Annales Univ. Sci. Budapest., Sect. Comp. 35 (2011) 223–238

ON THE SIMULTANEOUS NUMBER SYSTEMS OF GAUSSIAN INTEGERS

G. Nagy (Budapest, Hungary)

Dedicated to Professor Antal Járai on his 60th anniversary

Abstract. In this paper we show that there is no simultaneous number system of Gaussian integers with the canonical digit set. Furthermore we give the construction of a new digit set by which simultaneous number systems of Gaussian integers exist.

1. Introduction

K.-H. Indlekofer, I. Kátai and P. Racskó examined in [1], for what N_1, N_2 will $(-N_1, -N_2, \mathcal{A}_c)$ be a simultaneous number system, where $2 \leq N_1 < N_2$ are rational integers and $\mathcal{A}_c = \{0, 1, \ldots, |N_1||N_2| - 1\}$. The triple $(-N_1, -N_2, \mathcal{A}_c)$ is called a simultaneous number system if there exist $a_j \in \mathcal{A}_c$ $(j = 0, 1, \ldots, n)$ for all z_1, z_2 rational integers so that

$$z_1 = \sum_{j=0}^{n} a_j (-N_1)^j, \quad z_2 = \sum_{j=0}^{n} a_j (-N_2)^j.$$

In the first part of this article we examine the case of Gaussian integers with the canonical digit set (there exist no $Z_1, Z_2 \in \mathbb{Z}[i]$ for which $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system), and in the second part we give the construction

The Project is supported by the European Union and co-financed by the European Social Fund (grant agreement no. TAMOP 4.2.1./B-09/1/KMR-2010-0003).

https://doi.org/10.71352/ac.35.223

of a new digit set by which simultaneous number systems of Gaussian integers exist.

Let Z_1 and Z_2 be Gaussian integers and let $\mathcal{A}$ be a digit set. The triple $(Z_1, Z_2, \mathcal{A})$ is called a simultaneous number system if there exist $a_j \in \mathcal{A}$ ($j = 0, 1, \ldots, n$) for all $z_1, z_2 \in \mathbb{Z}[i]$ so that:

$$z_1 = \sum_{j=0}^{n} a_j Z_1^j, \quad z_2 = \sum_{j=0}^{n} a_j Z_2^j. \tag{1.1}$$

Statement 1.1. *If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system, then $Z_1 - Z_2$ is unit.*

Proof. Let (z_1, z_2) be an ordered pair which can be written in the form (1.1). We get:

$$z_1 - z_2 = \sum_{j=1}^{n} a_j \left(Z_1^j - Z_2^j \right).$$

It is easy to see, that $Z_1 - Z_2$ is the divisor of all terms of the right hand side of the equation, so it is the divisor of the left hand side of the equation as well. If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system, then every ordered pair (z_1, z_2) can be written in the form (1.1). This holds for $(z_1, z_1 - 1)$ as well. Hence we get that $Z_1 - Z_2$ is the divisor of 1, so it is unit. ■

Corollary 1.1. *If $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system of Gaussian integers, then $Z_1 - Z_2 \in \{\pm 1, \pm i\}$.*

2. The case of canonical digit set

Let $\mathcal{A}_c = \{0, 1, \ldots, |Z_1|^2 |Z_2|^2 - 1\}$. If we would like $(Z_1, Z_2, \mathcal{A}_c)$ to be a simultaneous number system, then Z_1 and Z_2 must be of the form $A \pm i$. Otherwise not every ordered pair (x, y) could be written in the form (1.1). Considering the previous Corollary we get that $(Z_1, Z_2, \mathcal{A}_c)$ can be a simultaneous number system, only if $Z_1 = A \pm i$ and $Z_2 = Z_1 \pm 1$. Similarly to the case of number systems of the Gaussian integers we get that $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system if and only if $(\overline{Z_1}, \overline{Z_2}, \mathcal{A}_c)$ is a simultaneous number system as well. Furthermore $(Z_1, Z_2, \mathcal{A}_c)$ is a simultaneous number system if and only if $(Z_2, Z_1, \mathcal{A}_c)$ is a simultaneous number system as well. Therefore it is enough to examine the case $Z_1 = A + i$ and $Z_2 = Z_1 - 1$.

Theorem 2.1. *$(Z_1, Z_2, \mathcal{A}_c)$ is not a simultaneous number system.*

Statement 2.1. *Let* $Z_1 = -A + i, \quad A \in \mathbb{Z}, \quad A > 0,\ Z_2 = Z_1 - 1,$ *and* $\mathcal{A}_c = \{0, 1, \ldots, |Z_1|^2|Z_2|^2 - 1\}$. *Then* $(Z_1, Z_2, \mathcal{A}_c)$ *is not a simultaneous number system.*

Proof of Statement 2.1. We will show that there are nontrivial periodic elements. If $a = (b, c) \in \mathbb{Z}[i] \times \mathbb{Z}[i]$ then let $d(a) \in \mathcal{A}_c$ be such that $d(a) \equiv$ $\equiv b\ (mod\ Z_1)$ and $d(a) \equiv c\ (mod\ Z_2)$. Furthermore let $J(a) = \left(\frac{b-d(a)}{Z_1}, \frac{c-d(a)}{Z_2}\right)$.

Let $B = \{1, 3, 4, 5, 6, 10, 11, 16\}$. If $A \in B$ then the structure of periodic elements of $(-A + i, -A - 1 + i, \mathcal{A}_c)$ or at least the values of transitions are different from the other cases.

If $A = 1$ then let $p_0 = (0, 0)$ and

$$
\begin{array}{lll}
p_1 = (2, 1), & p_2 = (2 + 2i, 2 + i), & p_3 = (3, 1), \\
p_4 = (-1 - i, 0), & p_5 = (i, 0), & p_6 = (3 + 2i, 2 + i), \\
p_7 = (4 + 2i, 3 + i), & p_8 = (-1 - 3i, -1 - i), & p_9 = (3i, 1 + i), \\
p_{10} = (-1, 0), & p_{11} = (3 + 3i, 2 + i), & p_{12} = (2 - i, 1), \\
p_{13} = (2 + 3i, 2 + i), & p_{14} = (5 + 2i, 3 + i), & p_{15} = (1 - i, 1), \\
p_{16} = (2 + 4i, 2 + i), & p_{17} = (3 - i, 1), & p_{18} = (1 + 2i, 2 + i), \\
p_{19} = (5 + 3i, 3 + i), & p_{20} = (-1 - 4i, -1 - i), & p_{21} = (2 + 6i, 3 + 2i), \\
p_{22} = (3 - 3i, -i), & p_{23} = (1 + 4i, 3 + 2i), & p_{24} = (5 + i, 2), \\
p_{25} = (-1 - 2i, 0), & p_{26} = (3 + 4i, 2 + i), & p_{27} = (5 + i, 3 + i), \\
p_{28} = (-2 - 3i, -1 - i), & p_{29} = (3 + 6i, 3 + 2i), & p_{30} = (5 - i, 2), \\
p_{31} = (-2 - i, 0), & p_{32} = (6 + 2i, 3 + i), & p_{33} = (-2 - 4i, -1 - i), \\
p_{34} = (4i, 1 + i), & p_{35} = (2 + 4i, 3 + 2i), & p_{36} = (2 - 2i, -i).
\end{array}
$$

Then

$$
\begin{array}{lllll}
J(p_0) = p_0, & J(p_1) = p_2, & J(p_2) = p_1, & J(p_3) = p_4, & J(p_4) = p_5, \\
J(p_5) = p_6, & J(p_6) = p_7, & J(p_7) = p_8, & J(p_8) = p_9, & J(p_9) = p_3, \\
J(p_{10}) = p_{11}, & J(p_{11}) = p_{12}, & J(p_{12}) = p_{10}, & J(p_{13}) = p_{14}, & J(p_{14}) = p_{15}, \\
J(p_{15}) = p_{13}, & J(p_{16}) = p_{17}, & J(p_{17}) = p_{18}, & J(p_{18}) = p_{19}, & J(p_{19}) = p_{20}, \\
J(p_{20}) = p_{21}, & J(p_{21}) = p_{22}, & J(p_{22}) = p_{23}, & J(p_{23}) = p_{24}, & J(p_{24}) = p_{25}, \\
J(p_{25}) = p_{16}, & J(p_{26}) = p_{27}, & J(p_{27}) = p_{28}, & J(p_{28}) = p_{29}, & J(p_{29}) = p_{30}, \\
J(p_{30}) = p_{31}, & J(p_{31}) = p_{26}, & J(p_{32}) = p_{33}, & J(p_{33}) = p_{34}, & J(p_{34}) = p_{32}, \\
J(p_{35}) = p_{36}, & J(p_{36}) = p_{35}, & & &
\end{array}
$$

furthermore $d(p_0) = 0$ and

$$\begin{array}{llllll} d(p_1)=6, & d(p_2)=4, & d(p_3)=1, & d(p_4)=0, & d(p_5)=5, & d(p_6)=9, \\ d(p_7)=0, & d(p_8)=2, & d(p_9)=3, & d(p_{10})=5, & d(p_{11})=4, & d(p_{12})=1, \\ d(p_{13})=9, & d(p_{14})=5, & d(p_{15})=6, & d(p_{16})=4, & d(p_{17})=6, & d(p_{18})=9, \\ d(p_{19})=0, & d(p_{20})=7, & d(p_{21})=2, & d(p_{22})=8, & d(p_{23})=7, & d(p_{24})=2, \\ d(p_{25})=5, & d(p_{26})=9, & d(p_{27})=0, & d(p_{28})=7, & d(p_{29})=7, & d(p_{30})=2, \\ d(p_{31})=5, & d(p_{32})=0, & d(p_{33})=2, & d(p_{34})=8, & d(p_{35})=2, & d(p_{36})=8. \end{array}$$

The structure of periodic elements is shown in Figure 1.

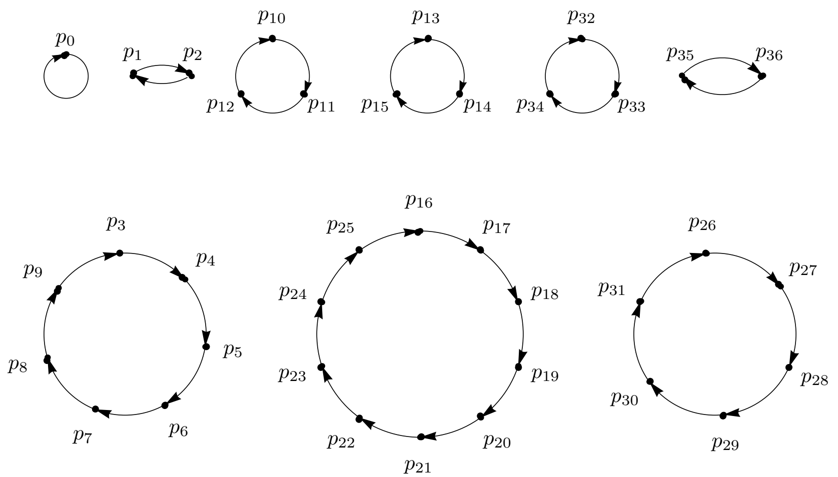


Figure 1. The structure of periodic elements of $(-1+i, -2+i, \mathcal{A}_c)$

If $A = 3$ then let $p_0 = (0, 0)$ and

$$\begin{array}{lll} p_1 = (9+4i, 7+2i), & p_2 = (43+13i, 34+8i), & p_3 = (-2-5i, -2i), \\ p_4 = (13+6i, 10+3i), & p_5 = (39+11i, 31+7i) & p_6 = (2-3i, 3-i), \\ p_7 = (47+15i, 37+9i), & p_8 = (-6-7i, -3-3i), & p_9 = (17+8i, 13+4i), \\ p_{10} = (35+9i, 28+6i), & p_{11} = (6-i, 6), & p_{12} = (5+2i, 4+i). \end{array}$$

Then

$$\begin{array}{lllll} J(p_0) = p_0, & J(p_1) = p_2, & J(p_2) = p_3, & J(p_3) = p_4, & J(p_4) = p_5, \\ J(p_5) = p_6 & J(p_6) = p_1, & J(p_7) = p_8, & J(p_8) = p_9, & J(p_9) = p_{10}, \\ J(p_{10}) = p_{11}, & J(p_{11}) = p_{12}, & J(p_{12}) = p_7, & & \end{array}$$

furthermore $d(p_0) = 0$ and

$$d(p_1) = 151, \quad d(p_2) = 32, \quad d(p_3) = 43, \quad d(p_4) = 141, \quad d(p_5) = 42,$$
$$d(p_6) = 33, \quad d(p_7) = 22, \quad d(p_8) = 53, \quad d(p_9) = 131, \quad d(p_{10}) = 52,$$
$$d(p_{11}) = 23, \quad d(p_{12}) = 161.$$

The structure of periodic elements is shown in Figure 2.

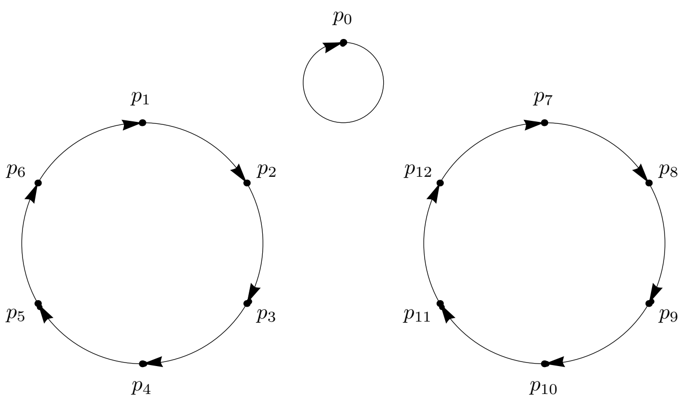


Figure 2. The structure of periodic elements of $(-3+i, -4+i, \mathcal{A}_c)$

If $A = 4$ then let

$$p_0 = (0,0), \quad p_1 = (55+14i, 45+9i), \quad p_2 = (58+11i, 49+8i),$$
$$p_3 = (63+13i, 53+9i), \quad p_4 = (9-i, 9).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_2, \quad J(p_2) = p_3, \quad J(p_3) = p_4, \quad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \quad d(p_1) = 298, \quad d(p_2) = 323, \quad d(p_3) = 98, \quad d(p_4) = 243.$$

The structure of periodic elements is shown in Figure 3a.

If $A = 5$ then let

$$p_0 = (0,0), \quad p_1 = (65+13i, 55+9i), \quad p_2 = (73+12i, 63+9i),$$
$$p_3 = (137+25i, 117+18i), \quad p_4 = (25, 24+i).$$

Then

$$J(p_0) = p_0, \qquad J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 442, \qquad d(p_2) = 783, \qquad d(p_3) = 262, \qquad d(p_4) = 363.$$

The structure of periodic elements is shown in Figure 3a.

If $A = 6$ then let

$$p_0 = (0,0), \qquad p_1 = (91+13i, 80+10i),$$
$$p_2 = (182+26i, 160+20i), \qquad p_3 = (229+38i, 198+28i),$$
$$p_4 = (44+i, 42+2i), \qquad p_5 = (139+25i, 119+18i),$$
$$p_6 = (253+38i, 221+29i), \qquad p_7 = (-28-11i, -20-7i).$$

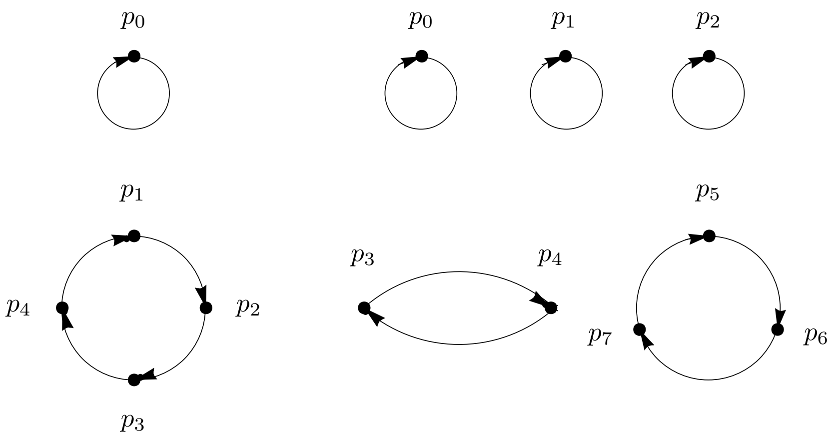
Then

$$J(p_0) = p_0, \qquad J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_4,$$
$$J(p_4) = p_3, \qquad J(p_5) = p_6, \qquad J(p_6) = p_7, \qquad J(p_7) = p_5,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 650, \qquad d(p_3) = 1300, \qquad d(p_3) = 494,$$
$$d(p_4) = 1456, \qquad d(p_5) = 1695, \qquad d(p_6) = 74, \qquad d(p_7) = 831.$$

The structure of periodic elements is shown in Figure 3b.



(a) $A \in \{4,5\}$ (b) $A = 6$

Figure 3. The structure of periodic elements of $(-A+i, -A-1+i, \mathcal{A}_c)$

If $A = 10$ then let $p_0 = (0, 0)$ and

$$p_1 = (439 + 45i, 399 + 37i), \qquad p_2 = (375 + 33i, 345 + 28i),$$
$$p_3 = (813 + 78i, 743 + 65i), \qquad p_4 = (-32 - 11i, -23 - 8i).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_2, \quad J(p_2) = p_3, \quad J(p_3) = p_4, \quad J(p_4) = p_1,$$

furthermore

$$d(p_0) = 0, \quad d(p_1) = 4222, \quad d(p_2) = 8583, \quad d(p_3) = 482, \quad d(p_4) = 4403.$$

The structure of periodic elements is shown in Figure 4a.

If $A = 11$ then let

$$p_0 = (0, 0), \qquad p_1 = (408 + 34i, 377 + 29i),$$
$$p_2 = (816 + 68i, 754 + 58i), \qquad p_3 = (1224 + 102i, 1131 + 87i),$$
$$p_4 = (1222 + 112i, 1121 + 94i), \qquad p_5 = (2 - 10i, 10 - 7i).$$

Then

$$J(p_0) = p_0, \quad J(p_1) = p_1, \quad J(p_2) = p_2, \quad J(p_3) = p_3, \quad J(p_4) = p_5, \quad J(p_5) = p_4,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 4930, \qquad d(p_2) = 9860,$$
$$d(p_3) = 14790, \qquad d(p_4) = 1234, \qquad d(p_5) = 13556.$$

The structure of periodic elements is shown in Figure 4b.

If $A = 16$ then let $p_0 = (0, 0)$ and

$$p_1 = (1105 + 65i, 1044 + 58i), \qquad p_2 = (2210 + 130i, 2088 + 116i),$$
$$p_3 = (3315 + 195i, 3132 + 174i), \qquad p_4 = (3586 + 226i, 3375 + 200i),$$
$$p_5 = (4370 + 259i, 4127 + 231i), \qquad p_6 = (-221 - 30i, -194 - 25i).$$

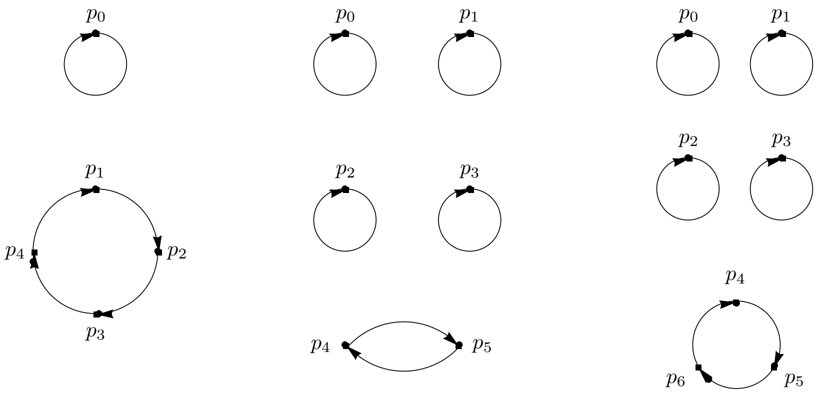
Then

$$J(p_0) = p_0, \qquad J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_3,$$
$$J(p_4) = p_5, \qquad J(p_5) = p_6, \qquad J(p_6) = p_4,$$

furthermore

$$d(p_0) = 0, \qquad d(p_1) = 18850, \qquad d(p_2) = 37700, \qquad d(p_3) = 56550,$$
$$d(p_4) = 73765, \qquad d(p_5) = 804, \qquad d(p_6) = 57381.$$

The structure of periodic elements is shown in Figure 4c.



(a) $A = 10$ (b) $A = 11$ (c) $A = 16$

Figure 4. The structure of periodic elements of $(-A+i, -A-1+i, \mathcal{A}_c)$

We can get the following connections with interpolation from examining a few examples:

Case 1. $A = 5k+1$. Let

$$a_{11} = 25k^3 + 40k^2 + 22k + 4, \qquad b_{11} = 5k^2 + 6k + 2,$$
$$a_{21} = 25k^3 + 35k^2 + 17k + 3, \qquad b_{21} = 5k^2 + 4k + 1,$$
$$a_{12} = 50k^3 + 80k^2 + 44k + 8, \qquad b_{12} = 10k^2 + 12k + 4,$$
$$a_{22} = 50k^3 + 70k^2 + 34k + 6, \qquad b_{22} = 10k^2 + 8k + 2,$$
$$a_{13} = 75k^3 + 120k^2 + 66k + 12, \qquad b_{13} = 15k^2 + 18k + 6,$$
$$a_{23} = 75k^3 + 105k^2 + 51k + 9, \qquad b_{23} = 15k^2 + 12k + 3,$$
$$a_{14} = 100k^3 + 160k^2 + 88k + 16, \qquad b_{14} = 20k^2 + 24k + 8,$$
$$a_{24} = 100k^3 + 140k^2 + 68k + 12, \qquad b_{24} = 20k^2 + 16k + 4,$$

and

$$p_1 = (a_{11} + b_{11}i, a_{21} + b_{21}i), \qquad p_2 = (a_{12} + b_{12}i, a_{22} + b_{22}i),$$
$$p_3 = (a_{13} + b_{13}i, a_{23} + b_{23}i), \qquad p_4 = (a_{14} + b_{14}i, a_{24} + b_{24}i).$$

Then

$$J(p_1) = p_1, \qquad J(p_2) = p_2, \qquad J(p_3) = p_3, \qquad J(p_4) = p_4,$$

furthermore

$$\begin{aligned}
d(p_1) &=, 125k^4 + 250k^3 + 195k^2 + 70k + 10,\\
d(p_2) &= 250k^4 + 500k^3 + 390k^2 + 140k + 20,\\
d(p_3) &= 375k^4 + 750k^3 + 585k^2 + 210k + 30,\\
d(p_4) &= 500k^4 + 1000k^3 + 780k^2 + 280k + 40.
\end{aligned}$$

CASE 2. $A = 5k + 2$. In this case $\mathcal{A}$ is not a suitable digit set since $((5k + 2)^2 + 1, (5k + 3)^2 + 1) = 5$.

CASE 3. $A = 5k + 3$. Let

$$\begin{aligned}
a_{11} &= 25k^3 + 115k^2 + 132k + 46, & b_{11} &= 5k^2 + 16k + 10,\\
a_{21} &= 25k^3 + 110k^2 + 117k + 38, & b_{21} &= 5k^2 + 14k + 7,\\
a_{12} &= 100k^3 + 235k^2 + 198k + 58, & b_{12} &= 20k^2 + 34k + 16,\\
a_{22} &= 100k^3 + 215k^2 + 168k + 47, & b_{22} &= 20k^2 + 26k + 10,\\
a_{13} &= 50k^3 + 155k^2 + 154k + 50, & b_{13} &= 10k^2 + 22k + 12,\\
a_{23} &= 50k^3 + 145k^2 + 134k + 41, & b_{23} &= 10k^2 + 18k + 8,\\
a_{14} &= 75k^3 + 195k^2 + 176k + 54, & b_{14} &= 15k^2 + 28k + 14,\\
a_{24} &= 75k^3 + 180k^2 + 151k + 44, & b_{24} &= 15k^2 + 22k + 9,
\end{aligned}$$

and

$$\begin{aligned}
p_1 &= (a_{11} + b_{11}i, a_{21} + b_{21}i), & p_2 &= (a_{12} + b_{12}i, a_{22} + b_{22}i),\\
p_3 &= (a_{13} + b_{13}i, a_{23} + b_{23}i), & p_4 &= (a_{14} + b_{14}i, a_{24} + b_{24}i).
\end{aligned}$$

Then

$$J(p_1) = p_2, \qquad J(p_2) = p_1, \qquad J(p_3) = p_4, \qquad J(p_4) = p_3,$$

furthermore

$$\begin{aligned}
d(p_1) &= 500k^4 + 1500k^3 + 1830k^2 + 1050k + 236,\\
d(p_2) &= 125k^4 + 750k^3 + 1245k^2 + 840k + 206,\\
d(p_3) &= 375k^4 + 1250k^3 + 1635k^2 + 980k + 226,\\
d(p_4) &= 250k^4 + 1000k^3 + 1440k^2 + 910k + 216.
\end{aligned}$$

Case 4. $A = 5k + 4$. Let

$$\begin{aligned}
a_{11} &= 25k^3 + 115k^2 + 162k + 72, & b_{11} &= 5k^2 + 16k + 12,\\
a_{21} &= 25k^3 + 110k^2 + 147k + 62, & b_{21} &= 5k^2 + 14k + 9,\\
a_{12} &= 50k^3 + 155k^2 + 169k + 64, & b_{12} &= 10k^2 + 22k + 13,\\
a_{22} &= 50k^3 + 145k^2 + 149k + 54, & b_{22} &= 10k^2 + 18k + 9,\\
a_{13} &= 100k^3 + 310k^2 + 323k + 113, & b_{13} &= 20k^2 + 44k + 25,\\
a_{23} &= 100k^3 + 290k^2 + 283k + 94, & b_{23} &= 20k^2 + 36k + 17,\\
a_{14} &= 75k^3 + 270k^2 + 316k + 121, & b_{14} &= 15k^2 + 38k + 24,\\
a_{24} &= 75k^3 + 255k^2 + 281k + 102, & b_{24} &= 15k^2 + 32k + 17,
\end{aligned}$$

furthermore

$$\begin{aligned}
p_1 &= (a_{11} + b_{11}i, a_{21} + b_{21}i), & p_2 &= (a_{12} + b_{12}i, a_{22} + b_{22}i),\\
p_3 &= (a_{13} + b_{13}i, a_{23} + b_{23}i), & p_4 &= (a_{14} + b_{14}i, a_{24} + b_{24}i).
\end{aligned}$$

Then

$$J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$\begin{aligned}
d(p_1) &= 250k^4 + 1000k^3 + 1590k^2 + 1180k + 341,\\
d(p_2) &= 500k^4 + 2000k^3 + 3030k^2 + 2070k + 541,\\
d(p_3) &= 375k^4 + 1750k^3 + 2985k^2 + 2230k + 621,\\
d(p_4) &= 125k^4 + 750k^3 + 1545k^2 + 1340k + 421.
\end{aligned}$$

Case 5. $A = 5k$. Let

$$\begin{aligned}
a_{11} &= 25k^3 + 40k^2 + 22k + 3, & b_{11} &= 5k^2 + 6k + 2,\\
a_{21} &= 25k^3 + 35k^2 + 17k + 2, & b_{21} &= 5k^2 + 4k + 1,\\
a_{12} &= 75k^3 + 45k^2 + 16k + 2, & b_{12} &= 15k^2 + 8k + 2,\\
a_{22} &= 75k^3 + 30k^2 + 11k + 2, & b_{22} &= 15k^2 + 2k + 1,\\
a_{13} &= 100k^3 + 85k^2 + 23k + 2, & b_{13} &= 20k^2 + 14k + 3,\\
a_{23} &= 100k^3 + 65k^2 + 13k + 2, & b_{23} &= 20k^2 + 6k + 1,\\
a_{14} &= 50k^3 + 80k^2 + 29k + 3, & b_{14} &= 10k^2 + 12k + 3,\\
a_{24} &= 50k^3 + 70k^2 + 19k + 2, & b_{24} &= 10k^2 + 8k + 1,
\end{aligned}$$

furthermore

$$p_1 =(a_{11} + b_{11}i, a_{21} + b_{21}i), \qquad p_2 =(a_{12} + b_{12}i, a_{22} + b_{22}i),$$
$$p_3 =(a_{13} + b_{13}i, a_{23} + b_{23}i), \qquad p_4 =(a_{14} + b_{14}i, a_{24} + b_{24}i).$$

Then

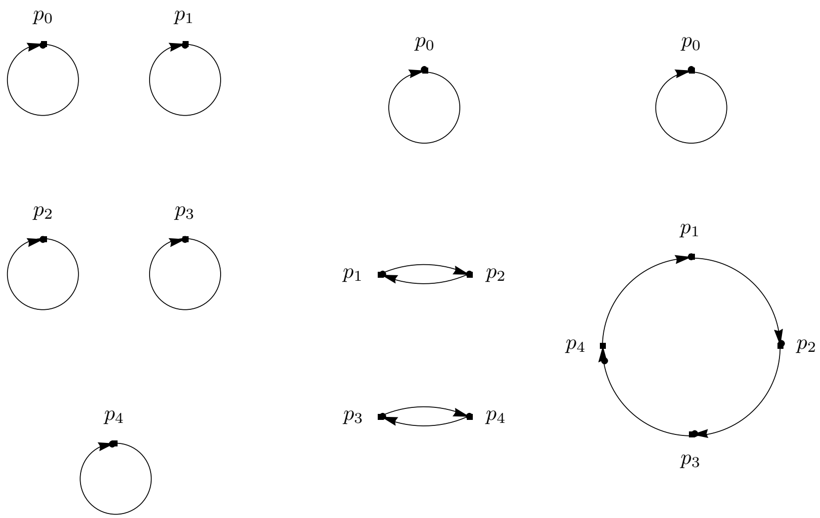
$$J(p_1) = p_2, \qquad J(p_2) = p_3, \qquad J(p_3) = p_4, \qquad J(p_4) = p_1,$$

furthermore

$$d(p_1) = 375k^4 + 250k^3 + 135k^2 + 40k + 5,$$
$$d(p_2) = 500k^4 + 500k^3 + 180k^2 + 40k + 5,$$
$$d(p_3) = 250k^4 + 500k^3 + 240k^2 + 50k + 5,$$
$$d(p_4) = 125k^4 + 250k^3 + 195k^2 + 50k + 5.$$

The statements can be verified by simple calculations.

The structure of periodic elements is shown in Figure 5. ■



(a) $A \equiv 1 \pmod 5$ (b) $A \equiv 3 \pmod 5$ (c) $A \equiv 0$ or $4 \pmod 5$

Figure 5. The structure of periodic elements of $(-A + i, -A - 1 + i, \mathcal{A}_c)$, if $A \notin B$

Conjecture 2.1. *There are no periodic elements other than the enumerated ones.*

If the conjecture is true, then if $A \notin B$, then the number of nontrivial periodic elements will be 4 and their structure depends on the remainder of A divided by 5. Namely:

- 4 pieces of loops
- 2 pieces of circles with the length of 2
- 1 piece of circle with the length of 4

Statement 2.2. *Let now* $Z_1 = A + i, \quad A \in \mathbb{Z}, \quad A > 0$, $Z_2 = Z_1 + 1$, *and* $\mathcal{A}_c = \{0, 1, \ldots, |Z_1|^2|Z_2|^2 - 1\}$. *Then* $(Z_1, Z_2, \mathcal{A}_c)$ *is not a simultaneous number system.*

Proof of Statement 2.2. If $A \equiv 2 \ (mod\ 5)$ then $\mathcal{A}_c$ is not a suitable digit set. Otherwise there would exist nontrivial periodic elements. Let

$$p = (-A^3 + A^2 - A + 1 + A^2i + i, -A^3 + 2A^2 - 2A + k^2i - 2Ai + 2i).$$

We get with simple calculations that in this case $J(p) = p$. ■

Proof of Theorem 2.1. Theorem 2.1 follows from Statement 2.1 and Statement 2.2 immediately. ■

We proved that $(Z_1, Z_2, \mathcal{A}_c)$ is not a simultaneous number system for all $Z_1, Z_2 \in \mathbb{Z}[i]$.

3. The case of the new digit set

With the help of K-type digit sets one can define such digit set by which simultaneous number systems of Gaussian integers exist.

Definition 3.1. Let $Z = a + bi$ and $t = |Z|^2$. Then let $E_\alpha^{(\varepsilon,\delta)}$ be the sets of those $d = k + li,\ k, l \in \mathbb{Z}$ for which

$$d\overline{Z} = (k + li)(a - bi) = (ka + bl) + (la - kb)i = r + si$$

satisfy the following conditions:

if $(\varepsilon, \delta) = (1, 1)$, then $r, s \in (-t/2, t/2]$,
if $(\varepsilon, \delta) = (-1, -1)$, then $r, s \in [-t/2, t/2)$,
if $(\varepsilon, \delta) = (-1, 1)$, then $r \in [-t/2, t/2), s \in (-t/2, t/2]$
if $(\varepsilon, \delta) = (1, -1)$, then $r \in (-t/2, t/2], s \in [-t/2, t/2)$.
We call the above constructed coefficient sets *K-type digit sets.*

The K-type digit set was used by G. Steidl in [2], by I. Kátai in [3] and by G. Farkas in [4], [5], [6], [7] and [8]. Now we use them to construct a new digit set by which simultaneous number systems of Gaussian integers exist.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be K-type digit sets belonging to given $Z_1, Z_2 \in \mathbb{Z}[i]$ Gaussian integers. Define $\mathcal{A}$ in the following way:

$$\mathcal{A} := \bigcup_{a_j \in \mathcal{A}_2} (\mathcal{A}_1 + a_j Z_1).$$

Theorem 3.1. *If $Z_1, Z_2 \in \mathbb{Z}[i]$ are such, that $Z_2 = Z_1 + \varepsilon$, where $\varepsilon \in \{\pm 1, \pm i\}$, $\mathcal{A}$ is as defined above and $|Z_1|$ is large enough, then $(Z_1, Z_2, \mathcal{A})$ is a simultaneous number system.*

Remarks.

$$\max_{a \in \mathcal{A}_1} |a| \leq \frac{|Z_1|}{\sqrt{2}}, \qquad \max_{a \in \mathcal{A}_2} |a| \leq \frac{|Z_1| + 1}{\sqrt{2}}.$$

$$M := \max_{a \in \mathcal{A}} |a| \leq \frac{|Z_1|}{\sqrt{2}} + \frac{|Z_1| + 1}{\sqrt{2}} |Z_1| = \frac{|Z_1|}{\sqrt{2}} (|Z_1| + 2).$$

Let $L_1 := \frac{M}{|Z_1| - 1}$, $L_2 := \frac{M}{|Z_2| - 1}$ and $L := \max(L_1, L_2)$. Then

$$L \leq \frac{\frac{|Z_1|}{\sqrt{2}} (|Z_1| + 2)}{|Z_1| - 2}.$$

Lemma 3.1. *If (z_1, z_2) is a periodic element, then $|z_1| \leq L_1$ and $|z_2| \leq L_2$.*

Lemma 3.2. *If $a \in \mathbb{Z}[i]$, $|a| \leq L$, then $a \in \mathcal{A}$.*

Lemma 3.3. *If $z_1 \neq z_2$, $|z_1|, |z_2| \leq L$ and $J(z_1, z_2) = (w_1, w_2)$, then $|w_1 - w_2| < |z_1 - z_2|$.*

Lemma 3.4. *For every $z_1, z_2 \in \mathbb{Z}[i]$ there exists $a \in \mathcal{A}$ such that $z_1 \equiv a \ (Z_1)$ and $z_2 \equiv a \ (Z_2)$.*

Proof of Lemma 3.1. The proof is similar to the proof for previous structures. ■

Proof of Lemma 3.2. $\mathcal{A}_2$ is K-type digit set. Therefore $\forall a \in \mathbb{Z}[i]$, if $|a| < \frac{|Z_2|}{2}$ then $a \in \mathcal{A}_2$. From the definition of $\mathcal{A}$ we get that if $|a| < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|$ then $a \in \mathcal{A}$. Consequently we have to solve the following inequality:

$$L < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|, \qquad \frac{\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)}{|Z_1|-2} < \left(\frac{|Z_2|}{2} - 1\right)|Z_1|,$$

$$\frac{|Z_1|(|Z_1|+2)}{\sqrt{2}(|Z_1|-2)} < \frac{|Z_1|-3}{2}|Z_1|, \qquad 2|Z_1|+4 < \sqrt{2}(|Z_1|^2 - 5|Z_1| + 6),$$

$$0 < |Z_1|^2 - 7|Z_1| + 2,$$

which is true, if $|Z_1| > \frac{7}{2} + \frac{1}{2}\sqrt{41} \approx 6,7.$ ■

Proof of Lemma 3.3.

$$\left|\frac{z_1 - a}{Z_1} - \frac{z_2 - a}{Z_2}\right| = \left|\frac{z_1 - a}{Z_1} - \frac{z_2 - a}{Z_1} + \frac{z_2 - a}{Z_1} - \frac{z_2 - a}{Z_1 + \varepsilon}\right| \leq$$

$$\leq \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{|\varepsilon(z_2 - a)|}{|Z_1(Z_1 + \varepsilon)|} = \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{|z_2 - a|}{|Z_1||Z_1 + \varepsilon|} \leq$$

$$\leq \frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{L + M}{|Z_1||Z_1 + \varepsilon|}.$$

Therefore we have to prove that if $|Z_1|$ is large enough, then

$$\frac{|(z_1 - a) - (z_2 - a)|}{|Z_1|} + \frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq |z_1 - z_2| = |(z_1 - a) - (z_2 - a)|,$$

or equivalently

$$\frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq |(z_1 - a) - (z_2 - a)|\left(1 - \frac{1}{|Z_1|}\right).$$

For this it is enough to prove that

$$\frac{L + M}{|Z_1||Z_1 + \varepsilon|} \leq 1 - \frac{1}{|Z_1|}.$$

Multiplying by $|Z_1|$ we get

$$\frac{L + M}{|Z_1| - 1} \leq |Z_1| - 1,$$

$$L + M \leq (|Z_1| - 1)^2.$$

Substituting L and M by their previous estimates we obtain

$$\frac{\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)}{|Z_1|-2}+\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)\le(|Z_1|-1)^2,$$

$$\frac{|Z_1|}{\sqrt{2}}(|Z_1|+2)\left(1+\frac{1}{|Z_1|-2}\right)\le(|Z_1|-1)^2.$$

Dividing by $|Z_1|^2$ leads to

$$\frac{1}{\sqrt{2}}\left(1+\frac{2}{|Z_1|}\right)\left(1+\frac{1}{|Z_1|-2}\right)\le\left(1-\frac{1}{|Z_1|}\right).$$

If $|Z_1|$ tends to infinity then the left hand side of the inequality tends to $\frac{1}{\sqrt{2}}$ and the right hand side tends to 1. Then the inequality holds if $|Z_1|$ is large enough,. The inequality is true, if $|Z_1|>4+\frac{5}{2}\sqrt{2}+\frac{1}{2}\sqrt{98+72\sqrt{2}}\approx 14,6$. ■

Proof of Lemma 3.4. Let $a_1\in\mathcal{A}_1$ and $a_2\in\mathcal{A}_2$ be such that $z_1\equiv a_1\ (Z_1)$ and $a_2\equiv\frac{a_1-z_2}{\varepsilon}\ (Z_2)$ hold. Then $a_1+a_2Z_1\in\mathcal{A}$ will be a suitable digit. ■

Proof of Theorem 3.1. The theorem follows from the lemmas immediately. ■

References

[1] **Indlekofer, K.-H., I. Kátai and P. Racskó,** Number systems and fractal geometry, *Probability Theory and Applications,* Kluwer Academic Publishers, The Netherlands, (1993), 319–334.

[2] **Steidl, G.,** On symmetric representation of Gaussian integers, *BIT,* **29** (1989), 563–571.

[3] **Kátai, I.,** Number systems in imaginary quadratic fields, *Annales Univ. Sci. Budapest., Sect. Comp.,* **14** (1994), 159–164.

[4] **Farkas, G.,** Number systems in real quadratic fields, *Annales Univ. Sci. Budapest., Sect. Comp.,* **18** (1999), 47–59.

[5] **Farkas, G.,** Digital expansion in real algebraic quadratic fields, *Matematica Pannonica Jannus Pannonius Univ. (Pécs),* **10/2** (1999), 235–248.

[6] **Farkas, G.,** Location and number of periodical elements in $\mathbb{Q}(\sqrt{2})$, *Annales Univ. Sci. Budapest., Sect. Comp.,* **20** (2001), 133–146.

[7] **Farkas, G.,** Periodic elements and number systems in $\mathbb{Q}(\sqrt{2})$, *Mathematical and Computer Modelling,* **38** (2003), 783–788.

[8] **Farkas, G. and A. Kovács** Digital expansion in $\mathbb{Q}(\sqrt{2})$, *Annales Univ. Sci. Budapest., Sect. Comp.,* **22** (2003), 83–94.

G. Nagy
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sétány 1/C
H-1117 Budapest, Hungary
`nagy@compalg.inf.elte.hu`