Annales Univ. Sci. Budapest., Sect. Comp. 35 (2011) 107–124

AN INTERPLAY BETWEEN JENSEN'S AND PEXIDER'S FUNCTIONAL EQUATIONS ON SEMIGROUPS

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Dedicated to Professor Antal Járai on his 60-th birthday

Abstract. Let $(S,+)$ and $(G,+)$ be two commutative semigroups. Assuming that the latter one is cancellative we deal with functions $f : S \longrightarrow G$ satisfying the Jensen functional equation written in the form

$$2f(x+y) = f(2x) + f(2y)\,.$$

It turns out that functions $f, g, h : S \longrightarrow G$ satisfying the functional equation of Pexider

$$f(x+y) = g(x) + h(y)$$

must necessarily be Jensen. The validity of the converse implication is also studied with emphasis placed on a very special Pexider equation

$$\varphi(x+y) + \delta = \varphi(x) + \varphi(y)\,,$$

where δ is a fixed element of G. Plainly, the main goal is to express the solutions of both: Jensen and Pexider equations in terms of semigroup homomorphisms.

Bearing in mind the algebraic nature of the functional equations considered, we were able to establish our results staying away from topological tools.

https://doi.org/10.71352/ac.35.107

1. Introduction

We will investigate the very classical functional equations of Jensen, i.e.

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2},$$

and of Pexider, i.e.

$$f(x+y) = g(x) + h(y),$$

where f, g, h are functions defined and assuming values in some abstract algebraic structures. These equations have very rich literature; the basic facts concerning that topic may be found (among others) in the well known monographs of J. Aczél [1] and M. Kuczma [2]. It is also commonly known that in the case where both the domain and the target spaces of functions considered are linear spaces, the general solution of the Jensen and of the Pexider equations may be expressed in terms of additive functions. Let us recall that a function a is called additive provided it satisfies the Cauchy functional equation

$$a(x+y) = a(x) + a(y).$$

In classical situations Jensen functions are represented as the sum of an additive map and a constant function. The same can be told about solutions of the Pexider equation. The question we are faced is: to what extent these representations remain valid and/or what kind of potentially new phenomena may occur while dealing with more abstract algebraic structures. In particular, regarding the Jensen equation, the category of not necessarily commutative groups was taken into account in the papers of C.T. Ng [3], [4] and H. Stetkaer [6]. In the present paper we will concentrate on semigroups as potential domains and codomains. In some cases, we try also to get rid of the 2-divisibility assumption dealing with a version of the Jensen equation which does not require the feasibility of such division. On the other hand, we try to keep the strictly algebraic character of our studies avoiding, in particular, any topological structures. This aspect distinguishes our approach from the one applied, for instance, in the paper of W. Smajdor [5]. The basic results from this paper will be generalized considerably just due to the fact that, bearing in mind the algebraic nature of the functional equations considered, we were able to stay away from topological tools.

2. Some lemmas

We start with a simpler case when the target space of functions considered is a group.

Lemma 1. *Let $(S,+)$ be a commutative semigroup and let $(G^*,+)$ be an Abelian group. Then a function $f : S \to G^*$ satisfies the Jensen functional equation*

$$2f(x+y) = f(2x) + f(2y), \qquad x, y \in S, \tag{1}$$

if and only if there exist an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f(2x) = A(x) + b, \quad x \in S, \qquad \textit{and} \qquad 2f(x) = A(x) + 2b, \quad x \in S+S.$$

Proof. Assume (1) and define a function $\varphi : S \to G^*$ by the formula

$$\varphi(x) := f(2x) - 2f(x), \quad x \in S.$$

Then by (1) we obtain

$$\begin{aligned} 2f(x+y+z) &= f(2(x+y)) + f(2z) = \varphi(x+y) + 2f(x+y) + f(2z) = \\ &= \varphi(x+y) + f(2x) + f(2y) + f(2z), \end{aligned}$$

as well as,

$$\begin{aligned} 2f(x+y+z) &= f(2x) + f(2(y+z)) = f(2x) + \varphi(y+z) + 2f(y+z) = \\ &= f(2x) + \varphi(y+z) + f(2y) + f(2z), \end{aligned}$$

for all $x, y, z \in S$, whence

$$\varphi(x+y) = \varphi(y+z), \qquad x, y, z \in S.$$

In particular, setting $z = y$, due to the commutativity of the binary law in S,

$$\varphi(2y) = \varphi(x+y) = \varphi(2x), \quad x, y \in S.$$

Therefore, $\varphi(t) \equiv \text{const} =: c$ on the set $S+S$. In view of (1) and the definition of φ, this implies

$$f(2x) + c + f(2y) + c = 2f(x+y) + c + c = f(2(x+y)) + c, \quad x, y \in S,$$

stating that the map $A(x) := f(2x) + c$, $x \in S$, is additive. By setting $b := -c$ we derive the first part of our assertion. For $x \in S+S$ one has $x = y+z$, $y, z \in \in S$ whence, by (1),

$$\begin{aligned} A(x) + 2b = A(y+z) + 2b = A(y) + b + A(z) + b = \\ = f(2y) + f(2z) = 2f(y+z) = 2f(x). \end{aligned}$$

This ends the proof of the necessity, and since the sufficiency is obvious, the proof is completed. ■

Corollary 1. *Let all the assumptions of* Lemma 1 *be satisfied. If, moreover, the division by* 2 *is uniquely performable in* $(G^*, +)$, *then* $f : S \to G^*$ *satisfies equation* (1) *if and only if there exist an additive map* $A^* : S \to G^*$ *and a constant* $b \in G^*$ *such that*

$$f(x) = \begin{cases} A^*(x) + b, & for \;\; x \in S+S \\ arbitrary, & on \;\; S \setminus (S+S). \end{cases}$$

Proof. By virtue of the second part of the assertion of Lemma 1 it suffices to put $A^*(x) := \frac{1}{2}A(x)$, $x \in S$. ■

Lemma 2. *Let all the assumptions of* Lemma 1 *be satisfied. If functions* $f, g, h : S \to G^*$ *satisfy the Pexider functional equation*

$$f(x+y) = g(x) + h(y), \quad x, y \in S, \tag{2}$$

then there exist an additive map $A : S \to G^*$ *and constants* $b, c \in G^*$ *such that*

$$(*) \qquad \begin{cases} f(2x) = A(x) + b, & x \in S; \\ 2g(x) = A(x) + b - c, & x \in S; \\ 2h(x) = A(x) + b + c, & x \in S\;; \\ 2f(x) = A(x) + 2b, & x \in S+S. \end{cases}$$

Conversely, every triple (f, g, h) *satisfying conditions* $(*)$ *yields a solution to the equation*

$$2f(x+y) = 2g(x) + 2h(y), \qquad x, y \in S. \tag{3}$$

Proof. (Necessity.) We shall first show that f satisfies (1). Indeed, for all $x, y \in S$ we have

$$\begin{aligned} 2f(x+y) &= f(x+y) + f(y+x) = g(x) + h(y) + g(y) + h(x) = \\ &= f(2x) + f(2y). \end{aligned}$$

On account of Lemma 1, there exists an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f(2x) = A(x) + b, \quad x \in S, \qquad \text{and} \quad 2f(x) = A(x) + 2b, \quad x \in S + S.$$

Since

$$g(x) + h(y) = f(x + y) = f(y + x) = g(y) + h(x), \qquad x, y \in S,$$

we get

$$h(x) - g(x) = h(y) - g(y) \equiv \text{const} =: c.$$

Consequently,

$$h(x) = g(x) + c, \quad x \in S\,,$$

and, therefore, for every $x, y \in S$ we have

$$f(x + y) = g(x) + h(y) = g(x) + g(y) + c,$$

whence

$$A(x) + b = f(2x) = 2g(x) + c, \quad x \in S,$$

and

$$2h(x) = 2g(x) + 2c = A(x) + b + c, \quad x \in S,$$

as claimed.

(Sufficiency.)

$$2g(x)+2h(y) = A(x)+b-c+A(x)+b+c = A(x+y)+2b = 2f(x+y), \quad x, y \in S,$$

which completes the proof. ∎

Corollary 2. *Let $(S, +)$ be a commutative semigroup and let $(G^*, +)$ be an Abelian group uniquely 2-divisible. Then the triple (f, g, h) of functions from S into G^* yields a solution to equation (2) if and only if*

$$f(x) = \begin{cases} A^*(x) + 2b^* & \text{for } x \in S + S \\ \text{arbitrary} & \text{on } S \setminus (S + S); \end{cases}$$

$$g(x) = A^*(x) + b^* - c^*, \quad x \in S;$$

$$h(x) = A^*(x) + b^* + c^*, \quad x \in S,$$

where $A^* : S \to G^*$ is additive and b^*, c^* are arbitrary constants from G^*.

Proof. In the light of Lemma 2 it suffices to put $A^* := \frac{1}{2}A$, $b^* := \frac{1}{2}b$, $c^* := \frac{1}{2}c$. ∎

3. Main results

In what follows, we shall apply these results to deal with the case where G is a cancellative semigroup.

Theorem 1. *Let $(S,+)$ be a commutative semigroup and let $(G,+)$ stand for an Abelian cancellative semigroup. A map $f : S \to G$ satisfies Jensen's functional equation* (1) *if and only if there exist elements $\beta,\ \gamma \in G$ such that*

$$\begin{cases} f(x+y)+\beta = f(x)+f(y)+\gamma & \textit{for } \ x,y \in 2S; \\ f(2x)+\beta = 2f(x)+\gamma & \textit{for } \ x \in S+S; \\ f \textit{ is arbitrary} & \textit{on } \ S \setminus (S+S)\ . \end{cases}$$

Proof. We embed the semigroup $(G,+)$ into a group $(G^*,+)$ of equivalence classes determined by the relation

$$(u,v) \sim (x,y) :\Longleftrightarrow u+y = v+x\,.$$

Clearly, we identify an element x from G with the class $[(2x,x)]$. Moreover, we have also

$$-[(x,y)] = [(y,x)], \quad \text{as well as} \quad 0 = [(x,x)]\,.$$

Finally, we put

$$f^*(x) := [(2f(x), f(x))], \quad x \in S.$$

Equation (1) may equivalently be written in the form

$$4f(x+y)+f(2x)+f(2y) = 2f(x+y)+2f(2x)+2f(2y), \qquad x,y \in S.$$

This allows us to write

$$\begin{aligned} 2f^*(x+y) \quad &= [(4f(x+y), 2f(x+y))] = \\ &= [(2f(2x)+2f(2y), f(2x)+f(2y))] = \\ &= f^*(2x)+f^*(2y). \end{aligned}$$

On account of Lemma 1 we infer that there exist an additive map $A : S \to G^*$ and a constant $b \in G^*$ such that

$$f^*(2x) = A(x)+b,\ x \in S, \qquad 2f^*(x) = A(x)+2b,\ x \in S+S.$$

Let $b = [(\beta,\gamma)]$. Then, for all $x,y \in S$, one has

$$f^*(2x+2y)+b = A(x+y)+2b = A(x)+A(y)+b+b = f^*(2x)+f^*(2y),$$

i.e.

$$[(2f(2x+2y)+\beta, f(2x+2y)+\gamma)] = [(2f(2x)+2f(2y), f(2x)+f(2y))],$$

whence

$$2f(2x+2y)+f(2x)+f(2y)+\beta = f(2x+2y)+2f(2x)+2f(2y)+\gamma, \quad x,y \in S,$$

i.e.

$$f(2x+2y)+\beta = f(2x)+f(2y)+\gamma, \quad x,y \in S$$

or, equivalently,

$$f(x+y)+\beta = f(x)+f(y)+\gamma, \quad \text{for all } x,y \in 2S.$$

Let now $x \in S+S$. Then $x = y+z,\ y,z \in S$ whence by (1):

$$\begin{aligned} 2f(x)+\gamma \quad &= 2f(y+z)+\gamma = f(2y)+f(2z)+\gamma = f(2y+2z)+\beta = \\ &= f(2x)+\beta, \end{aligned}$$

as claimed.

Clearly, equation (1) leaves the values of f on $S \setminus (S+S)$ undetermined.

(Sufficiency). Let $x,y \in S$. Then $x+y \in S+S$ and we have

$$f(2(x+y))+\beta = 2f(x+y)+\gamma \quad \text{and} \quad f(2x+2y)+\beta = f(2x)+f(2y)+\gamma,$$

whence

$$2f(x+y) = f(2x)+f(2y), \qquad x,y \in S.$$

This finishes the proof. ■

Corollary 3. *Let* $(S,+),(G,+)$ *and* f *be the same as in* Theorem 1. *Then the function*

$$a_f(x) := f(2x)+\beta+\gamma, \qquad x \in S,$$

enjoys the property

$$a_f(x+y)+2\beta = a_f(x)+a_f(y), \quad x,y \in S.$$

Proof.

$$\begin{aligned} a_f(x+y)+2\beta \quad &= f(2x+2y)+2\beta+\beta+\gamma = f(2x)+f(2y)+2\beta+2\gamma = \\ &= a_f(x)+a_f(y), \end{aligned}$$

for all $x,y \in S$. ■

Theorem 2. *Let* $(S,+)$ *be a commutative semigroup and let* $(G,+)$ *stand for an Abelian cancellative semigroup. If functions* $f,g,h : S \to G$ *satisfy the Pexider equation* (2), *then each of them satisfies the Jensen equation* (1). *Moreover, there exist a map* $\psi : S \to G$ *and constants* $\alpha,\beta,\gamma,\delta,\varepsilon \in G$ *such that*

$$\psi(x+y)+\varepsilon = 2f(x+y)+\alpha, \qquad x,y \in S, \tag{4}$$

$$\psi(x+y) = 2g(x+y)+\beta = 2h(x+y)+\gamma, \qquad x,y \in S, \tag{5}$$

and

$$\psi(x+y)+\delta = \psi(x)+\psi(y)\,, \quad x,y \in S. \tag{6}$$

Conversely, if $\alpha,\beta,\gamma,\delta,\varepsilon \in G$ *are arbitrary constants satisfying condition*

$$\beta+\gamma+\varepsilon = \alpha+\delta \tag{7}$$

and equalities (4), (5) *and* (6) *are fulfilled, then*

$$2f(2x+2y) = 2g(2x)+2h(2y), \quad x,y \in S. \tag{8}$$

Proof. Equation (2) implies that

$$2f(x+y) = f(x+y)+f(y+x) = g(x)+h(y)+g(y)+h(x) = f(2x)+f(2y),$$

for all $x,y \in S$, i.e. f satisfies Jensen equation (1). Therefore

$$f(2x)+f(2y) = 2f(x+y) = 2g(x)+2h(y), \quad x,y \in S.$$

Fix $u,v \in S$ arbitrarily and put $x = u+v$. Then

$$f(2u+2v)+f(2y) = 2g(u+v)+2h(y),$$

and by virtue of (2) we get

$$g(2u)+h(2v)+g(y)+h(y)+g(2v) = 2g(u+v)+2h(y)+g(2v),$$

whence also

$$g(2u)+g(2v)+f(2v+y) = 2g(u+v)+f(2v+y)$$

follows, i.e. $g(2u)+g(2v) = 2g(u+v)$. Analogously, we check that h is a Jensen function.

On account of Theorem 1, there exist constants $\beta_f, \gamma_f, \beta_g, \gamma_g, \beta_h, \gamma_h \in G$ such that

$$\varphi(x+y) + \beta_\varphi = \varphi(x) + \varphi(y) + \gamma_\varphi, \qquad x, y \in 2S, \tag{9}$$

and

$$\varphi(2x) + \beta_\varphi = 2\varphi(x) + \gamma_\varphi, \qquad x \in S + S, \tag{10}$$

where $\varphi \in \{f, g, h\}$. Let us define the functions $a_\varphi : S \to G,\ \varphi \in \{f, g, h\}$ by the formulas

$$a_\varphi(x) := \varphi(2x) + \beta_\varphi + \gamma_\varphi, \qquad x \in S.$$

Since φ is Jensen function we obtain by (10) that

$$a_\varphi(x+y) + 2\beta_\varphi = a_\varphi(x) + a_\varphi(y), \qquad x, y \in S. \tag{11}$$

According to (2) we have

$$\begin{aligned} a_g(x) + a_h(y) \quad &= g(2x) + \beta_g + \gamma_g + h(2y) + \beta_h + \gamma_h = \\ &= f(2x+2y) + \beta_g + \gamma_g + \beta_h + \gamma_h = \\ &= g(2y) + \beta_g + \gamma_g + h(2x) + \beta_h + \gamma_h = \\ &= a_g(y) + a_h(x), \end{aligned}$$

whence

$$a_g(x) + a_h(y) = a_g(y) + a_h(x), \quad x, y \in S.$$

Thus, there exist constants $\lambda, \mu \in G$ such that

$$a_g(x) + \lambda = a_h(x) + \mu, \quad x \in S. \tag{12}$$

Now, setting

$$\psi(x) := a_g(x) + \lambda = a_h(x) + \mu, \quad x \in S\,,$$

by virtue of (11), for all $x, y \in S$, we infer that

$$\psi(x)+\psi(y) = a_g(x)+\lambda+a_g(y)+\lambda = a_g(x+y)+2\beta_g+2\lambda = \psi(x+y)+2\beta_g+\lambda\,,$$

and it suffices to put $\delta := 2\beta_g + \lambda$ to obtain (6). It follows from (11), the definition of a_g and (10) that $\psi(x+y) = a_g(x+y)+\lambda = g(2(x+y))+\beta_g+\gamma_g+\lambda = = 2g(x+y)+2\gamma_g+\lambda$, for all $x, y \in S$, which coincides with the first equality in (5) on setting $\beta := 2\gamma_g + \lambda$. The other one may be derived similarly. Finally, by (4), (2) and (10)

$$\begin{aligned} \psi(x+y) + \delta + \beta_f \quad &= \psi(x) + \psi(y) + \beta_f = a_g(x) + \lambda + a_h(y) + \mu + \beta_f = \\ &= g(2x) + \beta_g + \gamma_g + \lambda + h(2y) + \beta_h + \gamma_h + \mu + \beta_f = \\ &= f(2(x+y)) + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu + \beta_f = \\ &= 2f(x+y) + \gamma_f + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu\,, \end{aligned}$$

and it sufficies to put $\varepsilon := \delta + \beta_f$ as well as $\alpha := \gamma_f + \beta_g + \gamma_g + \lambda + \beta_h + \gamma_h + \mu$ to arrive at (4).

Conversely, let $\alpha, \beta, \gamma, \delta, \varepsilon \in G$ be arbitrary constants satisfying (7) and assume that equalities (4), (5) and (6) are fulfilled. Then

$$\begin{aligned} 2g(2x) + 2h(2y) + \beta + \gamma + \varepsilon &= \psi(2x) + \psi(2y) + \varepsilon \\ &= \psi(2x + 2y) + \delta + \varepsilon = 2f(2x + 2y) + \alpha + \delta\,, \end{aligned}$$

which jointly with (7) implies (8) and finishes the proof. ■

As we see in our considerations the functional equation (6) (see also (11)) plays a crucial role. Thus the problem of solving this equation seems to be a basic one.

Theorem 3. *Let $(S, +)$ be a commutative semigroup and let $(G, +)$ stand for an Abelian cancellative semigroup. Given a fixed element $\delta \in G$, if a map $\psi : S \to G$ satisfies the equation*

$$\psi(x + y) + \delta = \psi(x) + \psi(y), \qquad x, y \in S, \tag{13}$$

then the set $S_\delta := \psi^{-1}(G + \delta)$ is either empty or $(S_\delta, +)$ yields a subsemigroup of $(S, +)$ and there exists a homomorphism $H : S_\delta \to G$ such that

$$\psi(x) = H(x) + \delta, \qquad x \in S_\delta. \tag{14}$$

If, moreover, there exists a $y_0 \in S$ such that $\psi(y_0) \in G + 2\delta$, then $S + y_0 \subset S_\delta$ and there exists an $\eta \in G$ such that

$$\psi(x) + \eta = H(x + y_0), \qquad x \in S. \tag{15}$$

In particular, such a representation takes place provided that ψ is a surjection from S onto G.

Proof. Assume that $S_\delta \neq \emptyset$ and take arbitrary $x, y \in S_\delta$. Then there exist $w, z \in G$ such that $\psi(x) = w + \delta$ and $\psi(y) = z + \delta$. By (13) we infer that

$$\psi(x + y) + \delta = \psi(x) + \psi(y) = w + \delta + z + \delta$$

whence

$$\psi(x + y) = w + z + \delta \in G + \delta.$$

This means that $x + y \in S_\delta$ and proves that $(S_\delta, +)$ forms a subsemigroup of $(S, +)$. It follows from the definition of S_δ that there exists a function $H : S_\delta \to G$ fulfilling equality (14). For all $x, y \in S_\delta$ we have

$$H(x + y) + 2\delta = \psi(x + y) + \delta = \psi(x) + \psi(y) = H(x) + \delta + H(y) + \delta.$$

which states that H is a homomorphism.

If for some $y_0 \in S$ we have $\psi(y_0) = \eta + 2\delta$ with some $\eta \in G$, then $y_0 \in S_\delta$. Consequently

$$\eta + 2\delta = \psi(y_0) = H(y_0) + \delta,$$

whence

$$H(y_0) = \eta + \delta \in G + \delta. \tag{16}$$

According to (13) we get

$$\psi(x + y_0) + \delta = \psi(x) + \psi(y_0) = \psi(x) + \eta + 2\delta, \quad x \in S,$$

and since G is cancellative,

$$\psi(x + y_0) = \psi(x) + \eta + \delta \in G + \delta, \qquad x \in S. \tag{17}$$

Therefore $x + y_0 \in S_\delta$, $x \in S$, or, equivalently,

$$S + y_0 \subset S_\delta.$$

On account of (14) we obtain

$$\psi(x + y_0) = H(x + y_0) + \delta, \qquad x \in S.$$

By virtue of (17) we get (15). It is easily seen that (15) takes place provided ψ is surjective. ∎

Corollary 4. *Let $(S, +)$ be a commutative semigroup and let $(G, +)$ stand for an Abelian cancellative monoid. Assume that $\psi : S \to G$ is a surjection of S onto G satisfying equation (13), $S_\delta := \psi^{-1}(G + \delta) \neq \emptyset$ and y_0 is a fixed element of S such that $\psi(y_0) \in G + 2\delta$. Then $(S_\delta, +)$ is a subsemigroup of $(S, +)$ and there exists a homomorphism H mapping S_δ into G such that*

$$\psi(x) = H(x + y_0), \quad x \in S,$$

and

$$H(S + y_0) = G, \qquad H(y_0) = \delta$$

Proof. Going back to the proof of Theorem 3, take $y_0 \in S$ such that $\psi(y_0) = 2\delta$ there. Then $\eta = 0$ and consequently $\psi(x) = H(x + y_0)$, $x \in S$, and $H(y_0) = \delta$. The equality $H(S + y_0) = G$ is obvious. ∎

Remark 1. Let $(S, +), (G, +)$ be the same as in Theorem 3. If $\psi : S \to G$ satisfies equation (13) and there exist $u, v \in S$ such that $\psi(u) = 2\psi(v)$, then the set $S_\delta = \psi^{-1}(G + \delta)$ is nonvoid.

In fact, $\psi(u) = 2\psi(v) = \psi(v) + \psi(v) = \psi(2v) + \delta \in G + \delta$.

Lemma 3. *Let $(S,+)$ be a commutative semigroup and let $(G,+)$ be an Abelian cancellative semigroup in which the division by 2 is uniquely performable. If $\psi : S \to G$ satisfies equation* (13)*, then for an arbitrary positive integer n and each $x \in S$ the following equality*

$$\psi(x) + \frac{1}{2^n}\delta = \frac{1}{2^n}\psi(2^n x) + \delta \tag{18}$$

holds true.

Proof. (Induction.) Putting $y = x$ in (13) we obtain

$$\psi(2x) + \delta = 2\psi(x), \qquad x \in S\,,$$

whence (18) follows immediately for $n = 1$. Assume (18) for a positive integer n and each $x \in S$. Then

$$\frac{1}{2}\psi(2x) + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta, \quad x \in S,$$

as well as

$$\frac{1}{2}\psi(2x) + \delta + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta + \delta, \quad x \in S.$$

Applying (18) for $n = 1$ we obtain

$$\psi(x) + \frac{1}{2}\delta + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \frac{1}{2}\delta + \delta$$

and, consequently,

$$\psi(x) + \frac{1}{2^{n+1}}\delta = \frac{1}{2^{n+1}}\psi(2^{n+1}x) + \delta\,,$$

which ends the proof. ■

Corollary 5. *Under the assumptions of* Lemma 3 *we have*

$$\psi(x) \in \bigcap_{n=1}^{\infty}\left(G + \left(1 - \frac{1}{2^n}\right)\delta\right), \quad x \in S.$$

Proof. Fix an $x \in S$ and a positive integer n. On account of Lemma 3 we have

$$\psi(x) + \frac{1}{2^n}\delta = \frac{1}{2^n}\psi(2^n x) + \frac{1}{2^n}\delta + \left(1 - \frac{1}{2^n}\right)\delta, \quad x \in S,\, n \in \mathbb{N},$$

whence

$$\psi(x) = \frac{1}{2^n}\psi(2^n x) + \left(1 - \frac{1}{2^n}\right)\delta, \quad x \in S,\, n \in \mathbb{N},$$

which finishes the proof. ■

Theorem 4. *Let $(S,+)$ be a commutative semigroup and let $(G,+)$ be a semigroup that is Abelian uniquely 2-divisible and cancellative. Assume that $\delta \in G$ is such that*

$$\bigcap_{n\in\mathbb{N}} \left(G + \left(1 - \frac{1}{2^n}\right)\delta\right) \subset G + \delta. \tag{19}$$

Then a map $\psi : S \to G$ satisfies (13) *if and only if there exists a homomorphism $H : S \to G$ such that*

$$\psi(x) = H(x) + \delta, \qquad x \in S.$$

Proof. It follows from (19) and Corollary 5, that

$$\psi(x) \in G + \delta, \qquad x \in S.$$

Therefore

$$\psi(x) = H(x) + \delta, \qquad x \in S,$$

where $H : S \to G$ is a function. Applying (19) we obtain

$$H(x+y) + 2\delta = \psi(x+y) + \delta = \psi(x) + \psi(y) = H(x) + \delta + H(y) + \delta,$$

which implies that $H(x+y) = H(x) + H(y)$, $x, y \in S$. Since the suffiency is obvious, the proof has been finished. ■

Theorem 5. *Let $(S,+), (G,+)$ be two commutative uniquely 2-divisible semigroups. Assume that $(G,+)$ is cancellative and such that condition* (19) *is fulfilled for every $\delta \in G$. Then $f : S \to G$ satisfies Jensen functional equation* (1) *if and only if there exists an additive function $H : S \to G$ such that*

$$f(x+y) = H(x) + f(y), \qquad x, y \in S.$$

Proof. By Theorem 1 there exist constants $\beta, \gamma \in G$ such that

$$f(x+y) + \beta = f(x) + f(y) + \gamma, \quad x, y \in 2S = S.$$

Putting $\psi(x) := f(x) + \gamma,\ x \in S,$ we note that

$$\psi(x+y) + \beta = f(x+y) + \gamma + \beta = f(x) + f(y) + 2\gamma = \psi(x) + \psi(y), \quad x, y \in S,$$

i.e. equation (13) is satisfied with $\delta = \beta$. On account of Theorem 4, there exists an additive map $H : S \to G$ such that

$$\psi(x) = H(x) + \beta, \quad x \in S.$$

Therefore

$$f(x) + \gamma = H(x) + \beta, \qquad x \in S\,,$$

and hence

$$f(x+y) + \beta = f(x) + f(y) + \gamma = H(x) + \beta + f(y), \quad x, y \in S\,,$$

yielding

$$f(x+y) = H(x) + f(y), \quad x, y \in S\,,$$

as claimed.

Conversely, for all $x, y \in S,$ one has

$$f(2x) + f(2y) = H(x) + f(x) + H(y) + f(y) = f(x+y) + f(y+x) = 2f(x+y)\,,$$

which completes the proof. ■

4. Generalizations of W. Smajdor's results

W. Smajdor [5] defines an *abstract convex cone* as a cancellative Abelian monoid $(G, +)$ provided that a map $[0, \infty) \times G \ni (\lambda, s) \to \lambda s \in G$ is given such that

$$1s = s,\ \lambda(\mu s) = (\lambda\mu)s,\ \lambda(s+t) = \lambda s + \lambda t,\ (\lambda + \mu)s = \lambda s + \mu s,$$

$$s, t \in G,\ \lambda, \mu \in [0, \infty).$$

Under the additional assumption that G is endowed with a complete metric ϱ such that

$$\varrho(s+t, s+t') = \varrho(t, t'),\ s, t, t' \in G, \quad \varrho(\lambda s, \lambda t) = \lambda\varrho(s, t),\ \lambda \in [0, \infty), s, t \in G\,,$$

W. Smajdor's main result (see Theorem 1 of [5]) states that any function f mapping an Abelian 2-divisible semigroup $(S, +)$ into $(G, +)$ satisfies the Jensen equation if and only if there exists an additive map $a : S \to G$ such that the equality $f(x+y) = a(x) + f(y)$ holds true for all $x, y \in S$.

The occurrence of a topology (actually: metric topology) in the target cone in Smajdor's theorem seems to be artificial bearing in mind the strictly algebraic nature of the problem considered. Our Theorem 5 generalizes her result by avoiding any topological structure in the target space. In fact, the only thing we need is to show that under W. Smajdor's assumptions condition (19), i.e. the inclusion

$$\bigcap_{n\in\mathbb{N}} \left(G + \left(1 - \frac{1}{2^n}\right)\delta \right) \subset G + \delta$$

is fulfilled for every δ from G. As a matter of fact, we shall achieve that with the aid of considerably weaker requirements.

Proposition. *Given a cancellative semigroup $(G, +)$ uniquely divisible by 2 and admitting a complete metric ϱ such that*

$$\varrho(x+z, y+z) = \varrho(x, y),\ \ x, y, z \in G, \quad \varrho(2x, 2y) = 2\varrho(x, y),\ x, y \in G\,,$$

there exists a neutral element 0 in G, i.e. $(G, +)$ is necessarily a monoid. Moreover, for every δ from G condition (19) holds true.

Proof. The binary law "+" has to be continuous; in fact, if

$$G \ni x_n \longrightarrow x_0 \in G \quad \text{and} \quad G \ni y_n \longrightarrow y_0 \in G,$$

then

$$\begin{aligned}\varrho(x_n + y_n, x_0 + y_0) &\le \varrho(x_n + y_n, x_n + y_0) + \varrho(x_n + y_0, x_0 + y_0) = \\ &= \varrho(y_n, y_0) + \varrho(x_n, x_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty\,.\end{aligned}$$

In particular the map $G \ni x \longrightarrow 2x \in G$ is continuous. Fix $\delta \in G$ arbitrarily. Then

$$\left(\frac{1}{2^n}\delta\right)_{n\in\mathbb{N}} \quad \text{is a Cauchy sequence}\,.$$

Indeed, for all positive integers n, k one has

$$\varrho\left(\frac{1}{2^{n+k}}\delta, \frac{1}{2^n}\delta\right) \le \sum_{j=0}^{k-1} \frac{1}{2^{n+j}}\varrho\left(\frac{1}{2}\delta, \delta\right) \le \frac{1}{2^{n-1}}\varrho\left(\frac{1}{2}\delta, \delta\right).$$

Since ρ is complete the sequence $(\frac{1}{2^n}\delta)_{n\in\mathbb{N}}$ converges to an $x_0 \in G$. Then also

$$2x_0 = 2\lim_{n\to\infty} \frac{1}{2^{n+1}}\delta = \lim_{n\to\infty} \frac{1}{2^n}\delta = x_0\,,$$

whence, for every $x \in G$, we get

$$x + x_0 = x + 2x_0 = (x + x_0) + x_0 \quad \text{and} \quad x_0 + x = 2x_0 + x = x_0 + (x_0 + x),$$

which, by means of the cancellativity assumption, states that x_0 is zero element in G.

Now, in order to show the inclusion (19), fix an arbitrary x from the intersection $\bigcap_{n\in\mathbb{N}}(G + (1 - \frac{1}{2^n})\delta)$. Then, for every $n \in \mathbb{N}$ one may find a $g_n \in G$ such that

$$x + \frac{1}{2^n}\delta = g_n + \delta .$$

Since the addition is continuous and the sequence $\left(\frac{1}{2^n}\delta\right)_{n\in\mathbb{N}}$ converges to the neutral element x_0, the sequence $(g_n + \delta)_{n\in\mathbb{N}}$ tends to x. Therefore, x belongs to $G + \delta$ since, obviously, the set $G + \delta$ is closed as a complete subspace of G. This completes the proof. ■

Remark 2. Condition (19) is automatically satisfied in any Abelian, uniquely 2-divisible group $(G, +)$. Actually, for any $\delta \in G$ the inclusion

$$G + \left(1 - \frac{1}{2^n}\right)\delta = G - \frac{1}{2^n}\delta + \delta \subset G + \delta$$

is satisfied for every $n \in \mathbb{N}$.

Another example of an Abelian, uniquely 2-divisible monoid in which condition (19) holds true reads as follows. Let $a : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function and let

$$G := \{x \in \mathbb{R} :\ a(x) \geq 0\}.$$

Equipped with the usual addition, the set G yields a commutative semigroup with 0 as the neutral element. For any $\delta \in G$ and for every $n \in \mathbb{N}$ we have

$$G + \left(1 - \frac{1}{2^n}\right)\delta = \left\{y \in \mathbb{R} :\ a(y) \geq \left(1 - \frac{1}{2^n}\right)a(\delta)\right\},$$

whence

$$\bigcap_{n\in\mathbb{N}}\left(G + \left(1 - \frac{1}{2^n}\right)\delta\right) = \bigcap_{n\in\mathbb{N}}\left\{y \in \mathbb{R} :\ a(y) \geq \left(1 - \frac{1}{2^n}\right)a(\delta)\right\} =$$

$$= \{y \in \mathbb{R} :\ a(y) \geq a(\delta)\} = G + \delta .$$

Noteworthy is the fact that in the case where $a(\delta) > 0$ the shift $G + \delta$ fails to coincide with G itself.

Finally, each uniquely 2-divisible topological monoid $(G,+;0)$ such that for every $\delta\in G$ the shift $G+\delta$ is closed and $\lim_{n\longrightarrow\infty}2^{-n}\delta=0$ enjoys the property (19) (cf. the proof of the Proposition).

The following example shows that, in general, condition (19) need not be fulfilled. Indeed, let $G=(0,\infty)$ and let $\delta>0$ be fixed. Then G equipped with the usual addition is a uniquely 2-divisible commutative semigroup and

$$\bigcap_{n\in\mathbb{N}}\left((0,\infty)+\left(1-\frac{1}{2^n}\right)\delta\right)=\bigcap_{n\in\mathbb{N}}\left(\left(1-\frac{1}{2^n}\right)\delta,\infty\right)=$$
$$=[\delta,\infty)\not\subset G+\delta=(\delta,\infty)\,.$$

We terminate this paper with the following generalization of Theorem 2 in [5] by W. Smajdor.

Theorem 6. *Let $(S,+),(G,+)$ be two commutative uniquely 2-divisible semigroups. Assume that $(G,+)$ is cancellative and such that condition (19) is fulfilled for every $\delta\in G$. If $f,g,h:S\to G$ fulfil the Pexider equation (2) then there exists a homomorphism $H:S\to G$ such that*

$$f(x+y)=H(x)+f(y),\ \ g(x+y)=H(x)+g(y),\ \ h(x+y)=H(x)+h(y),$$

for all $x,y\in S$.

Proof. On account of Theorem 2 we infer that f,g and h are Jensen functions. It follows from Theorem 5 that there exist additive functions H_f,H_g and H_h such that for all $x,y\in S$ the equalities

$$f(x+y)=H_f(x)+f(y),\ \ g(x+y)=H_g(x)+g(y),\ \ h(x+y)=H_h(x)+h(y),$$

hold true. Thus, for arbitrary $x,y\in S$ we have

$$\begin{aligned}H_f(x+y)+f(x+y)&=f(2x+2y)=g(x+y)+h(x+y)=\\&=H_g(x)+g(y)+H_h(y)+h(x)=\\&=H_g(x)+H_h(y)+f(x+y)\end{aligned}$$

which leads to

$$H_f(x+y)=H_g(x)+H_h(y),\quad x,y\in S.$$

Moreover,

$$H_g(x)+H_h(x)+2H_f(y)=H_f(2x)+2H_f(y)=2H_f(x+y)=2H_g(x)+2H_h(y),$$

whence

$$H_h(x)+2H_f(y)=H_g(x)+2H_h(y),\qquad x,y\in S.$$

Fix $y_0 \in S$ arbitrarily and put $\alpha := 2H_f(y_0)$, $\beta := 2H_h(y_0)$ to get the relationship

$$H_h(x) + \alpha = H_g(x) + \beta, \quad x \in S.$$

Similarly, by fixing an x_0 from S and setting $\gamma := \frac{1}{2}H_h(x_0)$, $\delta := \frac{1}{2}H_g(x_0)$ we arrive at

$$H_f(y) + \gamma = H_h(y) + \delta, \quad y \in S.$$

Now, with the aid of the embedding technics applied in the proof of Theorem 1, (we omit the details of that standard procedure) we deduce that the corresponding functions H_f^*, H_g^* and H_h^* mapping S into the group G^* are pairwise equal. This, in turn, forces the functions H_f, H_g and H_h to be pairwise equal, as well. Therefore, we finish the proof by setting $H := H_f = H_g = H_g$. ■

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