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MEAN VALUES OF MULTIPLICATIVE FUNCTIONS ON THE SET OF $\mathcal{P}_k + 1$, WHERE $\mathcal{P}_k$ RUNS OVER THE INTEGERS HAVING k DISTINCT PRIME FACTORS

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Dedicated to the 60th anniversary of Professor Antal Járai

Abstract. We investigate the limit behaviour of

$$\sum_{\substack{n \le x \\ n \in \mathcal{P}_k}} g(n+1)$$

as x tends to infinity where g is multiplicative with values in the unit disc and $\mathcal{P}_k$ runs over the integers having k distinct prime factors. We let k vary in the range $2 \le k \le \epsilon(x) \log\log x$ where $\epsilon(x)$ is an arbitrary function tending to zero as x tends to infinity.

Throughout this work n denotes a positive integer and $P(n)$, $p(n)$ denote the largest and the smallest prime factors of n, respectively. p, q with or without suffixes will always denote prime numbers. As usual, the number of primes up to x will be denoted by $\pi(x)$, and $\log_k x := \log(\log_{k-1} x)$ for all positive integers k where $\log_1 x = \log x$ means the natural logarithm of x. If

$$(1) \qquad n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}, \qquad p_1 < p_2 < \ldots < p_k, \quad r_i, i = 1, \ldots, k$$

are positive integers, p_i, $i = 1, \ldots, k$ are distinct primes then let $\omega(n) := k$. A typical integer n for which $\omega(n) = k$ will be denoted by π_k. We denote the set of integers having k distinct prime factors with $\mathcal{P}_k$, that is

$$\mathcal{P}_k := \{\pi_k \in \mathbb{N}\}.$$

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The set of integers in $\mathcal{P}_k$ up to x is denoted by $\mathcal{P}_k(x)$. We introduce the counting function for the set $\mathcal{P}_k$ in arithmetic progressions. If $(d, l) = 1$ then let

$$\pi_k(x, d, l) = \sum_{\substack{\pi_k \le x \\ \pi_k \equiv l \pmod d}} 1.$$

In the special case $d = l = 1$ we use $\pi_k(x)$ instead of $\pi_k(x, 1, 1)$.

An arithmetical function $g : \mathbb{N} \to \mathbb{C}$ is said to be *multiplicative* if $g(nm) = g(n)g(m)$ holds for all integers n, m with $(n, m) = 1$. It is called *additive* if $g(nm) = g(n)+g(m)$ for $(n, m) = 1$ and is called *strongly additive* if additionally $g(p^\alpha) = g(p)$ holds for all p and $\alpha \in \mathbb{N}$.

In the middle of the twentieth century Delange did some pioneering work concerning mean value estimations for multiplicative functions on the set $\mathbb{N}$. One of his results was the following (See [2])

Theorem (Delange)**.** *Let g be a multiplicative function with $|g(n)| \le 1$, satisfying*

$$\sum_p \frac{1 - \operatorname{Re} g(p)}{p} < \infty.$$

Then

$$\frac{1}{x} \sum_{n \le x} g(n) = \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m \ge 1} \frac{g(p^m)}{p^m}\right) + o(1)$$

as x tends to infinity.

Although this result provides sufficient condition for multiplicative functions to have zero mean value, the full description of such multiplicative functions was given by Wirsing [12] for real and by Halász [4] for complex multiplicative functions of modulus ≤ 1. The result of Halász extends Delange's theorem in the following way:

Theorem (Delange, Wirsing, Halász)**.** *Let g be a multiplicative function with $|g(n)| \le 1$, satisfying*

$$\sum_p \frac{1 - \operatorname{Re} g(p)p^{-i\tau}}{p} < \infty$$

for some real τ. Then

$$\frac{1}{x} \sum_{n \le x} g(n) = \frac{x^{i\tau}}{1 + i\tau} \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m \ge 1} \frac{g(p^m)}{p^{m(1+i\tau)}}\right) + o(1)$$

as x tends to infinity. On the other hand, if there is no such τ then

$$\frac{1}{x}\sum_{n\le x} g(n) = o(1) \quad (x\to\infty).$$

Kátai in [7, 8] began to investigate the mean behaviour of multiplicative functions on the set of shifted primes. Through the contribution of Hildebrand [6] and Timofeev [11] it turned out that the situation is basically different from the case of the whole set of natural numbers. Their result is

Theorem (Kátai, Hildebrand, Timofeev)**.** *Let g be a multiplicative function with $|g(n)| \le 1$ and suppose that there are a real τ and a primitive character χ_d modulo d for some modulus d such that*

$$\sum_p \frac{1-\operatorname{Re}\chi_d(p)f(p)p^{-i\tau}}{p}$$

converges. Then

$$\frac{1}{\pi(x)}\sum_{n\le x} f(p+1) = \frac{\mu(d)}{\varphi(d)}\,\frac{x^{i\tau}}{1+i\tau}\times$$

$$\times\prod_{\substack{p\le x\\ p\nmid d}}\left(1+\sum_{r\ge1}\frac{\chi_d(p^r)f(p^r)p^{-ri\tau}-\chi_d(p^{r-1})f(p^{r-1})p^{-(r-1)i\tau}}{\varphi(p^r)}\right)+o(1)$$

as $x\to\infty$, which is not necessarily $o(1)$ as x tends to infinity, if χ_d is a real character.

The main result of this paper is

Theorem 1. *Let $g(n)$ be a multiplicative function of modulus one, such that there are a primitive character χ (mod d) for some fixed d and a real τ such that*

$$\sum_p \frac{1-\operatorname{Re}\chi(p)g(p)p^{-i\tau}}{p}$$

converges. Let furthermore $\epsilon(x)$ be an arbitrary function tending to zero as x tends to infinity. Then

$$\pi_k(x)^{-1}\sum_{\substack{n\le x\\ \omega(n)=k}} g(n+1) =$$

$$= \frac{x^{i\tau}}{1+i\tau}\,\frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le x\\ p\nmid d}}\left(1-\frac{1}{p-1}+\sum_{\alpha\ge1}\frac{g(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right)+o(1) \qquad (x\to\infty)$$

uniformly for all k, if $1\le k\le \epsilon(x)\log\log x$.

We will use the method of [3] since as we deduce the results from the analogoue for $D\mathcal{P}+1$ where $\mathcal{P}$ denotes the set of primes.

Let

$$M(x,f,D) := \sum_{Dp+1\le x} f(Dp+1).$$

Theorem 2. *Let $f(n)$ be a multiplicative function of modulus 1. Let furthermore d be a positive integer. Suppose that there is a real τ such that the series*

$$\sum_p \frac{|\chi(p)f(p)p^{i\tau}-1|^2}{p} \tag{2}$$

converges for some primitive character $\chi \pmod d$. Let $0<\epsilon<1/2$. Then

$$\left(\pi\left(\frac{x-1}{D}\right)\right)^{-1} M(x,f,D) =$$
$$= \frac{x^{i\tau}}{1+i\tau}\frac{\mu(d)}{\varphi(d)} \prod_{\substack{p\le x\\ p\nmid dD}} \left(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right) + o(1) \quad (x\to\infty)$$

holds uniformly for all $x>2$ and $D\le x^{1/2-\epsilon}$ with $(d,D)=1$.

As an application of Theorem 2 we are able to analyze the mean behavior of multiplicative functions on the set $\mathcal{P}_k+1$ in some cases. We need the following

Lemma 1. *Let $\epsilon(x)\to 0$ as $x\to\infty$. Then there exist sequences $y_x\to\infty$, $\delta_x\to 0$ as $x\to\infty$ such that*

$$P(n) > x^{1-\delta_x}, \qquad y_x < p(n), \qquad n \text{ is square-free} \tag{3}$$

hold for all but $o(\pi_k(x))$ elements of $\mathcal{P}_k(x)$, uniformly for all

$$2\le k\le \epsilon(x)\log\log x \qquad \text{as } x\to\infty.$$

Proof. The following sets have zero relative density in $\mathcal{P}_k$.

1. If $A_1=\{n\in\mathcal{P}_k,\ n\le x:\ \exists\ p^2|n\}$, then we have

$$\#A_1 \le \sum_{\substack{p^\alpha\le x^{1/2}\\ \alpha\ge 2}} \pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha> x^{1/2}\\ \alpha\ge 2}} \frac{x}{p^\alpha} \ll \pi_k(x)\frac{k}{\log\log x}\sum_{\substack{p^\alpha\le x^{1/2}\\ \alpha\ge 2}}\frac{1}{p^\alpha} + \mathcal{O}(x^{3/4}).$$

Here we used that

$$\frac{\pi_{k-1}(x)}{\pi_k(x)} \sim \frac{k}{\log\log x}(\to 0) \quad (x\to\infty)$$

holds uniformly for $2 \le k \le \epsilon(x)\log\log x$. This is a direct consequence of the asymptotic estimation

$$\pi_k(x) = \frac{x}{\log x}\frac{\log\log^{k-1} x}{(k-1)!}\left(1+\mathcal{O}\left(\frac{1}{\log\log x}\right)\right), \tag{4}$$

which is uniform for $1 \le k \le \epsilon(x)\log\log x$ (see for example in [9]).

2. If $A_2 = \{n \in \mathcal{P}_k,\ n \le x : p(n) < y_x\}$, then we have

$$\#A_2 \le \sum_{\substack{p^\alpha \le x^{1/2}\\ p<y_x}} \pi_{k-1}\left(\frac{x}{p^\alpha}\right) + \sum_{\substack{p^\alpha > x^{1/2}\\ \alpha\ge 2}} \frac{x}{p^\alpha} \ll \pi_k(x)\frac{k}{\log\log x}\sum_{p<y_x}\frac{1}{p} + \mathcal{O}(x^{3/4}).$$

By means of these last two steps we can assume that $p(n) > y_x$, and n is square-free. Finally we have

$$\begin{aligned}
\sum_{\substack{\pi_k\le x\\ P(\pi_k)\le x^{1-\delta_x}}} 1 \ll & \sum_{\pi_k\le x^{1/2}} 1 + \sum_{\substack{x^{1/2}\le\pi_k\le x\\ P(\pi_k)\le x^{1-\delta_x}}} 1 \ll \\
\ll & x^{1/2} + \frac{1}{\log x}\sum_{\substack{x^{1/2}\le\pi_k\le x\\ P(\pi_k)\le x^{1-\delta_x}}} \log\pi_k \ll \\
\ll & \frac{1}{\log x}\sum_{p\le x^{1-\delta_x}} \pi_{k-1}\left(\frac{x}{p}\right)\log p + x^{1/2} \ll \\
\ll & \frac{x}{\log x}\frac{\log^{k-2}\log x}{(k-2)!}\sum_{p\le x^{1-\delta_x}}\frac{\log p}{p\log(x/p)} + x^{1/2} \ll \\
\ll & \frac{1}{\delta_x}\pi_k(x)\frac{k}{\log\log x}
\end{aligned}$$

and the proof is finished. ∎

Proof of Theorem 1. The case $k=1$ was proved by Kátai, Hildebrand and Timofeev, and is included in Theorem 2. Therefore we can suppose that $k\ge 2$. Let $U_k(x)$ be the set of those elements of $\mathcal{P}_k(x)$, for which (3) holds true. Let S_x be the set of those π_{k-1}, for which there exists at least one prime $p > P(\pi_{k-1})$ such that $\pi_{k-1}p \in U_k(x)$. Let $p^* = p_{\pi_{k-1}}$ be the smallest p with this property. Then $\pi_{k-1}p \in U_k(x)$ for all $p^* \le p \le \frac{x}{\pi_{k-1}}$. Using Lemma 1 we have that $\pi_{k-1} < x^{\lambda_x}$, with an appropriate $\lambda_x \to 0$, as x tends to infinity. Further,

$$P(\pi_{k-1}) < p, \qquad \text{and} \qquad p(\pi_{k-1}) > y_x,$$

where $y_x \to \infty$ as $x \to \infty$, slowly. We obtain

$$\tag{5} \begin{aligned} \sum_{\substack{n\le x\\ \omega(n)=k}} g(n+1) &= \sum_{\pi_{k-1}\in S_x} \sum_{p^*_{\pi_{k-1}}\le p\le \frac{x}{\pi_{k-1}}} g(\pi_{k-1}p+1)+o(\pi_k(x)) = \\ &= \sum_{\pi_{k-1}\in S_x} M(g,x,\pi_{k-1}) - \sum_{\pi_{k-1}\in S_x} \sum_{p\le p^*_{\pi_{k-1}}} g(\pi_{k-1}p+1)+o(\pi_k(x)) \end{aligned}$$

as $x \to \infty$.

Let

$$\psi(x,D) := \frac{x^{i\tau}}{1+i\tau}\frac{\mu(d)}{\varphi(d)} \prod_{\substack{p\le x\\ p\nmid dD}} \left(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right).$$

Note that using Lemma 1 we have $y_x \le p(\pi_{k-1})$, therefore in our case π_{k-1} and d are coprimes for large x. Furthermore,

$$\tag{6} \sum_{\pi_{k-1}\in S_x} \pi(p^*_{\pi_{k-1}}) \ll x^{1/2} + \sum_{\pi_{k-1}\in S_x} \sum_{P(\pi_{k-1})<p<p^*_{\pi_{k-1}}} 1$$

which, by the definition of S_x, equals $o(\pi_k(x))$ as x tends to infinity. Thus, the second sum on the most right hand side of (5) is $o(\pi_k(x))$. For the estimation of the first sum here we apply Theorem 2 and we deduce

$$\sum_{\substack{n\le x\\ \omega(n)=k}} g(n+1) = \sum_{\pi_{k-1}\in S_x} \psi(x,\pi_{k-1})\pi(\frac{x}{\pi_{k-1}})+o(\pi_k(x)) \quad (x\to\infty).$$

Defining $K(x,D)$ by the identity

$$\psi(x,1)=\psi(x,D)K(x,D),$$

such that

$$K(x,D) = \prod_{\substack{p\le x\\ p\mid D}} \left(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\right)$$

holds, we have that the left hand side of (5) equals

$$\psi(x,1)\sum_{\pi_{k-1}\in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) +$$

$$+\sum_{\pi_{k-1}\in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right)\psi(x,\pi_{k-1})[1-K(x,\pi_{k-1})]+o(\pi_k(x)) \quad (x\to\infty).$$

Since $y_x \le p(\pi_{k-1})$, and since

$$K(x, \pi_{k-1}) = \exp\Bigg[\sum_{\substack{p \le x \\ p | \pi_{k-1}}} \frac{f(p^\alpha)\chi(p^\alpha)p^{i\tau} - 1}{p} + \mathcal{O}\Bigg(\sum_{\substack{p \le x \\ p | \pi_{k-1}}} \frac{1}{p^2} \Bigg) \Bigg],$$

the right hand side of (5) equals

$$\psi(x,1) \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + o(1) \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + o(\pi_k(x)) \quad (x \to \infty).$$

By the same argument as in the estimation of (5) and then using (6) again we obtain

$$\pi_k^{-1}(x) \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \to 1 \quad (x \to \infty)$$

and the assertion follows. ∎

In order to show Theorem 2 we need an analogoue of the Turán–Kubilius inequality.

Lemma 2. *Let $0 \le \epsilon < 1$ and let $0 < \theta_x$ be an arbitrary sequence tending to zero as x tends to infinity. Let D be a positive integer, and let $x \ge 2D$. Let h be a real strongly additive function and*

$$h_x(n) = \sum_{\substack{p^\alpha || n \\ p \le (\frac{x-1}{D})^{1-\theta_x}}} h(p).$$

Then

$$\text{(7)} \qquad \frac{1}{\pi(\frac{x-1}{D})} \sum_{p \le (x-1)/D} \Bigg| h_x(Dp+1) - \sum_{\substack{q \le x \\ q \nmid D}} \frac{h(q)}{\varphi(q)} \Bigg|^2 \ll \frac{1}{\theta_x} \sum_{q \le x} \frac{|h(q)|^2}{q}$$

uniformly for all x and all $D \le x^\epsilon$.

Proof. With $x_D := (x-1)/D$ let

$$h_{1,x}(n) := \sum_{\substack{p^\alpha || n \\ p \le x_D^{1/8}}} h(p) \quad \text{and} \quad h_{2,x}(n) := \sum_{\substack{p^\alpha || n \\ x_D^{1/8} < p \le x_D^{1-\theta_x}}} h(p).$$

Further, define

$$A(y) := \sum_{\substack{p \le y \\ q \nmid D}} \frac{h(p)}{\varphi(p)} \quad \text{and} \quad B^2(y) := \sum_{p \le y} \frac{|h(p)|^2}{p}.$$

The left hand side of (7) is $\ll \Sigma_1+\Sigma_2+\Sigma_3$, where

$$\Sigma_1=\frac{1}{\pi(x_D)}\sum_{p\le x_D}|h_{1,x}(Dp+1)-A(x_D^{1/8})|^2,$$
$$\Sigma_2=\frac{1}{\pi(x_D)}\sum_{p\le x_D}|h_{2,x}(Dp+1)|^2,$$
$$\Sigma_3=\frac{1}{\pi(x_D)}\sum_{p\le x_D}|A(x)-A(x_D^{1/8})|^2.$$

Using the Cauchy–Schwarz inequality we have

$$\Sigma_3\ll\Bigg(\sum_{x_D^{1/8}\le p\le x}\frac{1}{p}\Bigg)\Bigg(\sum_{x_D^{1/8}\le p\le x}\frac{|h(p)|^2}{p}\Bigg)\ll\sum_{p\le x}\frac{|h(p)|^2}{p}.$$

In order to estimate Σ_2 note that a positive integer, $n\le x$, can have at most a bounded number of distinct prime divisors $q>x_D^{1/8}$. Thus, using the Brun–Titchmarsh inequality (Theorem I.4.9 in [10]) we deduce

$$\Sigma_2=\frac{1}{\pi(x_D)}\sum_{p\le x_D}\Bigg|\sum_{q|Dp+1}h_{2,x}(q)\Bigg|^2\ll\frac{1}{\pi(x_D)}\sum_{\substack{q\le x_D^{1-\theta_x}\\ q\nmid D}}|h(q)|^2\pi(x_D,q,l_{D,q})\ll$$
$$\ll\frac{x_D}{\pi(x_D)}\sum_{q\le x_D^{1-\theta_x}}\frac{|h(q)|^2}{q\log(\frac{x_D}{q})}\ll$$
$$\ll\frac{1}{\theta_x}\sum_{q\le x_D^{1-\theta_x}}\frac{|h(q)|^2}{q}.$$

Here we used that if $Dp+1=aq$ then there exists a unique residue class $l_{D,q}$ (mod q) such that $p\equiv l_{D,q}\pmod q$ holds.

It remains to estimate Σ_1. Performing the multiplications we obtain

$$\sum_{p\le x_D}\Big|h_{1,x}(Dp+1)-A(x_D^{1/8})\Big|^2=S_1-2S_2+S_3,$$

where

$$S_1=\sum_{p\le x_D}|h_{1,x}(Dp+1)|^2,$$
$$S_2=A(x_D^{1/8})\sum_{p\le x_D}h_{1,x}(Dp+1),$$
$$S_3=A(x_D^{1/8})^2\pi(x_D).$$

Further,

$$S_1=\sum_{p\le x_D}\Big(\sum_{q|Dp+1}h_{1,x}(q)\Big)^2=\sum_{\substack{q\le x_D\\ q\nmid D}}h_{1,x}^2(q)\pi(x_D,q,l_{D,q})+ \tag{8}$$
$$+\sum_{\substack{q_1,q_2\le x_D\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}h_{1,x}(q_1)h_{1,x}(q_2)\pi(x_D,q_1q_2,l_{D,q_1q_2}).$$

Since $h_{1,x}(q)=0$ for $q>x_D^{1/8}$, the Brun–Titchmarsh theorem is applicable and we deduce that the first term on the right hand side of (8) does not exceed $c\pi(x_D)B^2(x)$.

The second term on the right hand side of (8) equals

$$\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}h_{1,x}(q_1)h_{1,x}(q_2)\frac{\pi(x_D)}{\varphi(q_1q_2)}+ \tag{9}$$
$$+\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}h_{1,x}(q_1)h_{1,x}(q_2)\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\}.$$

Let T_1,T_2 be the sums in (9). We have

$$\frac{T_1}{\pi(x_D)}=A^2(x_D^{1/8})-\sum_{\substack{q_1\le x_D^{1/8}\\ q_1\nmid D}}\frac{h_{1,x}^2(q_1)}{\varphi^2(q_1)}=A^2(x_D^{1/8})+\mathcal{O}(B^2(x)).$$

For T_2 we use the Cauchy–Schwarz inequality to obtain

$$T_2^2\ll\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}\frac{h_{1,x}^2(q_1)}{\varphi(q_1)}\frac{h_{1,x}^2(q_2)}{\varphi(q_2)}\times$$
$$\times\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}\varphi(q_1q_2)\left\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right\}^2\ll$$
$$\ll B^4(x)\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}\varphi(q_1q_2)\left\{\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right\}^2.$$

Using the Brun–Titchmarsh inequality

$$T_2^2\ll B^4(x)\pi(x_D)\sum_{\substack{q_1,q_2\le x_D^{1/8}\\ q_1\ne q_2,\ q_1\nmid D,\ q_2\nmid D}}\left|\pi(x_D,q_1q_2,l_{D,q_1q_2})-\frac{\pi(x_D)}{\varphi(q_1q_2)}\right|,$$

and an application of the Bombieri–Vinogradov theorem (Chapter 28. in [1]) shows

$$T_2 \ll B^2(x)\frac{\pi(x_D)}{\log^A x_D},$$

where $A > 0$ is an arbitrary large costant. Since by the Cauchy–Schwarz inequality we have

$$A(y) = \sum_{\substack{q\le y\\ q\nmid D}} \frac{h(q)}{\varphi(q)} \ll \left(\sum_{q\le y}\frac{h^2(q)}{q}\right)^{1/2} \log\log^{1/2} y \ll B(y)\log\log^{1/2} y,$$

for $y \ge e^2$, in a similar way as in the estimation of T_2 we deduce

$$\begin{aligned} S_2 - A^2(x_D^{1/8})\pi(x_D) \ll & A(x_D^{1/8})B(x)\frac{\pi(x_D)}{\log^A x_D} \ll \\ \ll & B^2(x)\log\log x_D \frac{\pi(x_D)}{\log^A x_D} \ll \\ \ll & B^2(x)\pi(x_D), \end{aligned}$$

and the proof is finished. ■

Lemma 3. *Let D, q be two coprime positive integers and let $(l_{D,q} =)l_D$ be the unique residue class satisfying $Dl_D \equiv 1 \pmod q$. Let further $0 < \epsilon < 1/2$ and $x_D := (x-1)/D$ whenever $x > 2$ and let $a > \frac{1-2\epsilon}{1+2\epsilon}$. Then*

$$\sum_{\substack{q > x_D^a\\ q \text{ prime},\ q\nmid D}} q\pi^2(x_D, q, l_D) \ll \pi^2(x_D) \tag{10}$$

holds uniformly for all $x > 2$ and $D \le x^{1/2-\epsilon}$. The constant implied by $\ll$ depends on a.

Proof. The sum on the left hand side of (10) equals

$$\sum_{q > x_D^a} q \sum_{\substack{a_1 q = Dp_1+1\\ a_1\le x/q}} \sum_{\substack{a_2 q = Dp_2+1\\ a_2\le x/q}} 1 \le 2x \sum_{\substack{a_1 \le x x_D^{-a}\\ (a_1,D)=1}} \frac{1}{a_1} \sum_{\substack{a_2 < a_1\\ (a_2,D)=1}} \sum_{\substack{a_1 q = Dp_1+1\le x\\ a_2 q = Dp_2+1\le x}} 1. \tag{11}$$

Denote the inner sum by $(\Sigma(a_1,a_2) =)\Sigma$. It is nonempty only if $a_1 \equiv a_2 \pmod D$. Suppose, a_1, a_2 is fixed and

$$q = Dn + l_{a_1 D}.$$

Then

$$Dp_1 + 1 = a_1 Dn + a_1 l_{a_1 D}, \quad Dp_2 = a_2 Dn + a_2 l_{a_1 D}.$$

Thus, the primes we want to count in Σ satisfy

$$q = Dn + l_{a_1 D},$$
$$p_1 = a_1 n + t_{Da_1}, \quad p_2 = a_2 n + t_{Da_2},$$

where

$$a_1 l_{a_1 D} - Dt_{Da_1} = 1 \quad \text{and} \quad a_2 l_{a_1 D} - Dt_{Da_2} = 1.$$

It follows,

$$\Sigma \ll \#\Big\{ n \leq \frac{x_D}{a_1} : q = Dn + l_{a_1 D},\ p_1 = a_1 n + t_{Da_1},\ p_2 = a_2 n + t_{Da_2} \text{ primes} \Big\}.$$

Let

$$E = Da_1 a_2 (a_1 - a_2),$$

and let $\varrho(p)$ be the number of solutions of

$$(Dn + l_{a_1 D})(a_1 n + t_{Da_1})(a_2 n + t_{Da_2}) \equiv 0 \pmod{p}.$$

Since $E \leq x_D^A$ for some appropriate $A > 0$, by Theorem 5.7 of [5]

$$\Sigma \ll \frac{x_D}{a_1 \log^3 \frac{x_D}{a_1}} \prod_p (1 - \frac{\varrho(p) - 1}{p - 1})(1 - \frac{1}{p})^{-2}.$$

Noting that $(D, a_1 a_2) = 1$ we have

$$\varrho(p) = \begin{cases} 1 & \text{if } p|D,\ p|\frac{a_1 - a_2}{D} \text{ or } p|a_1,\ p|a_2 \\ 2 & \text{if } p|D,\ p\nmid \frac{a_1 - a_2}{D} \text{ or } p|a_1 a_2,\ p\nmid (a_1, a_2) \\ 3 & \text{otherwise.} \end{cases}$$

Now, making use of the inequality $\log(1 - z) = 1 + z + \mathcal{O}(z^2)$ which holds uniformly for all real numbers $|z| \leq 1/2$ we obtain

$$\prod_p \left(1 - \frac{\varrho(p) - 1}{p - 1}\right)\left(1 - \frac{1}{p}\right)^{-2} \ll$$
$$\ll \prod_{p|D} \left(1 + \frac{1}{p}\right) \prod_{p|\frac{a_1 - a_2}{D}} \left(1 + \frac{1}{p}\right) \prod_{p|a_1} \left(1 + \frac{2}{p}\right) \prod_{p|a_2} \left(1 + \frac{2}{p}\right).$$

Thus, the right hand side of (11) is at most

$$c\frac{x^2}{D}\prod_{p|D}\left(1+\frac{1}{p}\right)\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1^2\log^3\frac{x_D}{a_1}}\prod_{p|a_1}\left(1+\frac{2}{p}\right)\times$$

$$\times\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\prod_{p|\frac{a_1-a_2}{D}}\left(1+\frac{1}{p}\right)\prod_{p|a_2}\left(1+\frac{2}{p}\right).$$

Since $|ab|\le a^2+b^2$ holds for all real a,b we deduce

$$\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\prod_{p|\frac{a_1-a_2}{D}}\left(1+\frac{1}{p}\right)\prod_{p|a_2}\left(1+\frac{2}{p}\right)\ll$$

$$\ll\sum_{\substack{a_2\le a_1\\a_1\equiv a_2 \pmod D}}\left\{\sum_{d|\frac{a_1-a_2}{D}}\frac{2^{\omega(d)}\mu^2(d)}{d}+\sum_{d|a_2}\frac{4^{\omega(d)}\mu^2(d)}{d}\right\}\ll$$

$$\ll\sum_{d\le\frac{a_1}{D}}\frac{2^{\omega(d)}\mu^2(d)}{d}\sum_{\substack{a_2\le a_1\\a_2\equiv a_1 \pmod D\\\frac{a_1-a_2}{D}\equiv 0 \pmod d}}1+\sum_{\substack{d\le a_1\\(d,D)=1}}\frac{4^{\omega(d)}\mu^2(d)}{d}\sum_{\substack{a_2\le a_1\\a_2\equiv 0 \pmod d\\a_1\equiv a_2 \pmod D}}1\ll$$

$$\ll\frac{a_1}{D}.$$

Since $a>\frac{1-2\epsilon}{1+2\epsilon}$ and $a_1\le xx_D^{-a}$ we have $\log\frac{x_D}{a_1}\gg_a\log x\gg\log x_D$. Further,

$$\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1}\prod_{p|a_1}\left(1+\frac{2}{p}\right)=\prod_{\substack{p\le xx_D^{-a}\\p\nmid D}}\left(1+\frac{1}{p}\left(1+\frac{2}{p}\right)\right)\ll$$

$$\ll\prod_{p\le xx_D^{-a}}\left(1+\frac{1}{p}\right)\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}\ll$$

$$\ll\log x_D\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}.$$

Thus, the right hand side of (11) does not exceed

$$c\frac{x_D^2}{\log^3 x_D}\prod_{p|D}\left(1+\frac{1}{p}\right)\sum_{\substack{a_1\le xx_D^{-a}\\(a_1,D)=1}}\frac{1}{a_1}\prod_{p|a_1}\left(1+\frac{2}{p}\right)\ll\pi^2(x_D),$$

which proves the assertion. ∎

Proof of Theorem 2. First suppose that $\tau = 0$. We set $r = \log\log x$, and $x_D = \frac{x-1}{D}$. Let

$$K_D(x) := \{Dp + 1 \le x : p \text{ prime}\}.$$

We have

$$\#\{n \in K_D(x) \mid \exists q^2 | n,\ q > y\} \le$$

(12) $$\le \sum_{y<q<(\frac{x-1}{D})^a} \pi\left(\frac{x-1}{D}, q^2, l_q\right) + \frac{x-1}{D} \sum_{q \ge (\frac{x-1}{D})^a} \frac{1}{q^2} = \delta(y)\pi\left(\frac{x-1}{D}\right),$$

where $\delta(y) \to 0$ $(y \to \infty)$. Let f^* be a multiplicative function defined by

$$f^*(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p \le r \\ f(p), & \text{if } r < p \le x_D^{1-\vartheta_x} \\ \overline{\chi}(p), & \text{otherwise.} \end{cases}$$

Since $\chi(q) \ne 0$ for $q > d$, there exists a function $g(q) \in [-\pi, \pi)$ such that $f(q) = \chi(q)e^{ig(q)}$. By (12)

$$\left| \sum_{Dp+1 \le x} \{f(Dp+1) - f^*(Dp+1)\} \right| \le$$

$$\sum_{\substack{Dp+1 \le x \\ \exists q^2 | Dp+1,\ q > r}} 1 + \sum_{\substack{Dp+1 \le x \\ \exists q | Dp+1,\ q > x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| \le$$

$$\le \sum_{\substack{Dp+1 \le x \\ \exists q | Dp+1,\ q > x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| + o(\pi(x_D)) \quad (x \to \infty),$$

where

$$\tilde{g}(p^\alpha) = \begin{cases} g(p), & \text{if } x_D^{1-\vartheta_x} < q, \quad \alpha = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\substack{Dp+1 \le x \\ \exists q | Dp+1,\ q > x_D^{1-\vartheta_x}}} |e^{i\tilde{g}(Dp+1)} - 1| \le \sum_{\substack{Dp+1 \le x \\ \exists q | Dp+1,\ q > x_D^{1-\vartheta_x}}} |\tilde{g}(Dp+1)|$$

$$\le \sum_{x_D^{1-\vartheta_x} < q \le x} |g(q)| \pi(x_D, q, t_D),$$

where $(t_{D,q} =)t_D$ is the unique residue class satisfying

$$Dt_D \equiv -1 \pmod q.$$

Applying the Cauchy–Scwarz inequality then using Lemma 3 we obtain

$$\sum_{x_D^{1-\vartheta_x}<q\le x} |g(q)|\pi(x_D,q,t_D) \ll$$

$$\ll \left(\sum_{x_D^{1-\vartheta_x}<q\le x} \frac{g(q)^2}{q}\right)^{1/2} \left(\sum_{x_D^{1-\vartheta_x}<q\le x} q\pi^2(x_D,q,t_D)\right)^{1/2} \ll$$

$$\ll \pi(x_D)\left(\sum_{x_D^{1-\vartheta_x}<q\le x} \frac{g(q)^2}{q}\right)^{1/2}.$$

Noting that

$$|g(q)|^2 \ll |f(q)\overline{\chi}(q)-1|^2,$$

by (2) we obtain

$$\sum_{Dp+1\le x} \{f(Dp+1)-f^*(Dp+1)\} = o(\pi(x_D)) \quad (x\to\infty). \tag{13}$$

Let f_r be a further multiplicative function defined by

$$f_r(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p\le r \\ \overline{\chi}(p), & \text{if } r<p. \end{cases}$$

Next we give an alternative representation of $M(x,f_r,D)$. It can be written as follows

(14)

$$\sum_{Dp+1\le x} f_r(Dp+1) = \sum_{\substack{m\le x+1 \\ P(m)\le r \\ (D,m)=1}} f(m) \sum_{\substack{p\le x_D \\ p\equiv l_D \pmod m \\ (\frac{Dp+1}{m},\mathcal{P}(r))=1}} \overline{\chi}\left(\frac{Dp+1}{m}\right) + Err(x,r),$$

where

$$\mathcal{P}(r) := \prod_{p\le r} p,$$

and $(l_{D,m} =)l_D$ is the unique residue class satisfying

$$Dl_D \equiv -1 \pmod m,$$

and by (12)

$$Err(x,r) \ll \sum_{\substack{Dp+1\le x\\ \exists q^2 | Dp+1,\ r<q}} 1 = o(\pi(x_D)) \quad (x\to\infty).$$

Furthermore, $(\frac{Dp+1}{m}, \mathcal{P}(r)) = 1$. Hence, $\frac{Dp+1}{m}$ is always odd and there is at most one prime p satisfying $Dp+1 = m\frac{Dp+1}{m}$ if D and m have the same parity. The contribution of these integers to the sum on the right hand side of (14) is at most

$$\sum_{\substack{m\le x\\ P(m)\le r}} 1 \ll x\exp\left(-\frac{1}{2}\frac{\log x}{\log r}\right),$$

which inequality is well known in number theory (Theorem III.5.1 in [10]). The sum over the integers $m > e^r$ on the right hand side of (14) is at most

$$\sum_{\substack{e^r\le m\le\sqrt{x}\\ P(m)\le r}} \pi(x_D, m, l_D) + \sum_{\substack{\sqrt{x}\le m\le x\\ P(m)\le r}} \frac{x_D}{m} = \Sigma_1 + \Sigma_2.$$

Using the Brun–Titchmarsh theorem we obtain

$$\begin{aligned}\Sigma_1 \ll \pi(x_D) \sum_{\substack{e^r\le m\le\sqrt{x}\\ P(m)\le r}} \frac{1}{\varphi(m)} &\ll \frac{\pi(x_D)}{r} \sum_{\substack{m\le x\\ P(m)\le r}} \frac{\log m}{\varphi(m)} \ll\\ &\ll \frac{\pi(x_D)}{r} \sum_{p\le r}\sum_{\alpha} \log p^\alpha \sum_{\substack{mp^\alpha\le x\\ P(m)\le r,\ (m,p)=1}} \frac{1}{\varphi(p^\alpha m)} \ll\\ &\ll \frac{\pi(x_D)\log r}{r} \sum_{p\le r} \frac{\log p}{p} \ll\\ &\ll \pi(x_D)\frac{\log^2 r}{r}.\end{aligned}$$

Further, using the inequality $|\log(1-y) - y| \le 2y^2$, which is valid for all real

y with $|1-y| \le 1/2$ we have

$$\Sigma_2 \ll x_D x^{-1/8} \sum_{\substack{m\le x\\ P(m)\le r}} \frac{1}{m^{3/4}} \ll x_D x^{-1/4} \prod_{p\le r}\left(1-\frac{1}{p^{3/4}}\right)^{-1} \ll$$

$$\ll x_D x^{-1/4} \exp\left(\sum_{p\le r}\frac{1}{p^{3/4}}\right) \ll$$

$$\ll x_D x^{-1/4} e^r.$$

The inner sum on the right hand side of (14) equals

$$\sum_{\substack{Dp\le x\\ Dp\equiv -1 \pmod m}} \overline{\chi}_d\left(\frac{Dp+1}{m}\right) \sum_{\delta|(\frac{Dp+1}{m},\mathcal{P}(r))} \mu(\delta) =$$

$$= \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta) \sum_{\substack{Dp\le x\\ Dp+1\equiv 0 \pmod{\delta m}}} \overline{\chi}_d\left(\frac{Dp+1}{m}\right) =$$

$$= \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta) \sum_{\substack{b=1\\ (b,d)=1}}^{d} \overline{\chi}_d(b) J(x,m,\delta,b),$$

where

$$J_m(x,m,\delta,b) := \#\left\{p \le x_D \,:\, Dp+1\equiv 0 \pmod{\delta m},\ \tfrac{Dp+1}{m}\equiv b \pmod d\right\}.$$

Note that $J_m(x,m,\delta,b) \ll 1$ for all b with $(bm-1,d)\ne 1$. There is a unique $l_\delta \pmod d$ such that $\delta l_\delta \equiv b \pmod d$, therefore

$$Dp+1 = c\delta m \quad \text{and} \quad Dp+1 = mb+tdm,$$

implies

$$Dp+1 \equiv m l_\delta \delta \pmod{m\delta d}.$$

Thus,

$$J_m(x,m,\delta,b) = \#\{p\le x_D \,:\, Dp+1\equiv m\delta l_\delta \pmod{\delta dm}\}.$$

We arrive at

$$M(x,f_r,D) = \sideset{}{'}\sum_{\substack{m\le e^r\\ P(m)\le r}} f(m) \sum_{\substack{b=1\\ (b,d)=1\\ (bm-1,d)=1}}^{d} \overline{\chi}_d(b) \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta)\pi(x_D,\delta dm, m\delta l_\delta)+$$

$$+ o(\pi(x_D)) \quad (x\to\infty), \tag{15}$$

where Σ' indicates that m and D are of opposite parity. The right hand side of (15) equals

$$\sideset{}{'}\sum_{\substack{m\le e^r\\ P(m)\le r}} f(m) \sum_{\substack{b=1\\ (b,d)=1\\ (bm-1,d)=1}}^{d} \overline{\chi}_d(b) \sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \mu(\delta)\frac{\pi(x_D)}{\varphi(\delta dm)}+$$

$$+\mathcal{O}\left(\sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \sum_{\substack{m\le e^r\\ P(m)\le r}} \left|\pi(x_D,\delta dm, m\delta l_\delta)-\frac{\pi(x_D)}{\varphi(\delta dm)}\right|\right) = M + Err_2(x,r).$$

Applying the Cauchy–Schwarz inequality and then the Brun–Titchmarsh theorem we obtain that $Err_2^2(x,r)$ is at most

$$c\left(\sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}} 4^{\omega(\delta)} \max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|\right)^2 \ll$$

$$\ll \sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}} \frac{16^{\omega(\delta)}}{\varphi(\delta)} \sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}} \varphi(\delta) \max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|^2 \ll$$

$$\ll \prod_{p\le x}\left(1+\frac{16}{p}\right)\pi(x_D) \sum_{\delta\le\frac{\sqrt{x}}{\sqrt{D}\log^A x}} \max_{(l,\delta)=1}\left|\pi(x_D,\delta,l)-\frac{\pi(x_D)}{\varphi(\delta)}\right|,$$

which by the Bombieri–Vinogradov theorem does not exceed $\frac{\pi^2(x_D)}{\log^A x}$, where $A>0$ is an arbitrary large fixed constant.

Since

$$\varphi(\delta dm) = \delta dm \prod_{p|dm}\left(1-\frac{1}{p}\right) \prod_{\substack{p|\delta\\ p\nmid dm}}\left(1-\frac{1}{p}\right) = \varphi(dm)\delta \prod_{\substack{p|\delta\\ p\nmid dm}}\left(1-\frac{1}{p}\right),$$

we have

$$\sum_{\substack{\delta|\mathcal{P}(r)\\ (\delta,Dd)=1}} \frac{\mu(\delta)}{\varphi(\delta md)} = \frac{1}{\varphi(dm)} \prod_{\substack{p\le r\\ p\nmid Dd\\ p|dm}}\left(1-\frac{1}{p}\right) \prod_{\substack{p\le r\\ p\nmid Dd\\ p\nmid dm}}\left(1-\frac{1}{p-1}\right) =$$

$$= \frac{1}{\varphi(dm)} \prod_{\substack{p\le r\\ p\nmid Dd\\ p|m}}\left(1-\frac{1}{p}\right) \prod_{\substack{p\le r\\ p\nmid Dd\\ p\nmid m}}\left(1-\frac{1}{p-1}\right).$$

Further, by the inclusion-exclusion principle and by the orthogonality relation of the Dirichlet characters we have

$$\sum_{\substack{b=1\\(b(bm-1),d)=1}}^{d} \overline{\chi}(b) = \sum_{(b,d)=1} \overline{\chi}(b) \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \sum_{\chi \pmod k} \chi_k(bm).$$

Thus,

$$\frac{1}{\pi(x_D)} M = \sum_{(b,d)=1} \overline{\chi}(b) \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \sum_{\chi \pmod k} \chi_k(b) \times \tag{16}$$
$$\times \sideset{}{'}\sum_{\substack{m\\P(m)\le r}} \frac{f(m)\chi_k(m)}{\varphi(dm)} \prod_{\substack{p\le r\\ p\nmid Dd\\ p|m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\le r\\ p\nmid Dd\\ p\nmid m}} \left(1-\frac{1}{p-1}\right) + Err_3(r),$$

where

$$Err_3(r) \ll \sum_{\substack{m>e^r\\P(m)\le r}} \frac{1}{\varphi(dm)} \prod_{\substack{p\le r\\ p\nmid Dd\\ p|m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\le r\\ p\nmid Dd\\ p\nmid m}} \left(1-\frac{1}{p-1}\right) \ll$$
$$\ll \prod_{\substack{p\le r\\ p\nmid Dd}} \left(1-\frac{1}{p}\right) \sum_{\substack{m>e^r\\P(m)\le r}} \frac{1}{\varphi(m)} \ll$$
$$\ll \frac{\log^2 r}{r}.$$

Keeping in mind that m and D has opposite parity

$$\sideset{}{'}\sum_{\substack{m\\P(m)\le r}} \frac{f(m)\chi_k(m)}{\varphi(dm)} \prod_{\substack{p\le r\\ p\nmid Dd\\ p|m}} \left(1-\frac{1}{p}\right) \prod_{\substack{p\le r\\ p\nmid Dd\\ p\nmid m}} \left(1-\frac{1}{p-1}\right) \tag{17}$$

can be written as

$$\prod_{\substack{p\le r\\ p\nmid Dd}} \left(1-\frac{1}{p-1}+\sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right) \prod_{\substack{p\le r\\ p\nmid 2D\\ p|d}} \left(1+\sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right).$$

Thus, the first term on the right hand side of (16) equals

$$\sum_{\substack{b=1\\(b,d)=1}}^{d} \frac{\overline{\chi}(b)}{\varphi(d)} \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \times$$

$$(18) \qquad \times \sum_{\chi \pmod k} \chi_k(b) \prod_{\substack{p\le r\\ p\nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right) \times$$

$$\times \prod_{\substack{p\le r\\ p|d,\ p\nmid 2D}} \left(1 + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha}\right).$$

Since the character induced by $\chi_k \cdot \overline{\chi}$ is not the principal character if $\chi_k \neq \chi$ we obtain using Dirichlet's theorem in arithmetic progressions that

$$\sum_{z\le p\le r} \frac{|1-\chi_k \cdot \overline{\chi}(p)|^2}{p} \gg \log\left(\frac{\log r}{\log z}\right) \gg \log\left(\frac{\log_3 x}{\log_4 x}\right),$$

if $z = \log_3 x$. Here we used that $\chi_k \cdot \overline{\chi}(p)$ is at most a $\varphi(d)$-th root of unity. Further,

$$|\chi_k(p) f(p) - 1|^2 \gg |1 - \overline{\chi}(p)\chi_k(p)|^2 - |1 - \overline{\chi}(p) f(p)|^2,$$

therefore

$$\sum_{z\le p\le r} \frac{|1-\chi_k(p) f(p)|^2}{p} \gg$$

$$\gg \sum_{z\le p\le r} \frac{|1-\chi_k(p)\overline{\chi}(p)|^2}{p} + \mathcal{O}\left(\sum_{z\le p\le r} \frac{|1-\overline{\chi}(p) f(p)|^2}{p}\right) \gg$$

$$\gg \log\left(\frac{\log_3 x}{\log_4 x}\right) + o(1) \quad (x\to\infty).$$

Thus,

$$\Big|\prod_{\substack{p\le r\\ p\nmid dD}} (1 - \frac{1}{p-1} + \sum_{\alpha\ge 1} \frac{f(p^\alpha)\chi_k(p^\alpha)}{p^\alpha})\Big| \ll \Big|\exp\Big(\sum_{p\le r} \frac{f(p)\chi_k(p) - 1}{p}\Big)\Big| \ll$$

$$\ll \exp\Big(-\sum_{z\le p\le r} \frac{1 - \operatorname{Re} f(p)\chi_k(p)}{p}\Big) =$$

$$= o(1) \quad (x\to\infty).$$

Putting it back into (18) we deduce

$$\frac{1}{\pi(x_D)}M(x,f_r,D)=\frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le r\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big)\times \tag{19}$$
$$\times\prod_{\substack{p\le r\\ p\mid d,\ p\nmid 2D}}\Big(1+\sum_{\alpha\ge1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big)+o(1)\quad(x\to\infty).$$

Since $\chi(p^\alpha)=0$ for all $p\mid d$, introducing the notation

$$P(y):=\prod_{\substack{p\le y\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge1}\frac{f(p^\alpha)\chi(p^\alpha)}{p^\alpha}\Big),$$

we proved that

$$\pi(x_D)^{-1}M(x,f_r,D)=\frac{\mu(d)}{\varphi(d)}P(r)+o(1)\quad(x\to\infty). \tag{20}$$

Here we note that if (2) converges for $\tau=0$ then $1\ll|P(r)|\le1$. Now we can prove that

$$\frac{\mu(d)}{\varphi(d)}P(x_D)$$

is a good approximation of the sum $M(x,f,D)$. Now

$$\Big|\pi^{-1}(x_D)M(x,f,D)-\frac{\mu(d)}{\varphi(d)}P(x_D)\Big|\le$$
$$\le\Big|\pi^{-1}(x_D)M(x,f^*,D)-\pi^{-1}(x_D)M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big|+$$
$$+\pi(x_D)^{-1}|M(x,f^*,D)-M(x,f,D)\Big|+$$
$$+\Big|\frac{\mu(d)}{\varphi(d)}P(x_D)-\pi^{-1}(x_D)M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big|,$$

therefore by (13) and by (20) we have to show that

$$\pi^{-1}(x_D)\Big|M(x,f^*,D)-M(x,f_r,D)\frac{P(x_D)}{P(r)}\Big|=o(1)\quad(x\to\infty). \tag{21}$$

We note that, if $d<r$, then

$$|f^*(p^\alpha)|=|f_r(p^\alpha)|=1.$$

Hence there is a strongly additive function $g_r^*(p) \in (-\pi, \pi]$ with

$$f_r^*(n) = f^* \cdot \overline{f}_r(n) = e^{ig_r^*(n)}.$$

We note that if

$$p \le r, \quad \text{or} \quad p > x_D^{1-\vartheta_x}, \quad \text{then} \quad g_r^*(p) = 0.$$

By Lemma 2 we have

$$\sum_{Dp+1\le x} \Big| g_r^*(Dp+1) - \sum_{\substack{q\le x_D\\ q\nmid D}} \frac{g_r^*(q)}{q} \Big|^2 \ll \frac{1}{\vartheta_x} \pi(x_D) \sum_{p\le x_D} \frac{|g_r^*(p)|^2}{p}. \tag{22}$$

Let

$$A(x) := \sum_{\substack{p\le x_D\\ p\nmid D}} \frac{g_r^*(p)}{p}.$$

We obtain that the left hand side of (21) is at most

$$\frac{c}{\pi(x_D)} \Big| \sum_{Dp+1\le x} f^*(Dp+1) - f_r(Dp+1)\frac{P(x_D)}{P(r)} \Big| \ll$$

$$\ll \frac{1}{\pi(x_D)} \sum_{Dp+1\le x} \Big| f_r^*(Dp+1) - \frac{P(x_D)}{P(r)} \Big| \ll$$

$$\ll \frac{1}{\pi(x_D)} \sum_{Dp+1\le x} \big| f_r^*(Dp+1) - \exp[iA(x)] \big| + \Big| \exp[iA(x)] - \frac{P(x_D)}{P(r)} \Big| =$$

$$= \Sigma_1' + \Sigma_2'.$$

Using the Cauchy–Schwarz inequality again we obtain

$$\Sigma_1' = \pi(x_D)^{-1} \sum_{Dp+1\le x} \Big| \exp\big[i\big(g_r^*(Dp+1) - A(x)\big)\big] - 1 \Big| \le$$

$$\le \pi(x_D)^{-1/2} \Big(\sum_{Dp+1\le x} |g_r^*(Dp+1) - A(x)|^2 \Big)^{1/2}.$$

Thus, by (22) we deduce that Σ_1 is at most

$$\left(\frac{c}{\vartheta_x} \sum_{\substack{p\le x_D\\ p\nmid D}} \frac{|g_r^*(p)|^2}{p} \right)^{1/2}.$$

Further,

$$|g_r^*(p)|^2 \ll |f_r^*(p) - 1|^2 = |f(p) - f_r(p)|^2,$$

therefore

$$\Sigma_1' \ll \left(\frac{1}{\vartheta_x} \sum_{r \le p \le x} \frac{|\chi(p) f(p) - 1|^2}{p} \right)^{1/2},$$

which according to condition (2) tends to zero as $r \to \infty$ with a suitable choice of ϑ_x.

We have to estimate Σ_2'. It can be written as

$$\left| 1 - \prod_{\substack{r < p \le x_D \\ p \nmid D}} \Big(1 - \frac{1}{p-1} + \sum_{m \ge 1} \frac{f(p^m)\chi(p^m)}{p^m} \Big) \exp\Big(-i \sum_{\substack{r < p \le x_D \\ p \nmid D}} \frac{g_r^*(p)}{p} \Big) \right|,$$

which equals

$$\left| 1 - \exp\Big[\mathcal{O}\Big(\sum_{r < p \le x_D} \frac{|f(p)\chi(p) - 1|^2}{p} + \sum_{r < p} \frac{1}{p^2} \Big) \Big] \right|,$$

which again tends to zero as $x \to \infty$, such that (21) follows. Finally we note that $x^{1-\varepsilon} < x_D$, therefore we have

$$|P(x_D) - P(x)| \ll \left| \prod_{x_D < p \le x} \left(1 + \frac{f(p)\chi(p) - 1}{p} + \mathcal{O}\Big(\frac{1}{p^2}\Big) \right) - 1 \right| =$$

$$= \left| \exp\left(\sum_{x_D < p \le x} \frac{f(p)\chi(p) - 1}{p} + \mathcal{O}\Big(\frac{1}{p^2}\Big) \right) - 1 \right|,$$

which tends to zero as $x \to \infty$ inasmuch as

$$\begin{aligned} &\left| \sum_{x_D < p \le x} \frac{f(p)\chi(p) - 1}{p} \right| \ll \\ &\ll \left(\sum_{x_D < p \le x} \frac{1}{p} \right)^{1/2} \left(\sum_{x_D < p \le x} \frac{|f(p)\chi(p) - 1|^2}{p} \right)^{1/2} = o(1) \quad (x \to \infty). \end{aligned} \tag{23}$$

We proved Theorem 2 in the case $\tau = 0$.

Now consider the case of an arbitrary τ. We proved that

$$\pi(x_D)^{-1}M(x,f(n)n^{-i\tau},D) = \frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le x\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\Big)+ \\ + o(1) =: \psi(x)+o(1)$$

as $x\to\infty$. Using a summation by parts we obtain that

$$\sum_{Dp+1\le x} f(Dp+1) = x^{i\tau}\sum_{Dp+1\le x} f(Dp+1)(Dp+1)^{-i\tau} \\ - i\tau\int_2^x \sum_{Dp+1\le u} f(Dp+1)(Dp+1)^{-i\tau}u^{i\tau-1}\,\mathrm{d}u. \tag{24}$$

If $D<x^{\varepsilon}$, then $D<x^{\gamma\varepsilon'}$ with some other $\varepsilon<\varepsilon'<1$ and an appropriate $0\le\gamma<1$. Therefore the estimation

$$\pi\left(\frac{u-1}{D}\right)^{-1} M(u,f(n)n^{-i\tau},D) = \\ = \frac{\mu(d)}{\varphi(d)}\prod_{\substack{p\le u\\ p\nmid dD}}\Big(1-\frac{1}{p-1}+\sum_{\alpha\ge 1}\frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha}\Big)+o(1)\quad (x\to\infty)$$

remains valid in the range $x^\gamma<u<x$. Thus, we can estimate the integral on the right hand side of (24) in this range as

$$\int_{x^\gamma}^x \sum_{Dp+1\le u} f(Dp+1)(Dp+1)^{-i\tau}u^{i\tau-1}\,\mathrm{d}u = \\ = \frac{\mu(d)}{\varphi(d)}\int_{x^\gamma}^x \pi(u_D)\psi(u)u^{i\tau-1}\,\mathrm{d}u + o(1)\int_{x^\gamma}^x \frac{1}{D\log u}\,\mathrm{d}u\quad (x\to\infty). \tag{25}$$

Now if $x^\gamma\le u\le x$, then as in (23) we have

$$|\psi(x)-\psi(u)| = o(1)$$

as $x\to\infty$. Therefore the right hand side of (25) equals

$$\pi(x_D)\frac{x^{i\tau}}{1+i\tau}\,\frac{\mu(d)}{\varphi(d)}\,\psi(x)+o(\pi(x_D))\quad (x\to\infty).$$

Using the trivial bound

$$|M(u,f(n)n^{i\tau},D)|\le\pi(u_D),$$

we have that the integral on the right hand side of (24) in the range $2 \le u \le x^\gamma$ is not more than

$$\mathcal{O}\left(\frac{1}{D}\int\limits_{2D+1}^{x^\gamma} \frac{1}{\log(u/D)}\,\mathrm{d}u\right) \ll \int\limits_{2}^{x^\gamma/D} \frac{1}{\log(u)}\,\mathrm{d}u = o(\pi(x_D)) \quad (x \to \infty).$$

In summary we have

$$\sum_{Dp+1\le x} f(Dp+1) = \pi(x_D)\frac{x^{i\tau}}{1+i\tau}\,\frac{\mu(d)}{\varphi(d)}\,\psi(x) + o(\pi(x_D)) \quad (x \to \infty),$$

as asserted. ■

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