{3}{4}\nu]$ and

$$\begin{aligned}\sum_{k=1}^{n}G_n(x,y_{k,n})\left(\frac{x}{y_{k,n}}\right)^r\sqrt{\frac{y_{k,n}}{4n-y_{k,n}}}&=\\ =\sum_{y_{k,n}\le\frac{\nu}{4}}\dots+\sum_{\frac{\nu}{4}<y_{k,n}\le\frac{7}{8}\nu}\dots+\sum_{\frac{7}{8}\nu<y_{k,n}}\dots&=:\\ :=B_n^{(1)}(x)+B_n^{(2)}(x)+B_n^{(3)}(x).&\end{aligned} \tag{4.17}$$

If $x\in[\frac{\nu}{2},\frac{3}{4}\nu]$ and $y_{k,n}\le\frac{\nu}{4}$ then $|x-y_{k,n}|\sim n$ therefore by (4.7) and (4.9) we get

$$B_n^{(1)}(x)\le C_1\sum_{y_{k,n}\le\frac{\nu}{4}}\frac{1}{n}\frac{nn^2}{n+n^2n}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{k}}\left(\frac{n}{k}\right)^{2r}\frac{k}{n}\le$$

$$\le C_2\sum_{k=1}^{n}\frac{n^{2r-5/2}}{k^{2r-1/2}}\sim\begin{cases}\dfrac{\log n}{n}, & \text{if } r=\dfrac{3}{4}\\ n^{2r-5/2}, & \text{if } r>\dfrac{3}{4}\\ \dfrac{1}{n}, & \text{if } r<\dfrac{3}{4}\end{cases}$$

and this is bounded, if $r\le\frac{5}{4}$.

If $x\in[\frac{\nu}{2},\frac{3}{4}\nu]$ and $\frac{\nu}{4}\le y_{k,n}\le\frac{7}{8}\nu$ then

$$|x-y_{k,n}|\ge c_1\frac{|j^2-k^2|}{n}\ge c_2|j-k|$$

(see (4.4)) thus by (4.7) and (4.9) we have

$$B_n^{(2)}(x)\le C_1\sum_{\frac{\nu}{4}\le y_{k,n}\le\frac{7}{8}\nu}\frac{1}{n}\frac{nn^2}{n+|j-k|^2n}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\le C_2\sum_{k=1}^{n}\frac{1}{1+|j-k|^2},$$

i.e. this term is bounded for all r.

If $x \in [\frac{1}{2}\nu, \frac{3}{4}\nu]$ and $y_{k,n} \geq \frac{7}{8}\nu$ then $|x - y_{k,n}| \geq cn$ thus

$$B_n^{(3)}(x) \leq C_1 \sum_{\frac{7}{8}\nu \leq y_{k,n}} \frac{1}{n} \frac{nn^2}{n+n^2n} \frac{1}{\sqrt{n}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \leq$$

$$\leq \frac{C_2}{n^{7/4}} \int_{\frac{7}{8}\nu}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu - t}} dt \leq C_3 \frac{n^{3/4}}{n^{7/4}}$$

which means that this term is also bounded for all r.

Consequently there exists a constant $C > 0$ independent of x and n such that

$$B_n^{(1)}(x) + B_n^{(2)}(x) + B_n^{(3)}(x) \leq C \quad \left(x \in \left[\tfrac{1}{2}\nu, \tfrac{3}{4}\nu\right],\ n \in \mathbb{N}\right), \quad \text{if } r \leq \tfrac{5}{4}. \tag{4.18}$$

Case 3: Let $x \in [\frac{3}{4}\nu, y_{n,n}]$ and

$$\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} =$$

$$= \sum_{y_{k,n} \leq \frac{5\nu}{8}} \dots + \sum_{\frac{5\nu}{8} < y_{k,n} < y_{j-1,n}} \dots + \sum_{k=j-1}^{j+1} \dots + \sum_{y_{j+1,n} < y_{k,n} < \frac{x+y_{n,n}}{2}} \dots + \tag{4.19}$$

$$+ \sum_{\frac{x+y_{n,n}}{2} \leq y_{k,n}} \dots =:$$

$$=: D_n^{(1)}(x) + D_n^{(2)}(x) + D_n^{(3)}(x) + D_n^{(4)}(x) + D_n^{(5)}(x).$$

If $y_{k,n} \leq \frac{5}{8}\nu$ then $|x - y_{k,n}| \sim n$ and $|y_{j,n} - \nu| \geq c\sqrt[3]{n}$ therefore by (4.7) and (4.9) we get

$$D_n^{(1)}(x) \leq C_1 \sum_{y_{k,n} \leq \frac{5\nu}{8}} \frac{1}{n} \frac{nn^2}{n+n^2n} \frac{1}{\sqrt[4]{n|y_{j,n} - \nu|}} \frac{1}{\sqrt{k}} \left(\frac{n}{k}\right)^{2r} \frac{k}{n} \leq$$

$$\leq C_2 \sum_{k=1}^{n} \frac{n^{2r-7/3}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log n}{n^{5/6}}, & \text{if } r = \dfrac{3}{4} \\ n^{2r-7/3}, & \text{if } r > \dfrac{3}{4} \\ n^{-5/6}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

and this is bounded, if $r \leq \frac{7}{6}$.

If $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $\frac{5}{8}\nu \le y_{k,n} < y_{j-1,n}$ then

$$\nu^{1/3} + |x-\nu| + |y_{k,n}-\nu| \sim |y_{k,n}-\nu| \ge c|y_{j,n}-\nu|,$$

$$|y_{k,n}-\nu| \le cn,$$

$$x - y_{j-2,n} \ge \triangle y_{j-1,n} \sim \triangle y_{j,n} \sim \sqrt{\frac{n}{\nu - y_{j,n}}}$$

thus

$$D_n^{(2)}(x) \le C_1 \sum_{\frac{5\nu}{8} \le y_{k,n} < y_{j-1,n}} \frac{1}{n} \frac{n|y_{k,n}-\nu|^2}{n + (x-y_{k,n})^2 |y_{k,n}-\nu|} \times$$

$$\times \frac{1}{\sqrt[4]{n|y_{j,n}-\nu|}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} \le$$

$$\le C_2 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \sum_{\frac{5\nu}{8} \le y_{k,n} < y_{j-1,n}} \frac{\triangle y_{k,n}}{(x-y_{k,n})^2} \le$$

$$\le C_3 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \int_{\frac{5\nu}{8}}^{y_{j-2,n}} \frac{1}{(x-t)^2} dt \le C_3 \sqrt{\frac{n}{|y_{j,n}-\nu|}} \frac{1}{x - y_{j-2,n}} \le C_4,$$

which holds for all r.

Let us consider $D_n^{(3)}(x)$. Using that $\nu^{1/3} + |x-\nu| + |y_{j,n}-\nu| \sim |y_{j,n}-\nu|$ we get

$$\frac{1}{n} \frac{n|y_{j,n}-\nu|^2}{n + (x-y_{k,n})^2 |y_{j,n}-\nu|} \frac{1}{\sqrt{n|y_{j,n}-\nu|}} \sqrt{\frac{y_{j,n}}{4n - y_{j,n}}} \le$$
$$\le C_1 \frac{|y_{j,n}-\nu|^2}{n^{3/2}} \frac{\sqrt{n}}{|y_{j,n}-\nu|} \le C_2$$

hence $D_n^{(3)}(x)$ is bounded for all r.

If $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $y_{j+1,n} < y_{k,n} < \frac{x+y_{n,n}}{2}$ then

$$|y_{j,n} - y_{k,n}| \ge c_1 \frac{|j^2-k^2|}{n} \ge c_2|j-k|, \quad |x-\nu| \sim |y_{k,n}-\nu| \sim |y_{j,n}-\nu|$$

thus

$$D_n^{(4)}(x) \leq$$

$$\leq C_1 \sum_{y_{j+1,n} < y_{k,n} \leq \frac{x+y_{n,n}}{2}} \frac{1}{n} \frac{n|x-\nu|^2}{n+(y_{j,n}-y_{k,n})^2|x-\nu|} \frac{1}{\sqrt{n|x-\nu|}} \sqrt{\frac{n}{|x-\nu|}} \leq$$

$$\leq C_2 \sum_{k=j+1}^{n} \frac{1}{(j-k)^2} \leq C_3,$$

which holds for all r.

Finally let $x \in [\frac{3}{4}\nu, y_{n,n}]$ and $\frac{x+y_{n,n}}{2} \leq y_{k,n}$. Then

$$\nu^{1/3} + |x-\nu| + |y_{k,n}-\nu| \sim |x-\nu|, |x-y_{k,n}| \geq \frac{|x-\nu|}{2}, |y_{j,n}-\nu| \geq c\sqrt[3]{n}.$$

Thus

$$D_n^{(5)}(x) \leq C_1 \sum_{\frac{x+y_{n,n}}{2} \leq y_{k,n}} \frac{1}{n} \frac{n|x-\nu|^2}{n+(x-y_{k,n})^2|x-\nu|} \times$$

$$\times \frac{1}{\sqrt[4]{n|y_{j,n}-\nu|}} \frac{1}{\sqrt[4]{n|y_{k,n}-\nu|}} \sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} \leq$$

$$\leq C_2 \frac{1}{\sqrt{n}|y_{j,n}-\nu|^{5/4}} \sum_{\frac{x+y_{n,n}}{2} \leq y_{k,n}} \frac{\triangle y_{k,n}}{\sqrt[4]{\nu - y_{k,n}}} \leq$$

$$\leq \frac{C_3}{n^{11/12}} \int_{\frac{x+y_{n,n}}{2}}^{y_{n,n}} \frac{1}{\sqrt[4]{\nu-t}} dt \leq C_4 \frac{n^{3/4}}{n^{11/12}} \leq C_5$$

for all r.

Consequently there exists a constant $C > 0$ independent of x and n such that

$$\sum_{k=1}^{5} D_n^{(k)}(x) \leq C \quad \left(x \in \left[\tfrac{3}{4}\nu, y_{n,n}\right],\ n \in \mathbb{N}\right), \quad \text{if } r \leq \tfrac{7}{6}. \tag{4.20}$$

Case 4: Let $y_{n,n} \leq x \leq b_n(\gamma) \leq \nu + \sqrt[3]{\nu}$ and

$$\begin{aligned} &\sum_{k=1}^{n} G_n(x, y_{k,n}) \left(\frac{x}{y_{k,n}}\right)^r \sqrt{\frac{y_{k,n}}{4n-y_{k,n}}} = \\ &= \sum_{y_{k,n} \leq \frac{\nu}{2}} \ldots + \sum_{\frac{\nu}{2} < y_{k,n}} \ldots =: E_n^{(1)}(x) + E_n^{(2)}(x). \end{aligned} \tag{4.21}$$

If $y_{k,n} \leq \frac{\nu}{2}$ then (4.2), (4.7) and (4.9) yields

$$E_n^{(1)}(x) \leq C_1 \sum_{y_{k,n} \leq \frac{\nu}{2}} \frac{1}{n} \frac{n \cdot n^2}{n + n^2 \cdot n} \frac{1}{\sqrt[3]{n}} \frac{1}{\sqrt{k}} \left(\frac{n}{k}\right)^{2r} \frac{k}{n} \leq$$

$$\leq C_2 \sum_{k=1}^{n} \frac{n^{2r-7/3}}{k^{2r-1/2}} \sim \begin{cases} \dfrac{\log n}{n^{5/6}}, & \text{if } r = \dfrac{3}{4} \\ n^{2r-7/3}, & \text{if } r > \dfrac{3}{4} \\ n^{-5/6}, & \text{if } r < \dfrac{3}{4} \end{cases}$$

which is bounded if $r \leq \frac{7}{6}$.

Now let $\frac{\nu}{2} \leq y_{k,n} < y_{n,n}$ and $x \in [y_{n,n}, \nu + \sqrt[3]{\nu}]$. Then

$$|x - y_{k,n}| \geq c|y_{k,n} - \nu|.$$

Indeed, this is obvious if $x \geq \nu$. Moreover if $x \in [y_{n,n}, \nu]$ then by (4.2) and (4.3) we have

$$|y_{k,n} - \nu| = |x - y_{k,n}| + |x - \nu| \leq |x - y_{k,n}| + c_1 \sqrt[3]{n} \leq$$

$$\leq |x - y_{k,n}| + c_2|x - y_{n-1,n}| \leq c_3|x - y_{k,n}|.$$

Therefore

$$E_n^{(2)}(x) \leq C_1 \sum_{\frac{\nu}{2} \leq y_{k,n} < y_{n,n}} \frac{1}{n} \frac{n|y_{k,n} - \nu|^2}{n + |x - y_{k,n}|^2 |y_{k,n} - \nu|} \frac{1}{\sqrt[3]{n}} \times$$

$$\times \frac{1}{\sqrt[4]{n|y_{k,n} - \nu|}} \sqrt{\frac{y_{k,n}}{4n - y_{k,n}}} + C_2 \frac{1}{n} \frac{n n^{2/3}}{n} \frac{1}{\sqrt[3]{n}} \frac{1}{\sqrt[4]{n n^{1/3}}} \sqrt[3]{n} \leq$$

$$\leq C_3 n^{-7/12} \sum_{\frac{\nu}{2} \leq y_{k,n}} \frac{\triangle y_{k,n}}{|y_{k,n} - \nu|^{5/4}} + C_4 \leq$$

$$\leq C_5 n^{-7/12} \int_{\nu/2}^{y_{n,n}} \frac{1}{(\nu - t)^{5/4}} dt + C_6 \leq C_7.$$

From the above relations it follows that there exists a constant $C > 0$ independent of x and n such that

$$E_n^{(1)}(x) + E_n^{(2)}(x) \leq C \quad \big(x \in [y_{n,n}, b_n],\ n \in \mathbb{N}\big), \quad \text{if } r \leq \tfrac{7}{6}. \tag{4.22}$$

Combining (4.12)–(4.22) we get (4.11) so Lemma 4.1 is proved. ■

4.3. Finishing the proof. For the proof of the Theorem we use the Banach–Steinhaus theorem.

Lemma 4.1 states that the sequence of the norm of operators $\mathcal{F}_n$ $(n \in \mathbb{N})$ is uniformly bounded.

Now we show that (3.2) holds for every polynomial. It is enough to prove that for all fixed $j = 0, 1, 2, \dots$

$$\lim_{n\to+\infty} \left\| \Big(p_j(w_\alpha) - \sigma_n(p_j(w_\alpha), U_n(w_\alpha), \cdot)\Big)\sqrt{w_\gamma}\right\|_\infty = 0. \tag{4.23}$$

Using the quadrature formula for $\{p_j := p_j(w_\alpha)\}$ (see [12, Section 3.1]) we have

$$c_{l,n}(p_j) = \sum_{k=1}^{n} p_j(y_{k,n})p_l(y_{k,n})\lambda_{k,n} = \delta_{l,j}$$
$$\big(l, j = 0, 1, 2, \dots, n-1,\ n \in \mathbb{N}\big).$$

Thus

$$p_j - \sigma_n\big(p_j, U_n(w_\alpha)\big) = \left(1 - \frac{j}{n}\right) p_j,$$

which proves (4.23).

Since the polynomials are dense in the Banach space $\big(C_{\sqrt{w_\gamma}}, \|\cdot\|_{\sqrt{w_\gamma}}\big)$ (see Section 2.2) thus the conditions of the Banach–Steinhaus theorem hold, so the Theorem is proved. ∎

References

[1] **De Bonis, M.C., G. Mastroianni and M. Viggiano,** K-functionals, moduli of smoothness and weighted best approximation on the semiaxis, in: *Functions, Series, Operators; Alexits Memorial Conference; Budapest, August 9–14, 1999,* (eds.: L. Leindler et al), János Bolyai Mathematical Society, Budapest, 2002, pp. 181–211.

[2] **Erdős, P. and G. Halász,** On the arithmetic means of Lagrange interpolation, in: *Approximation Theory, Kecskemét (Hungary), 1990,* Colloq. Math. Soc. János Bolyai, **58**, North-Holland, Amsterdam, 1991, pp. 120–131.

[3] **Freud, G.,** On the convergence of a Lagrange interpolation process on infinite interval, *Mat. Lapok,* **18** (1967), 289–292 (in Hungarian).

[4] **Marcinkiewicz, J.,** Sur l'interpolation I, II, *Studia Math.,* **6** (1936), 1–17 and 67–81.

[5] **Mastroianni, G. and G.V. Milovanović,** *Interpolation Processes. Basic Theory and Applications,* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.

[6] **Mastroianni, G. and D. Occorsio,** Lagrange interpolation at Laguerre zeros in some weighted uniform spaces, *Acta Math. Hungar.,* **91(1–2)** (2001), 27–52.

[7] **Muckenhoupt B. and D.W. Webb,** Two-weight norm inequalities for Cesàro means of Laguerre expansions, *Trans. AMS,* **353** (2000), 1119–1149.

[8] **Névai, P.,** On Lagrange interpolation based on the roots of the Laguerre polynomials, *Mat. Lapok,* **22** (1971), 149–164 (in Hungarian).

[9] **Poiani, E.L.,** Mean Cesaro summability of Laguerre and Hermite series, *Trans. AMS,* **173** (1972), 1–31.

[10] **Stone, M.H.,** A generalized Weierstrass approximation theorem, *Studies in Math. Vol. 1.,* Studies in Modern Analysis, ed. R. C. Buck, The Math. Assoc. of Amer. (1962), 30–87.

[11] **Szabados, J.,** Weighted Lagrange and Hermite–Fejér interpolation on the real line, *J. of Inequal. and Appl.,* **1** (1997), 99–112.

[12] **Szegő, G.,** *Orthogonal Polynomials,* AMS Coll. Publ., Vol. 23, Providence, 1978.

[13] **Varma, A.K. and T.M. Mills,** On the summability of Lagrange interpolation, *J. Approx. Theory,* **9** (1973), 349–356.

[14] **Vértesi, P.,** On the Lebesgue function of weighted Lagrange interpolation I. (Freud-type weights), *Constr. Approx.,* **15** (1999), 355–367.

[15] **Vértesi, P.,** On the Lebesgue function of weighted Lagrange interpolation II, *J. Austral. Math. Soc.* (*Series A*)*,* **15** (1998), 145–162.

[16] **Vértesi, P.,** Classical (unweighted) and weighted interpolation, in: *A Panorama of Hungarian Mathematics in the Twentieth Century,* Bolyai Soc. Math. Stud., **14**, Springer, Berlin, 2006, 71–117.

[17] **Webb, D.W.,** Pointwise estimates of the moduli of Cesàro–Laguerre kernels, 2002 (manuscript).

L. Szili
Department of Numerical Analysis
Eötvös Loránd University
Pázmány P. sétány 1/C.
H-1117 Budapest, Hungary
`szili@ludens.elte.hu`