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FOURIER TRANSFORM FOR MEAN PERIODIC FUNCTIONS

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Dedicated to the 60th birthday of Professor Antal Járai

Abstract. Mean periodic functions are natural generalizations of periodic functions. There are different transforms - like Fourier transforms - defined for these types of functions. In this note we introduce some transforms and compare them with the usual Fourier transform.

1. Introduction

In this paper $\mathcal{C}(\mathbb{R})$ denotes the locally convex topological vector space of all continuous complex valued functions on the reals, equipped with the linear operations and the topology of uniform convergence on compact sets. Any closed translation invariant subspace of $\mathcal{C}(\mathbb{R})$ is called a *variety*. The smallest variety containing a given f in $\mathcal{C}(\mathbb{R})$ is called the *variety generated by* f and it is denoted by $\tau(f)$. If this is different from $\mathcal{C}(\mathbb{R})$, then f is called *mean periodic*. In other words, a function f in $\mathcal{C}(\mathbb{R})$ is mean periodic if and only if there exists a nonzero continuous linear functional μ on $\mathcal{C}(\mathbb{R})$ such that

$$f * \mu = 0 \tag{1}$$

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holds. In this case sometimes we say that f is *mean periodic with respect to* μ. As any continuous linear functional on $\mathcal{C}(\mathbb{R})$ can be identified with a compactly supported complex Borel measure on $\mathbb{R}$, equation (1) has the form

$$\int f(x-y)\,d\mu(y) = 0 \tag{2}$$

for each x in $\mathbb{R}$. The dual of $\mathcal{C}(\mathbb{R})$ will be denoted by $\mathcal{M}_C(\mathbb{R})$. As the convolution of two nonzero compactly supported complex Borel measures is a nonzero compactly supported Borel measure as well, all mean periodic functions form a linear subspace in $\mathcal{C}(\mathbb{R})$. We equip this space with the following topology. For each nonzero μ from the dual of $\mathcal{C}(\mathbb{R})$ let $V(\mu)$ denote the solution space of (1). Clearly, $V(\mu)$ is a variety and the set of all mean periodic functions is equal to the union of all these varieties. We equip this union with the inductive limit of the topologies of the varieties $V(\mu)$ for all nonzero μ from the dual of $\mathcal{C}(\mathbb{R})$. The locally convex topological vector space obtained in this way will be denoted by $\mathcal{MP}(\mathbb{R})$, the space of mean periodic functions.

An important class of mean periodic functions is formed by the exponential polynomials. We call a function of the form

$$\varphi(x) = p(x)\, e^{\lambda x} \tag{3}$$

an *exponential monomial*, where p is a complex polynomial and λ is a complex number. If $p \equiv 1$, then the corresponding exponential monomial $x \mapsto e^{\lambda x}$ is called an *exponential*. Exponential monomials of the form

$$\varphi_k(x) = x^k\, e^{\lambda x} \tag{4}$$

with some natural number k and complex number λ, are called *special exponential monomials*.

Linear combinations of exponential monomials are called *exponential polynomials*. To see that the special exponential monomial in (3) is mean periodic one considers the measure

$$\mu_k = (e^{\lambda}\,\delta_1 - \delta_0)^{k+1}\,, \tag{5}$$

where δ_y is the Dirac–measure concentrated at the number y for each real y, and the $k+1$-th power is meant in convolution-sense. It is easy to see that

$$\varphi_k * \mu_k = 0$$

holds. Sometimes we write 1 for δ_0.

Exponential polynomials are typical mean periodic functions in the sense that any mean periodic function f in $V(\mu)$ is the uniform limit on compact sets of a sequence of linear combinations of exponential monomials, which belong to $V(\mu)$, too. More precisely, the following theorem holds (see [9]).

Theorem 1 (L. Schwartz, 1947). *In any variety of $\mathcal{C}(\mathbb{R})$ the linear hull of all exponential monomials is dense.*

A similar theorem in $\mathcal{C}(R^n)$ fails to hold for $n \geq 2$ as it has been shown in [4] by D. I. Gurevich. Moreover, he gave examples for nonzero varieties in $\mathcal{C}(R^2)$ which do not contain nonzero exponential monomials at all. However, as it has been shown by L. Ehrenpreis in [1], Theorem 1 can be extended to varieties of the form $V(\mu)$ in $\mathcal{C}(R^n)$ for any positive integer n.

Another important result in [9] is the following (Théorème 7, on p. 881.):

Theorem 2. *In any proper variety of $\mathcal{C}(\mathbb{R})$ no special exponential monomial is contained in the closed linear hull of all other special exponential monomials in the variety.*

In other words, if a variety $V \neq \{0\}$ in $\mathcal{C}(\mathbb{R})$ is given, then for each special exponential monomial φ_0 in V there exists a measure μ in $\mathcal{M}_C(\mathbb{R})$ such that $\mu(\varphi_0) = 1$ and $\mu(\varphi) = 0$ for each special exponential monomial $\varphi \neq \varphi_0$ in V.

2. A mean operator for mean periodic functions

Based on Theorems 1 and 2 by L. Schwartz we introduced a mean operator on the space $\mathcal{MP}(\mathbb{R})$ in the following way (see also [10], pp. 64–65.).

For each x, y in $\mathbb{R}$ and f in $\mathcal{C}(\mathbb{R})$ let

$$\tau_y f(x) = f(x+y)\,,$$

and call $\tau_y f$ the *translate of f by y*. The continuous linear operator τ_y on $\mathcal{C}(\mathbb{R})$ is called *translation operator*. The operator τ_0 will be denoted by 1. Clearly, the continuous function f is a polynomial of degree at most k if and only if

$$(\tau_y - 1)^{k+1} f(x) = 0 \tag{6}$$

holds for each x, y in $\mathbb{R}$ and for $k = 0, 1, \ldots$. The set $\mathcal{P}(\mathbb{R})$ of all polynomials is a subspace of $\mathcal{MP}(\mathbb{R})$, which we equip with the topology inherited from $\mathcal{MP}(\mathbb{R})$.

Theorem 3. *The subspace $\mathcal{P}(\mathbb{R})$ is closed in $\mathcal{MP}(\mathbb{R})$.*

Proof. First we show that the set of the degrees of all polynomials in any proper variety is bounded from above. By the Taylor–formula it follows that the derivative of a polynomial is a linear combination of its translates, hence if a polynomial belongs to a variety then all of its derivatives belong to the same variety, too. Therefore, if the set of the degrees of all polynomials in a proper variety is not bounded from above, then all polynomials belong to this variety. But, in this case, by the Stone–Weierstrass–theorem, all continuous functions must belong to the variety, hence it cannot be proper.

Suppose now that $(p_i)_{i\in I}$ is a net of polynomials which converges in $\mathcal{MP}(\mathbb{R})$ to the continuous function f. By the definition of the inductive limit topology there exists a nonzero μ in $\mathcal{M}_c(\mathbb{R})$ such that p_i belongs to $V(\mu)$ for each i in I. By our previous consideration, for the degrees we have $\deg p_i \le k$ for some positive integer k. By (6) this means that

$$(\tau_y - 1)^{k+1} p_i(x) = 0$$

holds for each x, y in $\mathbb{R}$. This implies that the same holds for f, hence f is a polynomial of degree at most k, too. The theorem is proved. ■

Theorem 4. *There exists a unique continuous linear operator*

$$M : \mathcal{MP}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

satisfying the properties

1) $M(\tau_y f) = \tau_y M(f)$,
2) $M(p) = p$

for each f in $\mathcal{MP}(\mathbb{R})$, p in $\mathcal{P}(\mathbb{R})$ and y in $\mathbb{R}$.

Proof. First we prove uniqueness. By Theorem 1, it is enough to show that the properties of M determine M on the set of all special exponential monomials. Let $m \ne 1$ be any nonzero continuous complex exponential. Then we have

$$M(m) = M\big[m(-y)\tau_y m\big] = m(-y)M(\tau_y m) = m(-y)\tau_y M(m),$$

which implies that either $M(m) = 0$ or m is a polynomial. Hence $M(m) = 0$. Suppose that we have proved for $j = 0, 1, \ldots, k-1$ that

$$M\big[x^j m(x)\big] = 0$$

for any continuous complex exponential $m \ne 1$. Then we have

$$M\big[(x+y)^k m(x+y)\big] = M\Big[\sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} m(x)m(y)\Big] =$$

$$= \sum_{j=0}^{k} \binom{k}{j} y^{k-j} m(y) M\big[x^j m(x)\big] = m(y) M\big[x^k m(x)\big],$$

which implies, as above, that $M\big[x^k m(x)\big] = 0$. This proves the uniqueness.

In order to prove existence, first we notice, that, by Theorem 2, for any nonzero μ in $\mathcal{M}_c(\mathbb{R})$, the exponential 1 is not contained in the closed linear subspace of $\mathcal{C}(\mathbb{R})$ spanned by all special exponential monomials in $V(\mu)$ different from 1. This implies the existence of a measure μ_0 in $\mathcal{M}_c(\mathbb{R})$ such that $\mu_0(1) = 1$, further $\mu_0(\varphi) = 0$ for any special exponential monomial $\varphi \neq 1$ in $V(\mu)$.

From this fact it follows, that $x^k m(x) * \mu_0 = 0$ for each positive integer k and exponential $m \neq 1$ in $V(\mu)$, further $x^k * \mu_0 = x^k$. This shows, that $\varphi * \mu_0$ is a polynomial in $V(\mu)$ for any exponential polynomial φ in $V(\mu)$. On the other hand, as in the proof of Theorem 3, it follows that if a polynomial of degree n belongs to $V(\mu)$, then all the functions $1, x, x^2, \ldots, x^n$ also belong to $V(\mu)$. Hence, all polynomials in $V(\mu)$ have a degree smaller than some fixed positive integer. Now, if f is arbitrary in $V(\mu)$, then by Theorem 1, there exist exponential polynomials φ_i in $V(\mu)$ such that $f = \lim \varphi_i$. Then we have $f * \mu_0 = \lim(\varphi_i * \mu_0)$, hence also $f * \mu_0$ is a polynomial.

Suppose now, that f belongs also to some $V(\nu)$ with some nonzero ν. Then $f * \mu_0$ also belongs to $V(\nu)$, and it is a polynomial. Hence we have $f * \mu_0 = f * \mu_0 * \nu_0$. Similarly, $f * \nu_0 = f * \nu_0 * \mu_0$. Hence $f * \mu_0$ does not depend on the special choice of μ_0. On the other hand, each f in $\mathcal{MP}(\mathbb{R})$ is contained in some $V(\mu)$ with $\mu \neq 0$, and we can define

$$M(f) = f * \mu_0$$

with any μ_0 in $\mathcal{M}_c(\mathbb{R})$ satisfying the previous properties. The continuity and linearity of M follows from the definition, 1) follows from the properties of convolution, and 2) has been proved. ■

3. The Fourier transform

For each f in $\mathcal{C}(\mathbb{R})$ we define $\check{f}$ by the formula

$$\check{f}(x) = f(-x) \tag{7}$$

for any x in $\mathbb{R}$. It is obvious, that $f\check{m}$ is mean periodic for any f in $\mathcal{MP}(\mathbb{R})$ and for any continuous complex exponential m. Hence we may define $\hat{f}$ as follows:

$$\hat{f}(m) = M(f\check{m}) \tag{8}$$

for any nonzero continuous exponential m.

Theorem 5. *The map $f \mapsto \hat{f}$ defined above is linear and has the following properties:*

1) $\hat{p}(m) = 0$ *for* $m \neq 1$ *and* $\hat{p}(1) = p$,
2) $(p\,f)\hat{}\,(m) = p\hat{f}(m)$,
3) $(\tau_y f)\hat{}\,(m) = m(y)\tau_y\big(\hat{f}(m)\big)$,
4) $(\check{f})\hat{}\,(m) = \big[\hat{f}(\check{m})\big]\check{}$

for any f in $\mathcal{MP}(\mathbb{R})$, for any p in $\mathcal{P}(\mathbb{R})$ and for each y in $\mathbb{R}$, whenever $p\,f$ is mean periodic.

Proof. In the proof of Theorem 4 we have seen that $M(p\,m) = 0$ for each polynomial p and exponential $m \neq 1$. This means that if the exponential polynomial φ has the form

$$\varphi(x) = p_0(x) + \sum_{i=1}^{k} p_i(x)\, m_i(x) \tag{9}$$

for each real x, where k is a nonnegative integer, $p_0, p_1, \ldots, p_k$ are polynomials and $m_1, m_2, \ldots, m_k$ are different exponentials, then we have

$$M(\varphi) = p_0\,. \tag{10}$$

Clearly, this implies 1) – 4) for any exponential polynomial $f = \varphi$. Then, by Theorem 1, our statements follow for any mean periodic f. ■

Theorem 6 ("Uniqueness Theorem"). *For any f in $\mathcal{MP}(\mathbb{R})$, if $\hat{f} = 0$, then $f = 0$.*

Proof. From the previous theorem it follows by linearity and continuity, that $\hat{\varphi} = 0$ for all φ in $\tau(f)$. In particular, $\hat{\varphi} = 0$ for any exponential polynomial φ in $\tau(f)$, hence, by (9), we have that the only exponential polynomial in $\tau(f)$ is 0. Now our statement is a consequence of Theorem 1. ■

As the exponentials of the additive group of $\mathbb{R}$ can be identified with complex numbers, there is a one to one mapping between $\mathbb{C}$ and the set of all exponentials. Hence, instead of $\hat{f}(m)$ we can write $\hat{f}(\lambda)$, where λ is the unique complex number corresponding to the exponential m. By Theorem 5 the Fourier transform of the mean periodic function f is a polynomial-valued mapping $\hat{f}$, which is defined on $\mathbb{C}$, the set of complex numbers, having the properties listed in 5. On the other hand, the Fourier transformation $f \mapsto \hat{f}$ is an injective, linear mapping of $\mathcal{MP}(\mathbb{R})$ into the set of all polynomial-valued mappings of $\mathbb{C}$ into $\mathcal{P}(\mathbb{R})$, having the properties listed in 5. If f is a bounded mean periodic function, then $\tau(f)$ consists of bounded functions, in particular, each exponential is a character and each polynomial in $\tau(f)$ is constant. Hence, in this case $M(f)$ is a constant, and $\hat{f}(m)$ is constant, for each character m of $\mathbb{R}$. In particular, using the results in [8] we have the following theorem.

Theorem 7. *For any almost periodic f in $\mathcal{MP}(\mathbb{R})$, the function $\hat{f}$ coincides with the Fourier transform of f as an almost periodic function in the sense of Bohr.*

For exponential polynomials we have the following immediate "Inversion Theorem".

Theorem 8. *Let f be an exponential polynomial. Then*

$$f(x) = \sum_{\lambda \in \mathbb{C}} \hat{f}(\lambda)(x)\, e^{\lambda x} \tag{11}$$

holds for each x in $\mathbb{R}$.

For any mean periodic f we call the *spectrum of f* the set $sp(f)$ of all complex numbers λ for which the exponential $x \mapsto e^{\lambda x}$ belongs to the variety $\tau(f)$ generated by f. The following theorem is easy to prove.

Theorem 9. *A mean periodic function is a polynomial if and only if its spectrum is $\{0\}$. It is an exponential monomial if and only if its spectrum is a singleton and it is an exponential polynomial if and only if its spectrum is finite.*

4. The Carleman transform

As we have seen in the previous section it is possible to introduce a Fourier–like transform for mean periodic functions on $\mathbb{R}$ which enjoys several useful properties similar to the classical Fourier transform. However, this transform yields a polynomial-valued function, hence the role of classical Fourier coefficients are played by polynomials. The existence of this transform depends on the mean operator, which is a kind of mean value, but it takes polynomials as values, instead of numbers. The most important property of this mean — besides linearity and continuity — is that it commutes with translations: instead of translation invariance we have translation covariance, which — obviously — reduces to translation invariance in case of constant functions. The Fourier transform, based on this mean operator, can be realized in case of exponential polynomials as follows: if the exponential polynomial φ has the canonical representation (9) for each real x, where k is a nonnegative integer, $p_0, p_1, \dots, p_k$ are polynomials and $m_1, m_2, \dots, m_k$ are different exponentials, then the mean operator M takes the value p_0 on φ, and, more generally, the Fourier transform of φ at λ is the polynomial p_λ, which is the coefficient of the exponential $x \mapsto e^{\lambda x}$ in the canonical representation of φ. As spectral analysis and spectral synthesis hold in $\mathbb{R}$ by [9], heuristically, the support of $\hat{f}$ consists of those λ's which take part in the spectral analysis of f in the sense, that the corresponding exponentials $x \mapsto e^{\lambda x}$ belong to the spectrum of f, and the value $\hat{f}(\lambda) = M\big[f(x) \cdot e^{-\lambda x}\big]$, which is a polynomial, shows, to what content this exponential takes part in the reconstruction process of f from its spectrum: in the spectral synthesis of f.

As the existence of the Fourier transform introduced above is a result of a transfinite procedure, depending on Hahn–Banach-theorem, it is not clear how to determine the value of $\hat{f}$ at some complex number λ, how to compute it, if a general mean periodic function f is given, which is not necessarily an exponential polynomial. In other words, it is not clear how to compute the coefficients of the polynomial $\hat{f}(\lambda)$ for a general mean periodic function f. On the other hand, an "Inversion Theorem"-like result would be highly welcome, for which, as usual, different estimates on the "Fourier–like coefficients" were necessary.

In his fundamental work [6] (see also [5]) J. P. Kahane used another transform based on the Carleman transform (see [3]). Here we present the details.

Let f be a mean periodic function in $\mathcal{MP}(\mathbb{R})$ and we put

$$f^-(x) = \begin{cases} 0, & x \geq 0 \\ f(x), & x < 0\,. \end{cases} \tag{12}$$

As f is mean periodic, there exists a nonzero compactly supported Borel measure in $\mathcal{M}_c(\mathbb{R})$ such that

$$f * \mu = 0 \tag{13}$$

holds. Denote μ any of such measures and we put

$$g = f^- * \mu\,. \tag{14}$$

It is easy to see, that the support of g is compact (see [6], Lemma on p. 20). The *Carleman transform* of f is defined as

$$\mathcal{C}(f)(w) = \frac{\hat{g}(w)}{\hat{\mu}(w)} \tag{15}$$

for each w in $\mathbb{C}$ which is not a zero of $\hat{\mu}$. By the Paley–Wiener-theorem (see e.g. [11]) $\hat{g}$ and $\hat{\mu}$ are entire functions of exponential type, hence $\mathcal{C}(f)$ is meromorphic. Originally Carleman in [3] introduced this transform for functions which are not very rapidly increasing at infinity, but Kahane observed that it works also for mean periodic functions.

We present a simple example for the computation of this transform. Let

$$f(x) = x$$

for each x in $\mathbb{R}$. Then f is mean periodic and $\tau(f)$ is annihilated by the measure

$$\mu = (\delta_1 - 1)^2 .$$

The Fourier transform of μ is as follows:

$$\hat{\mu}(w) = \int e^{-iwx}\, d\mu(x) = \int e^{-iwx}\, d(\delta_1 - 1)^2(x) = \int e^{-iwx}\, d(\delta_2 - 2\delta_1 + 1)(x) =$$
$$= e^{-2iw} - 2e^{-iw} + 1 = (e^{-iw} - 1)^2$$

for each w in $\mathbb{C}$. The next step is to form the function f^- (see (12)). Hence, we have, by (14)

$$g(x) = \big(f^- * \mu\big)(x) = \int f^-(x-y)\, d\mu(y) = \int f^-(x-y)\, d(\delta_2 - 2\delta_1 + 1)(y) =$$
$$= f^-(x-2) + 2f^-(x-1) + f^-(x) =$$
$$= \begin{cases} 0, & x \geq 2 \\ x-2, & 2 > x \geq 1 \\ -x, & 1 > x \geq 0 \\ 0, & 0 > x . \end{cases}$$

The Fourier transform of g is

$$\hat{g}(w) = \int g(x) e^{-iwx}\, dx = \int_0^2 g(x) e^{-iwx}\, dx =$$
$$= \int_1^2 (x-2) e^{-iwx}\, dx - \int_0^1 x e^{-iwx}\, dx =$$
$$= -\frac{1}{iw} e^{-iw} - \frac{1}{(iw)^2}\Big(e^{-2iw} - e^{-iw}\Big) + \frac{1}{iw} e^{-iw} + \frac{1}{(iw)^2}\Big(e^{-iw} - 1\Big) =$$
$$= -\frac{1}{(iw)^2}(e^{-iw} - 1)^2 .$$

From this we have

$$\mathcal{C}(f)(w) = \frac{-\frac{1}{(iw)^2}(e^{-iw}-1)^2}{(e^{-iw}-1)^2} = -\frac{1}{(iw)^2}$$

for each w in $\mathbb{C}$ which is not a zero of $\hat{\mu}$.

At this moment one cannot see any relation between $\mathcal{C}(f)$ and $\hat{f}$. Consider another easy example. Let

$$f(x) = x^3\, e^{\lambda x}\,,$$

where x is real and λ is a complex number. In this case we can take

$$\mu = (e^{\lambda} - 1)^4\,,$$

and

$$\hat{\mu}(w) = \big(e^{-(iw-\lambda)} - 1\big)^4\,,$$

further

$$\hat{g}(w) = -\frac{3!}{(iw-\lambda)^4}\big(e^{-(iw-\lambda)} - 1\big)^4\,,$$

and finally

$$\mathcal{C}(f)(w) = \frac{\hat{g}(w)}{\hat{\mu}(w)} = -\frac{3!}{(iw-\lambda)^4}\,.$$

We shall see that there is an intimate relation between the Carleman transform and the Fourier transform of exponential monomials. First we need the following theorem.

Theorem 10. *For each x in $\mathbb{R}$ let*

$$f(x) = p(x)e^{\lambda x}\,, \tag{16}$$

where p is a polynomial and λ is a complex number. Then we have

$$\mathcal{C}(f)(w) = -\sum_{k=0}^{\infty}\frac{p^{(k)}(0)}{(iw-\lambda)^{k+1}}\,, \tag{17}$$

where the sum is actually finite.

Proof. Let

$$f_k(x) = x^k e^{\lambda x}$$

for each nonnegative integer k and complex number λ. Then f_k is mean periodic and $\tau(f)$ is annihilated by the finitely supported measure

$$\mu_k = (e^{\lambda}\delta_1 - 1)^{k+1}\,.$$

Indeed, we have for each x in $\mathbb{R}$

$$f_k * \mu_k(x) = \int f_k(x-y)\,d\mu(y) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} e^{\lambda j} (x-j)^k e^{\lambda x - \lambda j} =$$

$$= e^{\lambda x} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k = e^{\lambda x}\,(\tau_{-1} - 1)^{k+1} \varphi_k(x) = 0$$

by (6), where

$$\varphi_k(x) = x^k$$

for x in $\mathbb{R}$.

Let w be a complex number. For the sake of simplicity set

$$T = iw - \lambda\,.$$

The Fourier transform of μ_k at w in $\mathbb{C}$ is

$$\hat{\mu}_k(w) = \int e^{-iwx}\,d\mu_k(x) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} e^{\lambda j} e^{-iwj} = \left(e^{-T} - 1\right)^{k+1}.$$

As

$$f_k^-(x) = \begin{cases} 0, & x \geq 0 \\ f_k(x), & x < 0 \end{cases}$$

it follows for $l = 0, 1, \ldots, k$

$$g_k(x) = f_k^- * \mu_k(x) = \int f_k^-(x-y)\,d\mu_k(y) =$$

$$= \begin{cases} 0, & k+1 \leq x; \\ e^{\lambda x} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k, & l \leq x < l+1; \\ 0, & x < 0. \end{cases}$$

By definition, the Fourier transform of g_k at w in $\mathbb{C}$ is

$$\hat{g}_k(w) = \int e^{-iwx} g_k(x)\,dx = \sum_{l=0}^{k} \int_l^{l+1} e^{-iwx} g_k(x)\,dx =$$

$$= \sum_{l=0}^{k} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int_l^{l+1} (x-j)^k e^{-Tx}\,dx\,.$$

Using the fact, like above, that

$$\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k = 0\,,$$

we have

$$\hat{g}_k(w) = \sum_{l=0}^{k} \sum_{j=l+1}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= \sum_{l=0}^{k} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx -$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= \sum_{l=0}^{k} \int\limits_{l}^{l+1} \left[\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (x-j)^k \right] e^{-Tx}\, dx -$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$- \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= (-1)^k \sum_{j=0}^{k} \binom{k+1}{j} (-1)^j \sum_{l=j}^{k} \int\limits_{l}^{l+1} (x-j)^k e^{-Tx}\, dx =$$

$$= (-1)^k \sum_{j=0}^{k} \binom{k+1}{j} (-1)^j \int\limits_{j}^{k+1} (x-j)^k e^{-Tx}\, dx =$$

$$= - \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \int\limits_{j}^{k+1} (x-j)^k e^{-Tx}\, dx\,.$$

Integration by parts yields

$$\int_j^{k+1} (x-j)^k e^{-Tx}\,dx = \left[\frac{(x-j)^k e^{-Tx}}{-T}\right]_j^{k+1} + \frac{k}{T}\int_j^{k+1} (x-j)^{k-1} e^{-Tx}\,dx =$$

$$= \frac{(k+1-j)^k e^{-(k+1)T}}{-T} + \frac{k}{T}\int_j^{k+1} (x-j)^{k-1} e^{-Tx}\,dx\,,$$

for $k \geq 1$. Continuing this process we arrive at

$$\hat{g}(w) = \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{(k+1-j)^{k-i} e^{-(k+1)T}}{T^{i+1}} -$$

$$- \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \frac{k!}{T^{k+1}} e^{-jT} =$$

$$= \sum_{i=0}^{k} \frac{k!}{(k-i)!} \frac{1}{T^{i+1}} e^{-(k+1)T} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (k+1-j)^{k-i} -$$

$$- \frac{k!}{T^{k+1}} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} \left(e^{-T}\right)^j = -\frac{k!}{T^{k+1}} \left(e^{-T} - 1\right)^{k+1}.$$

Here we used again, that by (6)

$$\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} (k+1-j)^{k-i} = 0\,.$$

Returning to the original notation we have that

$$\mathcal{C}(f_k)(w) = -\frac{k!}{(iw-\lambda)^{k+1}}\,, \tag{18}$$

and this implies our statement. The theorem is proved. ∎

5. Relation between the Carleman transform and the Fourier transform

Using the initial of the name of Kahane here we introduce the K-mean of a mean periodic function f. In [6] it is proved that for a complex number λ the exponential monomial $x \mapsto p(x)e^{\lambda x}$ belongs to $\tau(f)$ if and only if λ is a pole of order at least n of $\mathcal{C}(f)$, where n is the degree of the polynomial p. As $\mathcal{C}(f)$ is meromorphic, each pole of it is of finite order. Consider the case $\lambda = 0$. If 0 is not a pole of $\mathcal{C}(f)$, then no nonzero polynomial belongs to $\tau(f)$. In particular, the function 1 does not belong to $\tau(f)$. In this case let $\mathcal{K}(f) = 0$, the zero polynomial. Suppose now that 0 is a pole of $\mathcal{C}(f)$. Let $n \geq 1$ denote the order of this pole, and define the polynomial $\mathcal{K}(f)$ of degree $n-1$ as follows: for each real x let

$$\mathcal{K}(f)(x) = -\sum_{k=0}^{n-1} \frac{i^{k+1}\, c_{k+1}}{k!}\, x^k \,, \tag{19}$$

where c_k denotes the coefficient of w^{-k} in the polar part of the Laurent series expansion of $\mathcal{C}(f)$ around $w = 0$ $(k = 0, 1, \ldots, n-1)$.

By Theorem 10. we have the following basic result.

Theorem 11. *For each polynomial p we have*

$$\mathcal{K}(p) = p \,. \tag{20}$$

Proof. Formula (18) gives the result with $\lambda = 0$ for the polynomial $x \mapsto x^k$ for each natural number k. The general case follows by linearity. ∎

Using again equation (18) and linearity we have the extension of the previous theorem.

Theorem 12. *Let φ be an exponential polynomial of the form* (9). *Then we have*

$$\mathcal{K}(\varphi) = p_0 \,. \tag{21}$$

Another basic property of the K-transform is expressed by the following theorem.

Theorem 13. *The K-transformation is a continuous linear mapping from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$, which commutes with all translations.*

Proof. By the definition of $\mathcal{C}(f)$ the K-transformation is clearly linear.

For the proof of continuity we remark that the mapping $f \mapsto f^-$ and hence also $f \mapsto g$ and $f \mapsto \mathcal{C}(f)$ are continuous on $\mathcal{MP}(\mathbb{R})$. Finally, the coefficients c_k of the Laurent expansion of $\mathcal{C}(f)$ can be expressed — by Cauchy's integral formulas — by path integrals which can be interchanged with taking uniform limits over compact sets. Hence the K-transformation is continuous from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$.

Let φ be an exponential polynomial of the form (9) and y be real number. Then, by Theorems 11. and 12., we have

$$\tau_y\, K(\varphi)(x) = K(\varphi)(x+y) = p_0(x+y) = K(\tau_y\varphi)(x)$$

for each real x. Hence the K-transformation commutes with all translations on the exponential polynomials. By the spectral synthesis result Theorem 1, exponential polynomials form a dense subset in $\tau(f)$ for each mean periodic f, hence, by continuity, the theorem is proved. ■

Our main theorem follows.

Theorem 14. *For each mean periodic function f we have*

$$\mathcal{K}(f) = M(f)\,. \tag{22}$$

Proof. In [10] we have shown (see Theorem 4.2.5 on p. 64) that linearity and continuity together with the property of commuting with translations and leaving polynomials fixed characterize the operator M among the mappings from $\mathcal{MP}(\mathbb{R})$ into $\mathcal{P}(\mathbb{R})$. As we have seen in the previous theorems the operator $\mathcal{K}$ shares these properties with M, hence they are identical. ■

6. Fourier series and convergence

In (11) we have seen that if f is an exponential polynomial, then we have the representation

$$f(x) = \sum_{\lambda\in\mathbb{C}} \hat{f}(\lambda)(x)\, e^{\lambda x}\,. \tag{23}$$

This is a finite sum because $\hat{f}(\lambda) = 0$ if λ does not belong to the spectrum of f, and the spectrum is finite. The question arises: if f is an arbitrary mean periodic function, does a similar - not necessarily finite - sum converge to f in some sense? The answer is clearly negative even in the case of periodic functions but still we can get a kind of convergence in a special class of measures.

The measure (or compactly supported distribution) μ is called *slowly decreasing* if there are constants $A, B, \varepsilon > 0$ such that

$$\max\{|\hat{\mu}(y)| : \, y\in\mathbb{R},\ |x-y| \le A\ln(2+|x|)\} \ge \varepsilon(1+|x|)^{-B}\,.$$

For instance, if $\hat{\mu}$ is a nonzero exponential polynomial, then μ is slowly decreasing.

We shall formulate a convergence theorem for another class of mean periodic functions, namely for C^∞-mean periodic functions. Let $\mathcal{E}(\mathbb{R})$ denote the space $C^\infty(\mathbb{R})$ with the usual topology of uniform convergence of all derivatives over compact subsets. This is a locally convex topological vector space and its dual is the space of all compactly supported distributions. If μ is a compactly supported distribution and f is in $\mathcal{E}(\mathbb{R})$ satisfying

$$f * \mu = 0\,,$$

then f is called *mean periodic with respect to* μ, or simply *mean periodic*. Now we can formulate a convergence theorem for Fourier series.

Theorem 15 (L. Ehrenpreis, 1960). *Let μ be a slowly decreasing compactly supported distribution and let f be a mean periodic function with respect to μ in $\mathcal{E}(\mathbb{R})$. Then there are finite disjoint subsets V_k $(k = 1, 2, \dots)$ of $sp(f)$ such that $\bigcup_k V_k = sp(f)$ and the series*

$$\sum_{k=1}^{\infty} \sum_{\lambda \in V_k} \hat{f}(\lambda)(x)\, e^{\lambda x}$$

converges to f in $\mathcal{E}(\mathbb{R})$.

We note that continuous mean periodic functions can be approximated very well by mean periodic functions in $\mathcal{E}(\mathbb{R})$. Indeed, let

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon}\chi\Big(\frac{x}{\varepsilon}\Big)\,,$$

where χ is a compactly supported C^∞ function. Then $f_\varepsilon = \chi_\varepsilon * f$ tends to f in $\mathcal{E}(\mathbb{R})$. Further f_ε satisfies the same equation as f:

$$f_\varepsilon * \mu = (\chi_\varepsilon * f) * \mu = \chi_\varepsilon * (f * \mu) = 0\,.$$

Hence the theory of continuous mean periodic functions can be reduced to the theory of C^∞-mean periodic functions.

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