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EXPONENTIAL UNITARY DIVISORS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. We say that d is an exponential unitary divisor of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{b_1} \cdots p_r^{b_r}$, where b_i is a unitary divisor of a_i, i.e., $b_i \mid a_i$ and $(b_i, a_i/b_i) = 1$ for every $i \in \{1, 2, \ldots, r\}$. We survey properties of related arithmetical functions and introduce the notion of exponential unitary perfect numbers.

1. Introduction

Let n be a positive integer. We recall that a positive integer d is called a unitary divisor of n if $d \mid n$ and $(d, n/d) = 1$. Notation: $d \mid_* n$. If $n > 1$ and has the prime factorization $n = p_1^{a_1} \cdots p_r^{a_r}$, then $d \mid_* n$ iff $d = p_1^{u_1} \cdots p_r^{u_r}$, where $u_i = 0$ or $u_i = a_i$ for every $i \in \{1, 2, \ldots, r\}$. Also, $1 \mid_* 1$.

Furthermore, d is said to be an exponential divisor (e-divisor) of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{e_1} \cdots p_r^{e_r}$, where $e_i \mid a_i$, for any $i \in \{1, 2, \ldots, r\}$. Notation: $d \mid_e n$. By convention $1 \mid_e 1$.

Let $\tau^*(n) := \sum_{d \mid_* n} 1$, $\sigma^*(n) := \sum_{d \mid_* n} d$ and $\tau^{(e)}(n) := \sum_{d \mid_e n} 1$, $\sigma^{(e)}(n) := \sum_{d \mid_e n} d$ denote, as usual, the number and the sum of the unitary divisors

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of n and of the e-divisors of n, respectively. These functions are multiplicative and one has

$$\tau^*(n) = 2^{\omega(n)}, \quad \sigma^*(n) = (1 + p_1^{a_1}) \cdots (1 + p_r^{a_r}), \tag{1}$$

$$\tau^{(e)}(n) = \tau(a_1) \cdots \tau(a_r), \quad \sigma^{(e)}(n) = \left(\sum_{d_1 | a_1} p_1^{d_1} \right) \cdots \left(\sum_{d_r | a_r} p_r^{d_r} \right), \tag{2}$$

where $\omega(n) := \sum_{p|n} 1$ is the number of distinct prime divisors of n, and $\tau(n) := \sum_{d|n} 1$ stands for the number of divisors of n.

Note that if n is squarefree, then $d \mid_* n$ iff $d \mid n$, and $\tau^*(n) = \tau(n)$, $\sigma^*(n) = \sigma(n) := \sum_{d|n} d$.

Closely related to the concepts of unitary and exponential divisors are the unitary convolution and the exponential convolution (e-convolution) of arithmetic functions defined by

$$(f \times g)(n) = \sum_{d \mid_* n} f(d) g(n/d), \quad n \geq 1, \tag{3}$$

and by $(f \odot g)(1) = f(1)g(1)$,

$$(f \odot g)(n) = \sum_{b_1 c_1 = a_1} \cdots \sum_{b_r c_r = a_r} f(p_1^{b_1} \cdots p_r^{b_r}) g(p_1^{c_1} \cdots p_r^{c_r}), \quad n > 1, \tag{4}$$

respectively.

The function $I(n) = 1$ $(n \geq 1)$ has inverses with respect to the unitary convolution and e-convolution given by $\mu^*(n) = (-1)^{\omega(n)}$ and $\mu^{(e)}(n) = \mu(a_1) \cdots \mu(a_r)$, $\mu^{(e)}(1) = 1$, respectively, where μ is the Möbius function. These are the unitary and exponential analogues of the Möbius function.

Unitary divisors (called block factors) and the unitary convolution (called compounding of functions) were first considered by R. Vaidyanathaswamy [23]. The current terminology was introduced by E. Cohen [1, 2]. The notions of exponential divisor and exponential convolution were first defined by M. V. Subbarao [15]. Various properties of arithmetical functions defined by unitary and exponential divisors, including the functions τ^*, σ^*, μ^*, $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and properties of the convolutions (3) and (4) were investigated by several authors.

A positive integer n is said to be unitary perfect if $\sigma^*(n) = 2n$. This notion was introduced by M. V. Subbarao and L. J. Warren [16]. Until now five unitary perfect numbers are known. These are $6 = 2 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$, $90 = 2 \cdot 3^2 \cdot 5$, $87\,360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ and the following number of 24 digits: $146\,361\,946\,186\,458\,562\,560\,000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$.

It is conjectured that there are finitely many such numbers. It is easy to see that there are no odd unitary perfect numbers.

An integer n is called exponentially perfect (e-perfect) if $\sigma^{(e)}(n) = 2n$. This originates from M. V. Subbarao [15]. The smallest e-perfect number is $36 = 2^2 \cdot 3^2$. If n is any squarefree number, then $\sigma^{(e)}(n) = n$, and $36n$ is e-perfect for any such n with $(n, 6) = 1$. Hence there are infinitely many e-perfect numbers. Also, there are no odd e-perfect numbers, cf. [14]. The squarefull e-perfect numbers under 10^{10} are: $2^2 \cdot 3^2$, $2^3 \cdot 3^2 \cdot 5^2$, $2^2 \cdot 3^3 \cdot 5^2$, $2^4 \cdot 3^2 \cdot 11^2$, $2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2$, $2^6 \cdot 3^2 \cdot 7^2 \cdot 13^2$, $2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2$, $2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13^2$. It is not known if there are infinitely many squarefull e-perfect numbers, see [4, p. 110].

For a survey on results concerning unitary and exponential divisors we refer to the books [10] and [12]. See also the papers [3, 5, 8, 9, 11, 13, 18, 19, 20] and their references.

M.V. Subbarao [15, Section 8] says: ,,We finally remark that to every given convolution of arithmetic functions, one can define the corresponding exponential convolution and study the properties of arithmetical functions which arise therefrom. For example, one can study the exponential unitary convolution, and in fact, the exponential analogue of any Narkiewicz-type convolution, among others."

While such convolutions were investigated by several authors, cf. [7, 6], it appears that arithmetical functions corresponding to the exponential unitary convolution mentioned above were not considered in the literature.

It is the aim of this paper to overcome this shortage. Combining the notions of e-divisors and unitary divisors we consider in this paper exponential unitary divisors (e-unitary divisors). We review properties of the corresponding τ, σ, μ and Euler-type functions. It turns out that the asymptotic behavior of these functions is similar to those of the functions $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and $\phi^{(e)}$ (the latter one will be given in Section 3). We define the e-unitary perfect numbers, which were not considered before, and state some open problems.

2. Exponential unitary divisors

We say that d is an exponential unitary divisor (e-unitary divisor) of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ if $d = p_1^{b_1} \cdots p_r^{b_r}$, where $b_i \mid_* a_i$, for any $i \in \{1, 2, \ldots, r\}$. Notation: $d \mid_{e*} n$. By convention $1 \mid_{e*} 1$.

For example, the e-unitary divisors of $n = p^{12}$, with p prime, are $d = p, p^3, p^4, p^{12}$, while its e-divisors are $d = p, p^2, p^3, p^4, p^6, p^{12}$.

Let $\tau^{(e)*}(n) := \sum_{d|_{e*}n} 1$ and $\sigma^{(e)*}(n) := \sum_{d|_{e*}n} d$ denote the number and the sum of the e-unitary divisors of n, respectively. It is immediate that these functions are multiplicative and we have

$$\tau^{(e)*}(n) = \tau^*(a_1)\cdots\tau^*(a_r) = 2^{\omega(a_1)+\ldots+\omega(a_r)},$$
$$\sigma^{(e)*}(n) = \left(\sum_{d_1|_* a_1} p_1^{d_1}\right)\cdots\left(\sum_{d_r|_* a_r} p_r^{d_r}\right). \tag{5}$$

If n is e-squarefree, i.e., $n = 1$ or $n > 1$ and all the exponents in the prime factorization of n are squarefree, then $d \mid_{e*} n$ iff $d \mid_e n$, and $\tau^{(e)*}(n) = \tau^{(e)}(n)$, $\sigma^{(e)*}(n) = \sigma^{(e)}(n)$.

Note that for any $n > 1$ the values $\tau^{(e)*}(n)$ and $\sigma^{(e)*}(n)$ are even.

The corresponding exponential unitary convolution (e-unitary convolution) is given by

$$(f \odot_* g)(1) = f(1)g(1),$$
$$(f \odot_* g)(n) = \sum_{\substack{b_1c_1=a_1\\(b_1,c_1)=1}} \cdots \sum_{\substack{b_rc_r=a_r\\(b_r,c_r)=1}} f(p_1^{b_1}\cdots p_r^{b_r})g(p_1^{c_1}\cdots p_r^{c_r}), \tag{6}$$

with the notation $n = p_1^{a_1}\cdots p_r^{a_r} > 1$.

The arithmetical functions form a commutative semigroup under (6) with identity μ^2. A function f has an inverse with respect to the e-unitary convolution iff $f(1) \neq 0$ and $f(p_1\cdots p_k) \neq 0$ for any distinct primes $p_1, \ldots, p_k$.

The inverse of the function $I(n) = 1$ $(n \geq 1)$ with respect to the e-unitary convolution is the function $\mu^{(e)*}(n) = \mu^*(a_1)\cdots\mu^*(a_r) = (-1)^{\omega(a_1)+\ldots+\omega(a_r)}$, $\mu^{(e)*}(1) = 1$.

These properties of convolution (6) are special cases of those of a more general convolution, involving regular convolutions of Narkiewicz-type, mentioned in the Introduction.

Remark. It is possible to define ,,unitary exponential divisors" (in the reverse order) in the following way. An integer d is a unitary exponential divisor (unitary e-divisor) of $n = p_1^{a_1}\cdots p_r^{a_r} > 1$ if $d \mid n$ and the integers d and n/d are exponentially coprime. This means that, denoting $d = p_1^{b_1}\cdots p_r^{b_r}$, we require d and n/d to have the same prime factors as n, i.e., $1 \leq b_i < a_i$, and $(b_i, a_i - b_i) = 1$ for any $i \in \{1, 2, \ldots, r\}$. This is fulfilled iff n is squarefull, i.e., $a_i \geq 2$ and $(b_i, a_i) = 1$ for every $i \in \{1, 2, \ldots, r\}$. Hence the number of unitary e-divisors of $n > 1$ is $\phi(a_1)\cdots\phi(a_r)$ (ϕ is Euler's function) or 0, depending on whether n is squarefull or not. We do not go into other details here. For exponentially coprime integers cf. [18].

3. Arithmetical functions defined by exponential unitary divisors

As noted before, the functions $\tau^{(e)*}$ and $\sigma^{(e)*}$ are multiplicative. Also, for any prime p, $\tau^{(e)*}(p) = 1$, $\tau^{(e)*}(p^2) = 2$, $\tau^{(e)*}(p^3) = 2$, $\tau^{(e)*}(p^4) = 2$, $\tau^{(e)*}(p^5) = 2$, ..., $\sigma^{(e)*}(p) = p$, $\sigma^{(e)*}(p^2) = p+p^2$, $\sigma^{(e)*}(p^3) = p+p^3$, $\sigma^{(e)*}(p^4) = p + p^4$, $\sigma^{(e)*}(p^5) = p + p^5$, ... Observe that the first difference compared with the functions $\tau^{(e)}$ and $\sigma^{(e)}$ occurs for p^4 (which is not e-squarefree).

The function $\tau^{(e)*}(n)$ is identical with the function $t^{(e)}(n)$, defined as the number of e-squarefree e-divisors of n and investigated by L. Tóth [20]. According to [20, Th. 4],

$$\sum_{n\le x} \tau^{(e)*}(n) = C_1 x + C_2 x^{1/2} + \mathcal{O}(x^{1/4+\varepsilon}), \tag{7}$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 := \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right), \tag{8}$$

$$C_2 := \zeta(1/2) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}}\right). \tag{9}$$

The error term of (7) was improved to $\mathcal{O}(x^{1/4})$ by Y.-F. S. Pétermann [11, Th. 1] showing that

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta(2s)}{\zeta(4s)} H(s), \quad \operatorname{Re} s > 1, \tag{10}$$

where $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\operatorname{Re} s > 1/6$.

For the maximal order of the function $\tau^{(e)*}$ we have

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)*}(n) \log\log n}{\log n} = \frac{1}{2} \log 2, \tag{11}$$

this is proved (for $t^{(e)}(n)$) in [20, Th. 5]. (11) holds also for the function $\tau^{(e)}$ instead of $\tau^{(e)*}$, cf. [15].

For the maximal order of the function $\sigma^{(e)*}$ we have

Theorem 1.

$$\limsup_{n\to\infty} \frac{\sigma^{(e)*}(n)}{n \log\log n} = \frac{6}{\pi^2} e^{\gamma}, \tag{12}$$

where γ is Euler's constant.

Proof. This is a direct consequence of the following general result of L. Tóth and E. Wirsing [22, Cor. 1]: Let f be a nonnegative real-valued multiplicative function. Suppose that for all primes p we have $\varrho(p) := \sup_{\nu \ge 0} f(p^\nu) \le$ $\le (1-1/p)^{-1}$ and that for all primes p there is an exponent $e_p = p^{o(1)}$ such that $f(p^{e_p}) \ge 1 + 1/p$. Then

$$\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^\gamma \prod_p \left(1-\frac{1}{p}\right) \varrho(p). \tag{13}$$

Apply this for $f(n) = \sigma^{(e)*}(n)/n$. Here $f(p) = 1$, $f(p^2) = 1 + 1/p$ and for $a \ge 2$, $f(p^a) \le \sigma^{(e)}(p^a)/p^a \le 1 + 1/p$. Hence $\varrho(p) = 1 + 1/p$ and we can choose $e_p = 2$ for all p. ■

(12) holds also for the function $\sigma^{(e)}$ instead of $\sigma^{(e)*}$. For the function $\mu^{(e)*}$ one has:

Theorem 2. (i) *The Dirichlet series of* $\mu^{(e)*}$ *is of the form*

$$\sum_{n=1}^{\infty} \frac{\mu^{(e)*}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} W(s), \quad \operatorname{Re} s > 1, \tag{14}$$

where $W(s) := \sum_{n=1}^{\infty} \frac{w(n)}{n^s}$ *is absolutely convergent for* $\operatorname{Re} s > 1/4$.

(ii)

$$\sum_{n\le x} \mu^{(e)*}(n) = C_3 x + \mathcal{O}(x^{1/2} \exp(-c(\log x)^{\Delta})), \tag{15}$$

where

$$C_3 := \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(-1)^{\omega(a)} - (-1)^{\omega(a-1)}}{p^a}\right), \tag{16}$$

and $\Delta = 9/25 - \varepsilon$ *for every* $\varepsilon > 0$, *where* $9/25 = 0.36$, *and* $c > 0$ *are constants*

Proof. A similar result was proved for the function $\mu^{(e)}$ in [20, Th. 2] (with the auxiliary Dirichlet series absolutely convergent for $\operatorname{Re} s > 1/5$). The same proof works out in case of $\mu^{(e)*}$. The error term can be improved assuming the Riemann hypothesis, cf. [20]. ■

The unitary analogue of Euler's arithmetical function, denoted by ϕ^* is defined as follows. Let $(k,n)_* := \max\{d \in \mathbb{N} : d \mid k, d \mid_* n\}$ and let

$$\phi^*(n) := \#\{k \in \mathbb{N} : 1 \le k \le n, (k,n)_* = 1\}, \tag{17}$$

which is multiplicative and $\phi^*(p^a) = p^a - 1$ for every prime power p^a $(a \geq 1)$. Why do we not consider here the greatest common unitary divisor of k and n? Because if we do so the resulting function is not multiplicative and its properties are not so close to those of Euler's function ϕ, cf. [21].

Furthermore, for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n are exponentially coprime, i.e., $d = p_1^{b_1} \cdots p_r^{b_r}$, where $1 \leq b_i \leq a_i$ and $(b_i, a_i) = 1$ for any $i \in \{1, \ldots, r\}$. By convention, let $\phi^{(e)}(1) = 1$. This is the exponential analogue of the Euler function, cf. [19]. Here $\phi^{(e)}$ is multiplicative and

$$\phi^{(e)}(n) = \phi(a_1) \cdots \phi(a_r), \quad n > 1. \tag{18}$$

We define the e-unitary Euler function in the following way: for $n = p_1^{a_1} \cdots p_r^{a_r} > 1$ let $\phi^{(e)*}(n)$ denote the number of divisors d of n such that $d = p_1^{b_1} \cdots p_r^{b_r}$, where $1 \leq b_i \leq a_i$ and $(b_i, a_i)_* = 1$ for any $i \in \{1, \ldots, r\}$. By convention, let $\phi^{(e)*}(1) = 1$. Then $\phi^{(e)*}$ is multiplicative and

$$\phi^{(e)*}(n) = \phi^*(a_1) \cdots \phi^*(a_r), \quad n > 1. \tag{19}$$

Theorem 3.

$$\sum_{n \leq x} \phi^{(e)*}(n) = C_4 x + C_5 x^{1/3} + \mathcal{O}(x^{1/4+\varepsilon}), \tag{20}$$

for every $\varepsilon > 0$, where C_4, C_5 are constants given by

$$C_4 := \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi^*(a) - \phi^*(a-1)}{p^a}\right), \tag{21}$$

(22)
$$C_5 := \zeta(1/3) \prod_p \left(1 + \frac{1}{p^{4/3}} + \sum_{a=5}^{\infty} \frac{\phi^*(a) - \phi^*(a-1) - \phi^*(a-3) + \phi^*(a-4)}{p^{a/3}}\right).$$

Proof. A similar result was proved for the function $\phi^{(e)}$ in [19, Th. 1], with error term $\mathcal{O}(x^{1/5+\varepsilon})$, improved to $\mathcal{O}(x^{1/5} \log x)$ by Y.-F. S. Pétermann [11, Th. 1]. The same proof works out in case of $\phi^{(e)*}$. ■

Theorem 4.

$$\limsup_{n \to \infty} \frac{\log \phi^{(e)*}(n) \log \log n}{\log n} = \frac{\log 4}{5}. \tag{23}$$

Proof. We apply the following general result given in [17]: Let F be a multiplicative function with $F(p^a) = f(a)$ for every prime power p^a, where f is positive and satisfies $f(n) = \mathcal{O}(n^\beta)$ for some fixed $\beta > 0$. Then

$$\limsup_{n\to\infty} \frac{\log F(n) \log\log n}{\log n} = \sup_{m\geq 1} \frac{\log f(m)}{m}. \tag{24}$$

Let $F(n) = \phi^{(e)*}(n)$, $f(a) = \phi^*(a)$, $L(m) = (\log f(m))/m$. Here $L(1) = L(2) = 0$, $L(3) = (\log 2)/3 \approx 0.231$, $L(4) = (\log 3)/4 \approx 0.274$, $L(5) = (\log 4)/5 \approx 0.277$, $L(6) = (\log 5)/6 \approx 0.268$, $L(7) = (\log 6)/7 \approx 0.255$, and $L(m) \leq (\log m)/m \leq (\log 8)/8 \approx 0.259$ for $m \geq 8$, using that $(\log m)/m$ is decreasing. This proves the result. ■

(23) holds also for the function $\phi^{(e)}$ instead of $\phi^{(e)*}$, cf. [19].

These results show that the asymptotic behavior of the functions $\tau^{(e)*}$, $\sigma^{(e)*}$, $\mu^{(e)*}$ and $\phi^{(e)*}$ is very close to those of the functions $\tau^{(e)}$, $\sigma^{(e)}$, $\mu^{(e)}$ and $\phi^{(e)}$.

This is confirmed also by the next result.

Theorem 5.

$$\begin{aligned} &\sum_{n\leq x} \frac{\tau^{(e)*}(n)}{\tau^{(e)}(n)} = \\ &= x \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)}/\tau(a) - 2^{\omega(a-1)}/\tau(a-1)}{p^a}\right) + \mathcal{O}\left(x^{1/4}\log x\right). \end{aligned} \tag{25}$$

A similar asymptotic formula, with the same error term, is valid also for the quotients $\sigma^{(e)*}(n)/\sigma^{(e)}(n)$ *and* $\phi^{(e)}(n)/\phi^{(e)*}(n)$ (*in the reverse order for the last one*).

Proof. This follows from the following general result, which may be known. Let g be a complex valued multiplicative function such that $|g(n)| \leq 1$ for every $n \geq 1$ and $g(p) = g(p^2) = g(p^3) = 1$ for every prime p. Then

$$\sum_{n\leq x} g(n) = x \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{g(p^a) - g(p^{a-1})}{p^a}\right) + \mathcal{O}\left(x^{1/4}\log x\right). \tag{26}$$

In order to obtain (26), which is similar to [20, Th. 1], let $h = g*\mu$ in terms of the Dirichlet convolution. Then h is multiplicative, $h(p) = h(p^2) = h(p^3) = 0$, $h(p^a) = g(p^a) - g(p^{a-1})$ and $|h(p^a)| \leq 2$ for every prime p and every $a \geq 4$.

Hence $|h(n)| \le \ell_4(n)2^{\omega(n)}$ for every $n \ge 1$, where $\ell_4(n)$ stands for the characteristic function of the 4-full integers. Note that

$$\ell_4(n)2^{\omega(n)} = \sum_{d^4 e = n} \tau(d)v(e), \tag{27}$$

where the function v is given by

$$\sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^{5s}} + \frac{2}{p^{6s}} + \frac{2}{p^{7s}} - \frac{1}{p^{8s}} - \frac{2}{p^{9s}} - \frac{2}{p^{10s}} - \frac{2}{p^{11s}}\right), \tag{28}$$

absolutely convergent for $\operatorname{Re} s > 1/5$. We obtain (26) by usual estimates, cf. the proof of [20, Th. 1]. ■

Note also, that $\mu^{(e)}(n)/\mu^{(e)*}(n) = |\mu^{(e)}(n)|$ is the characteristic function of the e-squarefree integers n. Asymptotic formulae for $|\mu^{(e)}(n)|$ were given in [24, Th. 2], [20, Th. 3].

4. Exponential unitary perfect numbers

We call an integer n exponential unitary perfect (e-unitary perfect) if $\sigma^{(e)*}(n) = 2n$.

If n is e-squarefree, then n is e-unitary perfect iff n is e-perfect. Consider the squarefull e-unitary perfect numbers. The first three such numbers given in Introduction, that is $36 = 2^2 \cdot 3^2$, $1\,800 = 2^3 \cdot 3^2 \cdot 5^2$ and $2\,700 = 2^2 \cdot 3^3 \cdot 5^2$ are e-squarefree, therefore also e-unitary perfect. It follows that there are infinitely many e-unitary perfect numbers.

The smallest number which is e-perfect but not e-unitary perfect is $17\,424 = 2^4 \cdot 3^2 \cdot 11^2$.

Theorem 6. *There are no odd e-unitary perfect numbers.*

Proof. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be an odd e-unitary perfect number. That is

$$\sigma^{(e)*}(p_1^{a_1}) \cdots \sigma^{(e)*}(p_r^{a_r}) = 2p_1^{a_1} \cdots p_r^{a_r}. \tag{29}$$

We can assume that $a_1, \ldots, a_r \ge 2$, i.e. n is squarefull (if $a_i = 1$ for an i, then $\sigma^{(e)*}(p_i) = p_i$ and we can simplify in (29) by p_i).

Now each $\sigma^{(e)*}(p_i^{a_i}) = \sum_{d \mid_* a_i} p_i^d$ is even, since the number of terms is $2^{\omega(a_i)}$, which is even.

From (29) we obtain that $r = 1$ and

$$\sigma^{(e)*}(p_1^{a_1}) = 2p_1^{a_1}. \tag{30}$$

Using that $a_1 \geq 2$ we have

$$2 = \frac{\sigma^{(e)*}(p_1^{a_1})}{p_1^{a_1}} \leq \frac{\sigma^{(e)}(p_1^{a_1})}{p_1^{a_1}} \leq 1 + \frac{1}{p_1} \leq 1 + \frac{1}{3} < 2, \tag{31}$$

which is a contradiction, and the proof is complete. ■

We state the following open problems.

Problem 1. Is there any e-unitary perfect number which is not e-squarefree, therefore not e-perfect?

Problem 2. Is there any e-unitary perfect number which is not divisible by 3?

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