ON MULTIPLICATIVE FUNCTIONS WITH SHIFTED ARGUMENTS

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Dedicated to Professor Antal Járai on his 60th anniversary

Abstract. It is proved that for given integers $a > 0$, $c > 0$, b, d with $ad - cb \neq 0$ there exists a constant $\eta > 0$ with the following property: If unimodular multiplicative functions g_1, g_2 satisfy $|g_1(p) - 1| < \eta$ and $|g_2(p) - 1| < \eta$ for all $p \in \mathcal{P}$, then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g_1(an+b) - \Gamma\, g_2(cn+d)| = 0$$

may hold with some $\Gamma \in \mathbb{C} \setminus \{0\}$ if $g_1(n) = g_2(n) = 1$ for all positive integers $n \in \mathbb{N}$, $(n, ac(ad-cb)) = 1$.

1. Introduction

An arithmetic function $g(n) \not\equiv 0$ is said to be multiplicative if $(n, m) = 1$ implies that

$$g(nm) = g(n)g(m)$$

and it is completely multiplicative if this relation holds for all positive integers n and m. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the class of all complex-valued multiplicative and completely multiplicative functions, respectively. A function g is said to be

The Project is supported by the European Union and co-financed by the European Social Fund (grant agreement no. TAMOP 4.2.1./B-09/1/KMR-2010-0003).

https://doi.org/10.71352/ac.35.035

unimodular if g satisfies the condition $|g(n)| = 1$ for all positive integers n. In the following we shall denote by $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

Let $\mathcal{A}, \mathcal{A}^*$ be the set of real valued additive and completely additive functions, respectively. As usual, let $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, integers, real and complex numbers, respectively. For each real number z we define $\| z \|$ as follows:

$$\| z \| = \min_{k \in \mathbb{Z}} | z - k | \,.$$

A. Hildebrand [1] proved the following

Theorem A. *There exists a positive constant δ with the following property. If $g \in \mathcal{M}^*(1)$ and $|g(p) - 1| \le \delta$ holds for every $p \in \mathcal{P}$, then either $g(n) = 1$ for all $n \in \mathbb{N}$ identically, or*

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| > 0.$$

By using the ideas of Hildebrand [1] and himself, I. Kátai [2] proved the following generalization:

Theorem B. *Let $g \in \mathcal{M}^*(1)$. There exist positive constants δ and $\beta < 1$ with the property: If*

$$\limsup_{x\to\infty} \sum_{x^\beta < p < x} \frac{|g(p) - 1|}{p} < \delta$$

and

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{\frac{x}{2} \le n \le x} |g(n+1) - g(n)| = 0,$$

then $g(n) = 1$ for all $n \in \mathbb{N}$ identically.

Our purpose in this paper is to prove the following

Theorem. *Let $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ with $ad - cb \neq 0$. There exists a constant $\eta > 0$ with the following property:*

If $g_1, g_2 \in \mathcal{M}(1)$, $|g_1(p) - 1| < \eta$ and $|g_2(p) - 1| < \eta$ for all $p \in \mathcal{P}$, then

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n \le x} |g_1(an+b) - \Gamma g_2(cn+d)| = 0$$

may hold with some $\Gamma \in \mathbb{C} \setminus \{0\}$ *if*

$$g_1(n) = g_2(n) = 1 \quad \textit{for all} \quad n \in \mathbb{N}, \ (n, ac(ad - cb)) = 1.$$

As a direct consequence we can formulate the next

Corollary. *Let* $a, c \in \mathbb{N}$, $b, d \in \mathbb{Z}$ *with* $ad - cb \neq 0$. *There exists a constant* $\eta > 0$ *with the following property:*

If $f_1, f_2 \in \mathcal{A}$, $\|f_1(p)\| < \eta$ *and* $\|f_2(p)\| < \eta$ *for all* $p \in \mathcal{P}$, *then*

$$\liminf_{x \to \infty} \frac{1}{x} \sum_{n \le x} \|f_1(an + b) - f_2(cn + d) - \Delta\| = 0$$

may hold with some $\Delta \in \mathbb{R}$ *if*

$$\|f_1(n)\| = \|f_2(n)\| = 0 \quad \textit{for all} \quad n \in \mathbb{N}, \ (n, ac(ad - cb)) = 1.$$

We note that I. Kátai [2] has conjectured that if

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \|f(n + 1) - f(n)\| = 0,$$

then there is a real number $\lambda \in \mathbb{R}$ such that

$$\|f(n) - \lambda \log n\| = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

This conjecture remains open.

2. Lemmata

N. M. Timofeev [3] proved the following assertion (see [3], Lemma 1):

Lemma 1. *Suppose that* $f_1(n)$ *and* $f_2(n)$ *are multiplicative with* $|f_1(n)| \le 1$ *and* $|f_2(n)| \le 1$ *that satisfy the condition*

$$\sum_{p \le x} \big(|f_1(p) - 1| + |f_2(p) - 1|\big) \frac{\log p}{p} \le \varepsilon(x) \log x, \tag{2.1}$$

where $\varepsilon(x)$ *is a decreasing function that approaches zero as* $x \to \infty$, *but* $\varepsilon(x)\sqrt{\log x}$ *approaches infinity as* $x \to \infty$, *and let* $a > 0,\ b,\ c > 0,\ d,\ a_j,\ b_j,\ \delta_j$ $(j = 1, 2)$ *be integers with*

$$a = \delta_1 a_1, \quad b = \delta_1 b_1, \quad c = \delta_2 a_2, \quad d = \delta_2 b_2,$$

$$(a_1, b_1) = 1, \quad (a_2, b_2) = 1, \quad \Delta = a_1 b_2 - a_2 b_1 \neq 0.$$

Then

$$\frac{1}{x} \sum_{n \le x} f_1(an + b) f_2(cn + d) = \prod_{p \le x} \omega_p(f_1, f_2) + O\left(\sqrt{\varepsilon(x)}\right), \tag{2.2}$$

where for $p \not| a_1 a_2 \Delta$

$$\omega_p(f_1, f_2) = \left(1 - \frac{2}{p}\right) f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right) +$$
$$+ \sum_{r=1}^{\infty} \frac{1}{p^r}\left(1 - \frac{1}{p}\right)\left[f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right) + f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right)\right];$$

if $p | a_1$, *but* $p \not| (a_1, a_2)$, *then*

$$\omega_p(f_1, f_2) = \left[f_2\left(p^{\alpha_p(\delta_2)}\right) + \sum_{r=1}^{\infty} f_2\left(p^{r+\alpha_p(\delta_2)}\right)\frac{1}{p^r}\right]\left(1 - \frac{1}{p}\right) f_1\left(p^{\alpha_p(\delta_1)}\right);$$

if $p | a_2$, *but* $p \not| (a_1, a_2)$, *then*

$$\omega_p(f_1, f_2) = \left[f_1\left(p^{\alpha_p(\delta_1)}\right) + \sum_{r=1}^{\infty} f_1\left(p^{r+\alpha_p(\delta_1)}\right)\frac{1}{p^r}\right]\left(1 - \frac{1}{p}\right) f_2\left(p^{\alpha_p(\delta_2)}\right);$$

if $p | \Delta$, *but* $p \not| a_1 a_2$, *then*

$$\omega_p(f_1, f_2) = \left(1 - \frac{1}{p}\right)\Bigg[\sum_{0 \le r \le \alpha_p(\Delta) - 1} f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right)\frac{1}{p^r} +$$
$$+ f_1\left(p^{\alpha_p(\Delta)+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\Delta)+\alpha_p(\delta_2)}\right)\left(1 - \frac{1}{p}\right)^{-1}\left(1 - \frac{2}{p}\right) +$$
$$+ \sum_{r \ge 1} \frac{1}{p^{r+\alpha_p(\Delta)}}\Bigg(f_1\left(p^{r+\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)+\alpha_p(\Delta)}\right) +$$
$$+ f_1\left(p^{\alpha_p(\delta_1)+\alpha_p(\Delta)}\right) f_2\left(p^{r+\alpha_p(\delta_2)}\right)\Bigg)\Bigg];$$

if $p | (a_1, a_2)$, *then*

$$\omega_p(f_1, f_2) = f_1\left(p^{\alpha_p(\delta_1)}\right) f_2\left(p^{\alpha_p(\delta_2)}\right).$$

Here $\alpha_p(n)$ *is the largest integer* α *such that* p^α *divides* n.

Analyzing the proof of Lemma 1, one can see that it remains true in the following form:

Lemma 1′. *Assume that in the notations of Lemma 1, instead of* (2.1)

$$\sum_{p\le x}\Big(|f_1(p)-1|+|f_2(p)-1|\Big)\frac{\log p}{p}\le \delta\log x \tag{2.3}$$

if $x>x_0(\delta)$. *Then*

$$\limsup_{x\to\infty}\left|\frac{1}{x}\sum_{n\le x} f_1(an+b)f_2(cn+d)-\prod_{p\le x}\omega_p(f_1,f_2)\right|\le C\sqrt{\delta}, \tag{2.4}$$

where C *is a constant that may depend only on* $a,\ b,\ c,\ d$.

3. Proof of the theorem

Assume that the conditions of Theorem hold and

$$\sum_{n\le x_\nu}|g_1(an+b)-\Gamma g_2(cn+d)|<\varepsilon_\nu x_\nu, \tag{3.1}$$

where $\varepsilon_\nu\searrow 0$, $x_\nu\nearrow\infty$. From (3.1) it is clear that $|\Gamma|=1$ and

$$\sum_{n\le x_\nu}|\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|<\varepsilon_\nu x_\nu.$$

Since

$$|1-z|^2=2(1-\text{Re } z)\le 2|1-z| \quad\text{when}\quad |z|=1,$$

we have

$$\sum_{n\le x_\nu}|\,\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|^2\le 2\sum_{n\le x_\nu}|\overline{\Gamma}g_1(an+b)\overline{g}_2(cn+d)-1|<2\varepsilon_\nu x_\nu,$$

which implies

$$\text{Re } 2\overline{\Gamma}\sum_{n\le x_\nu} g_1(an+b)\overline{g}_2(cn+d)\ge 2(1-\varepsilon_\nu)x_\nu. \tag{3.1$'$}$$

Let us apply Lemma 1′ with $f_1=g_1$, $f_2=\overline{g}_2$ and $\delta=2\eta$. We obtain that

$$\prod_{p\le x}|\omega_p(g_1,\overline{g}_2)|\ge 1-C\sqrt{\delta}. \tag{3.2}$$

Assume that δ is small, $C\sqrt{\delta} < 1$. Then, from (3.2), we have

$$\sum_{p\in\mathcal{P}}\Big(1-|\omega_p(g_1,\overline{g}_2)|^2\Big) < \infty.$$

If $(p, ac\Delta) = 1$, then $\alpha_p(\delta_1) = \alpha_p(\delta_2) = 0$ and

$$\omega_p(g_1,\overline{g}_2) = \left(1-\frac{2}{p}\right) + \left(1-\frac{1}{p}\right)\frac{1}{p}\big(g_1(p)+g_2(p)\big) + O\left(\frac{1}{p^2}\right) = 1+\xi_p,$$

where

$$\xi_p = \frac{1}{p}\big[(g_1(p)-1)+(g_2(p)-1)\big] + O\left(\frac{1}{p^2}\right).$$

Therefore

$$|\omega_p(g_1,\overline{g}_2)|^2 = 1+\xi_p+\overline{\xi}_p+|\xi_p|^2,$$

and so

$$\sum_{p\in\mathcal{P}}\big(1-|\omega_p(g_1,\overline{g}_2)|^2\big) = 2\mathrm{Re}\left\{\sum_{p\in\mathcal{P}}\frac{1-g_1(p)}{p}+\sum_{p\in\mathcal{P}}\frac{1-g_2(p)}{p}\right\} + O(1).$$

Since

$\mathrm{Re}\,(1-g_1(p)) \geq 0$, $\mathrm{Re}\,(1-g_2(p)) \geq 0$ and $|1-z|^2 = 2(1-\mathrm{Re}\,z)$ when $|z| = 1$,

therefore

(3.3)
$$\sum_{p\in\mathcal{P}}\frac{|1-g_j(p)|^2}{p} < \infty, \quad j = 1, 2.$$

Let

$$\sigma_j(x) = \sum_{\sqrt{x}\leq p\leq x}\frac{|1-g_j(p)|^2}{p}.$$

From (3.3) we have

$$\sum_{l=0,1,\ldots}\sigma_j(x^{1/2^l}) < c,$$

where c is a constant. Since

$$\sum_{p\leq x}\frac{1}{p} = \log\log(x) + C + O\left(\frac{1}{\log x}\right) \quad \text{where } \ C = 0.2615\ldots,$$

by applying Cauchy's inequality, we have

$$\sum_{\sqrt{x}\leq p\leq x}\frac{|1-g_j(p)|\log p}{p} \leq \log x \sum_{\sqrt{x}\leq p\leq x}\frac{1}{\sqrt{p}}\frac{|1-g_j(p)|}{\sqrt{p}} \leq$$

$$\le \log x \left(\sum_{\sqrt{x} \le p \le x} \frac{1}{p} \right)^{1/2} \left(\sum_{\sqrt{x} \le p \le x} \frac{|1 - g_j(p)|^2}{p} \right)^{1/2} \le c_1 \log x \sqrt{\sigma_j(x)}.$$

Therefore

$$\sum_{2 \le p \le x} \frac{|1 - g_j(p)| \log p}{p} \le c_1 \sum_{2^l \le \log x} \left(\log x^{1/2^l} \right) \sqrt{\sigma_j(x/2^l)} = c_1 \log x \Theta_j(x),$$

where

$$\Theta_j(x) = \sum_{2^l \le \log x} \frac{\sqrt{\sigma_j(x/2^l)}}{2^l}.$$

It is clear that $\Theta_j(x) \to 0$ $(x \to \infty)$. Let

$$\varepsilon_j(y) = \max_{x \ge y} \Theta_j(x) \quad \text{and} \quad \epsilon(y) = \epsilon_1(y) + \epsilon_2(y).$$

Thus (2.1) holds with this $\epsilon(x)$.

From (3.1)$'$ and (2.2) with $f_1 = g_1$ and $f_2 = \overline{g}_2$, we obtain that

$$\mathrm{Re}\, \overline{\Gamma} \prod_{p \in \mathcal{P}} \omega_p(g_1, \overline{g}_2) = 1,$$

which implies that

$$|\omega_p(g_1, \overline{g}_2)| = 1 \quad \text{for all} \quad p \in \mathcal{P}$$

and

$$\prod_{p \in \mathcal{P}} \omega_p(g_1, \overline{g}_2) = \Gamma.$$

It is clear that if $(p, ac\Delta) = 1$, then $\alpha_p(\delta_1) = \alpha_p(\delta_2) = 0$ (in the notations of Lemma 1), and so

$$\omega_p(g_1, \overline{g}_2) = \left(1 - \frac{2}{p}\right) + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{1}{p^r} \left(g_1(p^r) + \overline{g}_2(p^r) \right). \tag{3.4}$$

Let

$$\lambda_p = \sum_{r=1}^{\infty} \frac{1}{p^r} \left(g_1(p^r) + \overline{g}_2(p^r) \right).$$

It is clear that $|\lambda_p| \le \frac{2}{p-1}$, and one can check from (3.4) that $|\omega_p(g_1, \overline{g}_2)| < 1$, if $g_1(p^r) + \overline{g}_2(p^r) \ne 2$ for at least one r.

Thus we have $g_1(p^r) = g_2(p^r) = 1$ if $p \nmid a_1 a_2 \Delta$, $p > \max(\delta_1, \delta_2)$.

The proof of our theorem is completed. ■

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