A CHARACTERIZATION OF THE RELATIVE ENTROPIES

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Dedicated to Professor Antal Járai on his sixtieth birthday

Abstract. In this note we give a characterization of a family of relative entropies on open domain depending on a real parameter α, which is based on recursivity and semisymmetry. In cases $\alpha = 1$ and $\alpha = 0$ we use a weak regularity assumption additionally while in the other cases no regularity assumptions are made at all.

1. Introduction and preliminaries

Throughout this paper $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ will denote the sets of all positive integers, real numbers, and positive real numbers, respectively. For all $2 \leq n \in \mathbb{N}$ let

$$\Gamma_n^\circ = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; p_i \in \mathbb{R}_+, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} p_i = 1 \right\}$$

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and

$$\Gamma_n = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; p_i \geq 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} p_i = 1 \right\}.$$

Furthermore, for a fixed $\alpha \in \mathbb{R}$, define the function $D_n^\alpha(\cdot|\cdot) : \Gamma_n^\circ \times \Gamma_n^\circ \to \mathbb{R}$ by

$$D_n^\alpha(p_1, \ldots, p_n | q_1, \ldots, q_n) = -\sum_{i=1}^{n} p_i \ln_\alpha \left(\frac{q_i}{p_i} \right), \tag{1.1}$$

where

$$\ln_\alpha(x) = \begin{cases} \dfrac{x^{1-\alpha} - 1}{1 - \alpha}, & \text{if } \alpha \neq 1 \\ \ln(x), & \text{if } \alpha = 1 \end{cases} \qquad (x > 0).$$

The sequence (D_n^α) is called the Shannon relative entropy (or Kullback–Leibler entropy or Kullback's directed divergence) if $\alpha = 1$, and the Tsallis relative entropy if $\alpha \neq 1$, respectively. (D_n^1) is introduced and extensively discussed in Kullback [12] and Aczél–Daróczy [2], respectively. For $0 \leq \alpha \neq 1$, (D_n^α) was introduced and discussed in Shiino [15], Tsallis [17], and Rajagopal–Abe [14] from physical point of view, and in Furuichi–Yanagi–Kuriyama [8] and Furuichi [7] from mathematical point of view, respectively. In [7] and also in Hobson [9], several fundamental properties of (D_n^α) are listed and it is proved that some of them together determine (D_n^α), up to a constant factor.

In this note, we follow the method of the basic references [2] and Ebanks–Sahoo–Sander [6] of investigating characterization problems of information measures. We prove a characterization theorem similar to those of [9] and [7], and we point out that the regularity conditions (say, continuity) can be avoided if $\alpha \notin \{0, 1\}$, and can essentially be weakened if $\alpha \in \{0, 1\}$.

In what follows, a sequence (I_n) of real-valued functions $I_n, (n \geq 2)$ on $\Gamma_n^\circ \times \Gamma_n^\circ$ or on $\Gamma_n \times \Gamma_n$ is called a *relative information measure* on the open or closed domain, respectively. In the closed domain case, however, the expressions $\frac{0}{0+0}$, $\frac{0}{0+\ldots+0}$, 0^α, $0^{1-\alpha}$, $\ln_\alpha \frac{0}{0}$ can appear. Therefore, throughout the paper, the conventions

$$\frac{0}{0+0} = \frac{0}{0+\ldots+0} = 0^\alpha = 0^{1-\alpha} = \ln_\alpha \frac{0}{0} = 0$$

are always adapted (see also [3]).

Our characterization theorem for the Shannon and the Tsallis relative entropies will be based on the following two properties.

Definition 1.1. Let $\alpha \in \mathbb{R}$. The relative information measure (I_n) is α–*recursive* on the open or closed domain, if for any $n \geq 3$ and

$$(p_1, \ldots, p_n),\ (q_1, \ldots, q_n) \in \Gamma_n^\circ \ \text{ or } \ \Gamma_n,$$

respectively, the identity

$$I_n\left(p_1, \ldots, p_n | q_1, \ldots, q_n\right) =$$
$$= I_{n-1}\left(p_1 + p_2, p_3, \ldots, p_n | q_1 + q_2, q_3, \ldots, q_n\right) +$$
$$+ (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right)$$

holds. We say that (I_n) is 3-*semisymmetric* on the open or closed domain, if

$$I_3\left(p_1, p_2, p_3 | q_1, q_2, q_3\right) = I_3\left(p_1, p_3, p_2 | q_1, q_3, q_2\right)$$

is fulfilled for all $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \Gamma_3^\circ$ or Γ_3, respectively.

The following lemma shows how the initial element of an α–recursive relative information measure (I_n) determines (I_n) itself.

Lemma 1.2. *Let $\alpha \in \mathbb{R}$ and assume that the relative information measure (I_n) is α–recursive on the open domain and define the function $f :]0,1[^2 \to \mathbb{R}$ by*

$$f(x, y) = I_2(1 - x, x | 1 - y, y) \qquad \big(x, y \in]0, 1[\,\big).$$

Then, for all $n \geq 3$ and for arbitrary, $(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n^\circ$

$$I_n(p_1, \ldots, p_n | q_1, \ldots, q_n) =$$
$$= \sum_{i=2}^{n} (p_1 + p_2 + \ldots + p_i)^\alpha (q_1 + q_2 + \ldots + q_i)^{1-\alpha} \times$$
$$\times f\left(\frac{p_i}{p_1 + p_2 + \ldots + p_i}, \frac{q_i}{q_1 + q_2 + \ldots + q_i}\right)$$

holds.

Proof. The proof runs by induction on n. If we use the α–recursivity of (I_n) and the definition of the function f, we obtain that

$$I_3(p_1, p_2, p_3 | q_1, q_2, q_3) =$$
$$= I_2(p_1 + p_2, p_3 | q_1 + q_2, q_3) + (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} \times$$
$$\times I_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) =$$
$$= \sum_{i=2}^{3} (p_1 + \ldots + p_i)^\alpha (q_1 + \ldots + q_i)^{1-\alpha} f\left(\frac{p_i}{p_1 + \ldots p_i}, \frac{q_i}{q_1 + \ldots + q_i}\right)$$

is fulfilled for all $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \Gamma_3^\circ$, that is, the statement is true for $n = 3$. Assume now that the statement holds for some $3 < n \in \mathbb{N}$. We will prove that in this case the proposition holds also for $n + 1$. Let $(p_1, \ldots, p_{n+1}), (q_1, \ldots, q_{n+1}) \in \Gamma_{n+1}^\circ$ be arbitrary. Then the α–recursivity and the induction hypothesis together imply that

$$I_{n+1}(p_1, \ldots, p_{n+1}|q_1, \ldots, q_{n+1}) = I_n(p_1 + p_2, \ldots, p_{n+1}|q_1 + q_2, \ldots, q_{n+1})+$$

$$+(p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right) =$$

$$= \sum_{n=3}^{n+1} ((p_1 + p_2) + p_3 \ldots + p_i)^\alpha ((q_1 + q_2) + p_3 + \ldots + q_i)^{1-\alpha} \times$$

$$\times f \left(\frac{p_i}{(p_1 + p_2) + \ldots + p_i}, \frac{q_i}{(q_1 + q_2) + \ldots + q_i} \right) +$$

$$+(p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} I_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right) =$$

$$= \sum_{i=2}^{n+1} (p_1 + p_2 + \ldots + p_i)^\alpha (q_1 + q_2 + \ldots + q_i)^{1-\alpha} \cdot$$

$$\cdot f \left(\frac{p_i}{p_1 + p_2 + \ldots + p_i}, \frac{q_i}{q_1 + q_2 + \ldots + q_i} \right),$$

that is, the statement holds also for $n + 1$, which ends the proof. ■

2. The characterization

We begin with the following

Theorem 2.1. *For any $\alpha \in \mathbb{R}$ the relative entropy (D_n^α) is an α–recursive relative information measure on the open domain.*

Proof. In the proof, we will use the identities

$$\ln_\alpha(xy) = \ln_\alpha(x) + \ln_\alpha(y) + (1 - \alpha) \ln_\alpha(x) \ln_\alpha(y),$$

$$\ln_\alpha \left(\frac{1}{x} \right) = -x^{\alpha-1} \ln_\alpha(x)$$

several times, which hold for all $\alpha \in \mathbb{R}$ and $x, y \in \mathbb{R}_+$. Let $n \geq 3$ and

$$(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n^\circ$$

be arbitrary. Then

$$(p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} D_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \middle| \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right) =$$

$$= (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} \times$$

$$\times \left(-\frac{p_1}{p_1 + p_2} \ln_\alpha \left(\frac{p_1 + p_2}{q_1 + p_2} \frac{q_1}{p_1} \right) - \frac{p_2}{p_1 + p_2} \ln_\alpha \left(\frac{p_1 + p_2}{q_1 + q_2} \frac{q_2}{p_2} \right) \right) =$$

$$= (p_1 + p_2)^\alpha (q_1 + q_2)^{1-\alpha} \left(-\ln_\alpha \left(\frac{p_1 + p_2}{q_1 + q_2} \right) + \left(1 + (1 - \alpha) \ln_\alpha \left(\frac{p_1 + p_2}{q_1 + q_2} \right) \right) \times \right.$$

$$\left. \times \left(-\frac{p_1}{p_1 + p_2} \ln_\alpha \left(\frac{q_1}{p_1} \right) - \frac{p_2}{p_1 + p_2} \ln_\alpha \left(\frac{q_2}{p_2} \right) \right) \right) =$$

$$= (p_1 + p_2) \ln_\alpha \left(\frac{q_1 + q_2}{p_1 + p_2} \right) + \left[\left(\frac{q_1 + q_2}{p_1 + p_2} \right)^{1-\alpha} - (1 - \alpha) \ln_\alpha \left(\frac{q_1 + q_2}{p_1 + p_2} \right) \right] \times$$

$$\times \left[-p_1 \ln_\alpha \frac{q_1}{p_1} - p_2 \ln_\alpha \frac{q_2}{p_2} \right] =$$

$$= (p_1 + p_2) \ln_\alpha \left(\frac{q_1 + q_2}{p_1 + p_2} \right) - p_1 \ln_\alpha \left(\frac{q_1}{p_1} \right) - p_2 \ln_\alpha \left(\frac{q_2}{p_2} \right) =$$

$$= D_n(p_1, \ldots, p_n | q_1, \ldots, q_n) - D_{n-1}(p_1 + p_2, \ldots, p_n | q_1 + q_2 + \ldots, q_n).$$

Therefore the relative entropy (D_n^α) is α–recursive, indeed. ■

Obviously (D_n^α) is 3-semisymmetric, and for arbitrary $\gamma \in \mathbb{R}$, (γD_n^α) is α–recursive and 3-semisymmetric, as well. Before dealing with the converse we need two lemmas about *logarithmic* functions. A function $\ell : \mathbb{R}_+ \to \mathbb{R}$ is logarithmic if $\ell(xy) = \ell(x) + \ell(y)$ for all $x, y \in \mathbb{R}_+$. If a logarithmic function ℓ is bounded above or below on a set of positive Lebesgue measure then $\ell(x) = c \ln(x)$ for all $x \in \mathbb{R}_+$ with some $c \in \mathbb{R}$ (see [11], Theorem 5 and Theorem 8 on pages 311, 312). The concept of *real derivation* will also be needed. The function $d : \mathbb{R} \to \mathbb{R}$ is a real derivation if it is *additive*, i.e. $d(x+y) = d(x)+d(y)$ for all $x, y \in \mathbb{R}$, and satisfies the functional equation $d(xy) = xd(y) + yd(x)$ for all $x, y \in \mathbb{R}$. It is somewhat surprising that there are non-identically zero real derivations (see [11], Theorem 2 on page 352). If d is a real derivation then the function $x \mapsto \frac{d(x)}{x}$, $x \in \mathbb{R}_+$ is logarithmic. Therefore it is easy to see that the real derivation is identically zero if it is bounded above or below on a set of positive Lebesgue measure.

Lemma 2.2. *Suppose that the logarithmic function $\ell : \mathbb{R}_+ \to \mathbb{R}$ satisfies the equality*

$$x\ell(x) + (1-x)\ell(1-x) = 0 \qquad \big(x \in]0,1[\big). \tag{2.1}$$

Then there exists a real derivation $d : \mathbb{R} \to \mathbb{R}$ such that

$$x\ell(x) = d(x) \qquad (x \in \mathbb{R}_+). \tag{2.2}$$

Proof. Let $x, y \in \mathbb{R}_+$. Then, by (2.1) and by using the properties of the logarithmic function, we have that

$$\begin{aligned} 0 = \frac{x}{x+y}\ell\left(\frac{x}{x+y}\right) + \frac{y}{x+y}\ell\left(\frac{y}{x+y}\right) = \\ = \frac{x}{x+y}\big(\ell(x) - \ell(x+y)\big) + \frac{y}{x+y}\big(\ell(y) - \ell(x+y)\big) = \\ = \frac{1}{x+y}\big(x\ell(x) + y\ell(y) - (x+y)\ell(x+y)\big). \end{aligned}$$

This shows that the function $x \mapsto x\ell(x)$, $x \in \mathbb{R}_+$ is additive on $\mathbb{R}_+$. Hence, by the well-known extension theorem (see e.g. [11], Theorem 1 on page 471), there exists an additive function $d : \mathbb{R} \to \mathbb{R}$ such that (2.2) holds. Since ℓ is logarithmic, this implies that $d(xy) = xd(y) + yd(x)$ holds for all $x, y \in \mathbb{R}_+$. On the other hand, d is odd. Therefore this equation holds also for all $x, y \in \mathbb{R}$, that is, d is a real derivation. ■

Lemma 2.3. *Suppose that $\ell : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function and the function g_0 defined on the interval $]0,1[$ by*

$$g_0(x) = x\ell(x) + (1-x)\ell(1-x)$$

is bounded on a set of positive Lebesque measure. Then there exist a real number β and a real derivation $d : \mathbb{R} \to \mathbb{R}$ such that

$$x\ell(x) + \beta x \ln(x) = d(x) \qquad (x \in \mathbb{R}_+). \tag{2.3}$$

Proof. Define the function g on the interval $[0,1]$ by $g(0) = g(1) = 0$, and for $x \in]0,1[$ by

$$g(x) = \begin{cases} -\dfrac{g_0(x)}{\ell(2)}, & \text{if } \ell(2) \neq 0 \\ g_0(x) - x\log_2(x) - (1-x)\log_2(1-x), & \text{if } \ell(2) = 0. \end{cases}$$

Then g is a symmetric information function (see [2], (3.5.33) Theorem on page 100) which, by our assumption, is bounded on a set of positive Lebesgue measure. Therefore, applying a theorem of Diderrich [5], we obtain that

$$g(x) = -x\log_2(x) - (1-x)\log_2(1-x) \qquad \big(x \in]0,1[\big).$$

For a short proof of Diderrich's theorem see also [13] in which an idea of Járai [10] proved to be very efficient. Taking into consideration the definition of g and applying Lemma 2.2, we get (2.3) with suitable $\beta \in \mathbb{R}$. ■

Now we are ready to prove our main result.

Theorem 2.4. *Let $\alpha \in \mathbb{R}$, (I_n) be an α-recursive and 3-semisymmetric relative information measure on the open domain, and*

$$f(x,y) = I_2(1-x, x|1-y, y) \qquad \big(x, y \in]0,1[\big).$$

Furthermore, suppose that

$$I_2(p_1, p_2|p_1, p_2) = 0 \qquad \big((p_1, p_2) \in \Gamma_2\big). \tag{2.4}$$

If $\alpha \notin \{0,1\}$ then $(I_n) = (\gamma D_n^\alpha)$ for some $\gamma \in \mathbb{R}$.

If $\alpha = 1$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot, v)$ is bounded on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded above or below on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^1)$ for some $\gamma \in \mathbb{R}$.

And finally, if $\alpha = 0$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot, v)$ is bounded above or below on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^0)$ for some $\gamma \in \mathbb{R}$.

Proof. Applying Theorem 4.2.3. on page 87 of [6] with $M(x,y) = x^\alpha y^{1-\alpha}$, $x, y \in \mathbb{R}_+$, and taking into consideration Lemma 1.2.12. on page 16 of [6], (see also [1]), we have that

$$I_n(p_1, \dots, p_n|q_1, \dots, q_n) = bp_1^\alpha q_1^{1-\alpha} + c\sum_{i=2}^{n} p_i^\alpha q_i^{1-\alpha} - b \tag{2.5}$$

in case $\alpha \notin \{0,1\}$,

$$I_n(p_1, \dots, p_n|q_1, \dots, q_n) = \sum_{i=1}^{n} p_i(\ell_1(p_i) + \ell_2(q_i)) + c(1-p_1) \tag{2.6}$$

in case $\alpha = 1$, and

$$I_n(p_1, \ldots, p_n | q_1, \ldots, q_n) = \sum_{i=1}^{n} q_i(\ell_1(p_i) + \ell_2(q_i)) + c(1 - q_1) \tag{2.7}$$

in case $\alpha = 0$ for all $n \geq 2, (p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n^\circ$ with some $b, c \in \mathbb{R}$ and logarithmic functions $\ell_1, \ell_2 : \mathbb{R}_+ \to \mathbb{R}$.

Now we utilize our further conditions on (I_n). In case $\alpha \notin \{0, 1\}$, (2.5) with $n = 2$ and (2.4) imply that $0 = bp_1 + cp_2 - b$ for all $(p_1, p_2) \in \Gamma_2$ whence $b = c$ follows. Thus, by (2.5), we obtain that $(I_n) = (\gamma D_n^\alpha)$ with $\gamma = (\alpha - 1)^{-1}$. In case $\alpha = 1$, (2.6) with $n = 2$ and (2.4) imply that

$$0 = p_1\ell(p_1) + p_2\ell(p_2) + c(1 - p_1) \qquad \big((p_1, p_2) \in \Gamma_2\big),$$

where $\ell = \ell_1 + \ell_2$. Therefore $c = 0$, and, by Lemma 2.2 we get that $x\ell_2(x) = -x\ell_1(x) + d_1(x)$ for all $x \in \mathbb{R}_+$ and for some real derivation $d_1 : \mathbb{R} \to \mathbb{R}$. Thus

$$f(x, y) = x\ell_1\left(\frac{x}{y}\right) + (1 - x)\ell_1\left(\frac{1 - x}{1 - y}\right) + \left(\frac{x}{y} - \frac{1 - x}{1 - y}\right) d_1(y) \quad \big(x, y \in]0, 1[\big).$$

Since the function $f(\cdot, v)$ is bounded on a set of positive Lebesque measure, we get that the function $x \mapsto x\ell_1(x) + (1 - x)\ell_1(1 - x)$, $x \in]0, 1[$ has the same property. Thus, by Lemma 2.3,

$$x\ell_1(x) + \beta x \ln(x) = d_2(x) \qquad (x \in \mathbb{R}_+)$$

for some $\beta \in \mathbb{R}$ and derivation $d_2 : \mathbb{R} \to \mathbb{R}$. Hence

$$f(x, y) = -\beta x \ln\left(\frac{x}{y}\right) - \beta(1 - x) \ln\left(\frac{1 - x}{1 - y}\right) - \left(\frac{x}{y} - \frac{1 - x}{1 - y}\right)(d_2(y) - d_1(y))$$
$$\big(x, y \in]0, 1[\big).$$

$f(u, \cdot)$ is bounded above or below on a set of positive Lebesgue measure for some $u \in]0, 1[$ thus the derivation $d_2 - d_1$ has the same property, so $d_2 - d_1 = 0$. Therefore

$$f(x, y) = -\beta x \ln\left(\frac{x}{y}\right) - \beta(1 - x) \ln\left(\frac{1 - x}{1 - y}\right) \qquad \big(x, y \in]0, 1[\big)$$

and the statement follows from Lemma 1.2 with a suitable $\gamma \in \mathbb{R}$. The case $\alpha = 0$ can be handled similarly by interchanging the role of the distributions $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ and of the logarithmic functions ℓ_1 and ℓ_2, respectively. ■

3. Connections to known characterizations

In this section we discuss the connection between our characterization theorem and other statements known from the literature in this subject. Here we deal especially with the results of Hobson [9] and Furuichi [7] which were the main motivations of our paper. They considered the relative information measures on closed domain thus the comparison is not obvious.

We begin with some definitions.

Definition 3.1. The relative information measure (I_n) on the closed domain is said to be *expansible*, if

$$I_{n+1}(p_1, \ldots, p_n, 0 | q_1, \ldots, q_n, 0) = I_n(p_1, \ldots, p_n | q_1, \ldots, q_n)$$

is satisfied for all $n \geq 2$ and $(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \Gamma_n$, and it is called *decisive*, if $I_2(1, 0|1, 0) = 0$. Let $\alpha \in \mathbb{R}$ be arbitrarily fixed. We say that the relative information measure (I_n) satisfies the *generalized additivity* on the closed (resp. open) domain if for all $n, m \geq 2$ and for arbitrary

$$(p_{1,1}, \ldots, p_{1,m}, \ldots, p_{n,1}, \ldots, p_{n,m}),\ (q_{1,1}, \ldots, q_{1,m}, \ldots, q_{n,1}, \ldots, q_{n,m}) \in \\ \in \Gamma_{nm} \quad (\text{or } \Gamma^{\circ}_{nm})$$

$$I_{nm}(p_{1,1}, \ldots, p_{1,m}, \ldots, p_{n,1}, \ldots, p_{n,m} | q_{1,1}, \ldots, q_{1,m}, \ldots, q_{n,1}, \ldots, q_{n,m}) = \\ = I_n(P_1, \ldots, P_n | Q_1, \ldots Q_n) + \sum_{i=1}^{n} P_i^{\alpha} Q_i^{1-\alpha} I_m\left(\frac{p_{i,1}}{P_i}, \ldots, \frac{p_{i,m}}{P_i} | \frac{q_{i,1}}{Q_i}, \ldots, \frac{q_{i,m}}{Q_i}\right)$$

is fulfilled, where $P_i = \sum_{j=1}^{m} p_{i,j}$ and $Q_i = \sum_{j=1}^{m} q_{i,j}$, $i = 1, \ldots, n$.

A lengthy but simple calculation shows that the relative information measure (D_n^{α}) fulfills all of the above listed criteria. As well as Hobson [9] and Furuichi [7], we would like to investigate the converse direction. More precisely, the question is whether the generalized additivity property determines (D_n^{α}) up to a multiplicative constant. In general this is not true. Let us observe that in case we consider the generalized additivity on the open domain Γ_n° then this property is insignificant for I_n if n is a prime. Nevertheless, on the closed domain this property is well–treatable. In this case we can prove the following.

Lemma 3.2. *If the relative information measure (I_n) on the closed domain is expansible and satisfies the general additivity property with a certain $\alpha \in \mathbb{R}$, then it is also decisive and α–recursive.*

Proof. Firstly, we will show that the generalized additivity and the expansibility imply that the relative information measure (I_n) is decisive. Indeed, if we use the generalized additivity with the choice $n = m = 2$ and $(p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4) = (1, 0, 0, 0)$, then we get that

$$I_4(1, 0, 0, 0|1, 0, 0, 0) = I_2(1, 0|1, 0) + I_2(1, 0|1, 0)$$

holds. On the other hand, (I_n) is expansible, therefore

$$I_4(1, 0, 0, 0|1, 0, 0, 0) = I_2(1, 0|1, 0).$$

Thus $I_2(1, 0|1, 0) = 0$ follows, so (I_n) is decisive.

Now we will prove the α–recursivity of (I_n).
Let $(r_1, \ldots, r_n), (s_1, \ldots, s_n) \in \Gamma_n$ and use the generalized additivity with the following substitution

$$p_{1,1} = r_1, \quad p_{1,2} = r_2, \quad p_{i,1} = r_{i+1}, i = 2, \ldots, n-1, \quad p_{i,j} = 0 \quad \text{otherwise}$$

and

$$q_{1,1} = s_1, \quad q_{1,2} = s_2, \quad q_{i,1} = s_{i+1}, i = 2, \ldots, n-1, \quad q_{i,j} = 0 \quad \text{otherwise}$$

to derive

$$I_{nm}(r_1, r_2, 0, \ldots, 0, r_3, 0, \ldots, 0, r_n, 0, \ldots, 0|s_1, s_2, 0, \ldots, 0, s_3, 0, \ldots, 0, s_n, 0, \ldots, 0) =$$

$$= I_n(r_1 + r_2, r_3, \ldots, r_n, 0|s_1 + s_2, s_3, \ldots, s_n, 0)+$$

$$+(r_1 + r_2)^{\alpha}(s_1 + s_2)^{1-\alpha} I_2 \left(\frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2} \middle| \frac{s_1}{s_1 + s_2}, \frac{s_2}{s_1 + s_2} \right) +$$

$$+ \sum_{j=3}^{n} r_j^{\alpha} q_j^{1-\alpha} I_m(1, 0, \ldots, 0|1, 0, \ldots, 0).$$

After using that (I_n) is expansible and decisive, we obtain the α–recursivity. ∎

In view of Theorem 2.4. and Lemma 3.2. the following characterization theorem follows easily.

Theorem 3.3. *Let* $\alpha \in \mathbb{R}$, (I_n) *be an expansible and* 3*-semisymmetric relative information measure on the closed domain which also satisfies the generalized additivity property on* Γ_n *with the parameter* α *and let* $f(x, y) = I_2(1 - x, x|1 - y, y)$, $x, y \in]0, 1[$. *Additionally, suppose that*

$$\text{(3.1)} \qquad I_2(p_1, p_2|p_1, p_2) = 0. \qquad \big((p_1, p_2) \in \Gamma_2\big)$$

If $\alpha \notin \{0,1\}$ then $(I_n) = (\gamma D_n^\alpha)$ for some $\gamma \in \mathbb{R}$.

If $\alpha = 1$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot,v)$ is bounded on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded above or below on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^1)$ for some $\gamma \in \mathbb{R}$.

And finally, if $\alpha = 0$ and there exists a point $(u,v) \in]0,1[^2$ such that the function $f(\cdot,v)$ is bounded above or below on a set of positive Lebesgue measure and the function $f(u,\cdot)$ is bounded on a set of positive Lebesgue measure then $(I_n) = (\gamma D_n^0)$ for some $\gamma \in \mathbb{R}$.

Finally, we remark that the essence of Theorems 2.4. and 3.3. is that, in case $\alpha \notin \{0,1\}$, the algebraic properties listed in these theorems determine the information measure (D_n^α) up to a multiplicative constant *without any regularity assumption*. Furthermore, if $\alpha \in \{0,1\}$, then the mentioned algebraic properties with a really mild regularity condition determine (D_n^α) up to a multiplicative constant.

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