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ON THE THEOREM OF H. DABOUSSI OVER THE GAUSSIAN INTEGERS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. Some analogues of the theorem of Daboussi over the set of Gaussian integers are investigated.

1. Introduction

Let $c, c_1, c_2, \ldots, K, K_1, K_2, \ldots$ be positive constants, not necessarily the same at every occurrence. Let $\mathcal{M}$ be the set of complex valued multiplicative functions and $\mathcal{M}_1$ be the set of those $g \in \mathcal{M}$ for which additionally $|g(n)| \leq 1$ $(n \in \mathbb{N})$ holds as well. Let $e(\alpha) := e^{2\pi i \alpha}$.

A famous theorem of H. Daboussi published in the paper written jointly with H. Delange in [2] asserts that

$$\sup_{f \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{n \leq x} f(n) e(n\alpha) \right| = \varrho_x \to 0 \qquad (x \to \infty), \tag{1.1}$$

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whenever α is an irrational number. This famous theorem has been generalized in different aspects in [1], [3]–[20]. In [2] the following assertion was proved:

Let S be an arithmetical function satisfying the following conditions:

(i) *S is almost-periodic B^1,*

(ii) *the Fourier series of S is $\lambda + \sum \lambda_\nu e(\alpha_\nu n)$, where all the α_ν are irrational.*

Then, as x tends to infinity, we have

$$\sup_{f\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{n\le x} f(n)S(n) - \frac{1}{\lambda}\sum f(n)\right| \le \varrho_x(S),$$

$\varrho_x(S) \to 0$ as $(x \to \infty)$.

In [20] the following theorem is proved.

Let $k \ge 1$ be fixed, $J_1, \dots, J_k \subseteq [0,1)$ be such sets which are the union of finitely many intervals. Let $P_1(x), \dots, P_k(x)$ be non-constant real valued polynomials,

$$Q_{m_1,\dots,m_k}(x) = m_1P_1(x) + \dots + m_kP_k(x)$$

for $m_1, \dots, m_k \in \mathbb{Z}$.

Assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has at least one irrational coefficient for every $m_1, \dots, m_k \in \mathbb{Z}$, except when $m_1 = \dots = m_k = 0$.

Let

$$S := \{n \mid n \in \mathbb{N}, \quad \{P_l(n)\} \in J_l, \quad l = 1, \dots, k\}.$$

Let λ be the Lebesgue measure.

Theorem A. *Under the conditions stated for $P_1, \dots, P_k, J_1, \dots, J_k$ we have*

$$\sup_{g\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{\substack{n\le x\\ n\in S}} g(n) - \frac{\lambda(J_1)\dots\lambda(J_k)}{x}\sum_{n\le x} g(n)\right| = \tau_x, \tag{1.2}$$

$\tau_x \to 0$ as $x \to \infty$.

By using the same method and Theorem B we can prove

Theorem 1. *Let $J_1, \dots, J_k, P_1, \dots, P_k, S$ be as above. Let P be a non-constant real valued polynomial.*

Let $R_{m_0,m_1,\dots,m_k}(x) = m_0P(x) + Q_{m_1,\dots,m_k}(x)$. Assume that

$$R_{m_0,m_1,\dots,m_k}(x) - R_{m_0,m_1,\dots,m_k}(0)$$

has at least one irrational coefficient for every $m_0, m_1, \dots, m_k$ *except the case when* $m_0 = m_1 = \ldots = m_k = 0$.

Then

$$\sup_{g\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{\substack{n\le x\\ n\in S}} g(n)e(P(n))\right| = \varrho_x \to 0, \qquad \text{as} \quad x\to\infty. \tag{1.3}$$

ϱ_x *may depend on* S *and on* P.

Theorem B. (See [7].) (1.3) *is true, if* $S = \mathbb{N}$.

Applying Theorem A for $g(n) = 1$ we obtain that

$$\frac{1}{x}\#\{n \le x \mid n \in S\} \to \lambda(J_1)\ldots\lambda(J_k).$$

From Theorem 1, by using Weyl's criterion for uniformly distributed sequences we get

Theorem 2. *Let* $J_1, \dots, J_k, P, P_1, \dots, P_k, S$ *as in Theorem* 1. *Let* $\mathcal{A}$ *be the set of additive arithmetical functions,* $S = \{t_1, t_2, \dots\}$, $t_j < t_{j+1}$ $(j = 1, 2, \dots)$, $\xi_n(f) := f(t_n) + P(t_n)$ $(n = 1, \dots)$,

$$\Delta_N(f \mid S) :=$$

$$:= \sup_{[\alpha,\beta)\subseteq[0,1)} \left|\frac{1}{N}\#\{\xi_n(f) \mod 1 \in [\alpha,\beta], n\in N\} - (\beta-\alpha)\right|. \tag{1.4}$$

Then

$$\sup_{f\in\mathcal{A}} \Delta_N(f|S) = \varrho_N \to 0 \qquad \text{as} \quad N\to\infty. \tag{1.5}$$

ϱ_N *may depend on* S.

Let $\mathcal{N}_k$ be the set of the integers the number of the prime power factors of which is k. Let $N_k(x)$ be the size of $n \le x$, $n \in \mathcal{N}_k$. In our paper [10] we proved

Theorem C. *Let* $0 < \delta(< 1)$ *be an arbitrary constant, and* α *be an irrational number. Then*

$$\lim_{x\to\infty} \sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta} \sup_{f\in\mathcal{M}_1} \frac{1}{N_k(x)}\left|\sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} f(m)e(m\alpha)\right| = 0. \tag{1.6}$$

The proof depends on an important assertion due to Dupain, Hall, Tenenbaum [4], namely that

$$\sup_{\frac{k}{\log\log x}\leq 2-\delta} \frac{1}{N_k(x)}\left|\sum_{\substack{m\leq x\\ m\in\mathcal{N}_k}} e(m\alpha)\right| \to 0 \qquad \text{as} \quad x\to\infty. \tag{1.7}$$

Theorem 3.

1.) *Let $P(n)=\alpha n$, $P_j(n)=\alpha_j n$, $(j=1,\dots,k)$, $J_1,\dots,J_k$ and S as earlier. Assume that $m\alpha+m_1\alpha_1+\dots+m_k\alpha_k$ is irrational for every nontrivial choice of $m,m_1,\dots,m_k$. Let $S_k(x)=\#\{n\leq x\mid n\in\mathcal{N}_k,\ n\in S\}$.*

Then

$$\lim_{x\to\infty}\sup_{\delta\leq\frac{k}{\log\log x}\leq 2-\delta}\ \sup_{f\in\mathcal{M}_1}\frac{1}{S_k(x)}\left|\sum_{\substack{n\leq x\\ n\in\mathcal{N}_k\cap S}} f(n)e(n\alpha)\right| = 0. \tag{1.8}$$

2.) *Let $P_1,\dots,P_k,J_1,\dots,J_k$ and S as earlier. Assume that $m_1\alpha_1+\dots+m_k\alpha_k$ is irrational for every nontrivial choice of $m_1,\dots,m_k$. Then*

$$\lim_{x\to\infty}\sup_{\delta\leq\frac{k}{\log\log x}\leq 2-\delta}\ \sup_{f\in\mathcal{M}_1}\left|\frac{1}{S_k(x)}\sum_{\substack{n\leq x\\ n\in\mathcal{N}_k\cap S}} f(n) - \frac{1}{N_k(x)}\sum_{\substack{n\leq x\\ n\in\mathcal{N}_k}} f(n)\right| = 0. \tag{1.9}$$

Since the Theorems 1, 2, 3 can be deduced from already published papers by the method used in [20], we omit the proofs of them. In the next section we formulate and prove Theorem 4.

2.

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, $\mathbb{Z}^*[i]=\mathbb{Z}[i]\setminus\{0\}$ be the multiplicative group of nonzero Gaussian integers.

Let χ be such an additive character on $\mathbb{Z}[i]$, for which $\chi(1)=e(A)$, $\chi(i)=e(B)$. Let $\mathcal{K}_1$ be the set of multiplicative functions $g:\mathbb{Z}^*[i]\to\mathbb{C}$ satisfying $|g(\alpha)|\leq 1$ $(\alpha\in\mathbb{Z}^*[i])$. Let W be the union of finitely many convex bounded domain in $\mathbb{C}$. In our paper [11] written jointly with N.L. Bassily and J.-M. De Koninck we proved

Theorem D. *Assume that at least one of A or B is irrational. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \frac{1}{|xW|} \left| \sum_{\beta\in xW} g(\beta)\chi(\beta) \right| = 0. \tag{2.1}$$

Let $I = [0,1) \times [0,1)$, $S = S_1 \cup \ldots \cup S_r \subseteq I$, where S_j are domains the boundary of which is a rectifiable continuous curve for every j. For some small $\Delta > 0$ let

$$\begin{aligned} S^{(-\Delta)} &= \{(u,v) \mid [u-\Delta, u+\Delta] \times [v-\Delta, v+\Delta] \subseteq S\}, \\ S^{(+\Delta)} &= \{(u,v) \mid [u-\Delta, u+\Delta] \times [v-\Delta, v+\Delta] \cap S \neq 0\}. \end{aligned}$$

Let

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) \in S \\ 0, & \text{if } (x,y) \in I \setminus S, \end{cases} \tag{2.2}$$

and let us extend the definition of f over $\mathbb{R}^2$ by

$$f(x+k, y+l) = f(x,y) \qquad (k,l \in \mathbb{Z}).$$

Let $\sum_{m,n\in\mathbb{Z}} a_{m,n} e(mx+ny)$ be the Fourier-series of $f(x,y)$. Let $\Delta > 0$ be so small that $S^{(+\Delta)} \subseteq I$, and

$$f_\Delta(x,y) := \frac{1}{(2\Delta)^2} \int_{-\Delta}^{\Delta} \int_{-\Delta}^{\Delta} f(x+u) f(y+v)\, \mathrm{d}u\, \mathrm{d}v. \tag{2.3}$$

Since

$$\kappa(n) := \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} e(nu)\ \mathrm{d}u = \frac{1}{4\pi i n\Delta}(e(n\Delta) - e(-n\Delta))$$

if $n \neq 0$, and $\kappa(0) = 1$, therefore the Fourier coefficients $b_{m,n}$ of f_Δ are

$$b_{m,n} = a_{m,n} \kappa(m) \cdot \kappa(n).$$

Assume that for some $\delta > 0$,

$$|a_{m,n}| \le c \left(\frac{1}{1+|m|^\delta} \right) \left(\frac{1}{1+|n|^\delta} \right), \tag{2.4}$$

c is a constant. Thus

$$|b_{m,n}| \le |a_{m,n}| \min\left(1, \frac{2}{|m|\Delta}\right) \min\left(1, \frac{2}{|n|\Delta}\right). \tag{2.5}$$

It is clear that $f_\Delta(u,v)=1$ if $(u,v)\in S^{(-\Delta)}$, and $f_\Delta(u,v)=0$ if $(u,v)\in \in I\setminus S^{(+\Delta)}$.

Let $z=u+iv\in\mathbb{C}$. The fractional part of z is defined as $\{z\}=\{u\}+i\{v\}$.

Theorem 4. *Let $\gamma_j=\xi_j+i\eta_j$ $(j=1,\dots,k)$ be distinct nonzero numbers, $\mathcal{T}=\{\beta\mid\beta\in\mathbb{Z}[i],\{\gamma_j\beta\}\in S,\ j=1,\dots,k\}$. Assume that S satisfies the conditions stated above. Assume that $\xi_1,\dots,\xi_k,\eta_1,\dots,\eta_k$ are linearly independent over $\mathbb{Q}$. Then*

$$\lim_{x\to\infty}\sup_{g\in\mathcal{K}_1}\left|\frac{1}{|xW|}\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}}g(\beta)-\frac{a_{0,0}^k}{|xW|}\sum_{\beta\in xW}g(\beta)\right|=0. \tag{2.6}$$

Here $a_{0,0,}=\lambda(S)=$Lebesgue measure of S.

Theorem 5. *Let $S,\gamma_j,\mathcal{T}$ be as above, $\chi(u+iv)=e(Au+Bv)$. Let $\mathcal{L}$ be the lattice $\{m_1\xi_1+\dots+m_k\xi_k+n_1\eta_1+\cdots n_k\eta_k\}$. Assume that either $nA\notin\mathcal{L}$ for $n\in\mathbb{Z}\setminus\{0\}$ or $nB\notin\mathcal{L}$ for $n\in\mathbb{Z}\setminus\{0\}$. Then*

$$\lim_{x\to\infty}\sup_{g\in\mathcal{K}_1}\left|\frac{1}{|xW|}\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}}g(\beta)\chi(\beta)\right|=0. \tag{2.7}$$

Proof of Theorem 4. First we observe that

$$\begin{aligned}\#\{\beta\in xW\mid\{\gamma_j\beta\}\in S^{(+\Delta)}\setminus S^{(-\Delta)}\}\le\\ \le c_1\lambda(S^{(+\Delta)}\setminus S^{(-\Delta)})\lambda(xW),\end{aligned} \tag{2.8}$$

and that $\lambda(S^{(+\Delta)}\setminus S^{(-\Delta)})\le c_2\Delta$. c_2 may depend on S. Let $F(u+iv)= =f(u,v)$, $F_\Delta(u+iv)=f_\Delta(u,v)$. In this notation

$$\begin{aligned}\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}}g(\beta)&=\sum_{\beta\in xW}g(\beta)F(\beta\gamma_1)\dots F(\beta\gamma_k)=\\ &=\sum_{\beta\in xW}g(\beta)F_\Delta(\beta\gamma_1)\dots F_\Delta(\beta\gamma_k)+\mathcal{O}(\Delta\lambda(xW)).\end{aligned} \tag{2.9}$$

Let K be so large that

$$\sum_{n\in\mathbb{Z}}\sum_{|m|\ge K}|b_{m,n}|+\sum_{|n|\ge K}\sum_{m}|b_{m,n}|\le\Delta. \tag{2.10}$$

Since $\sum b_{m,n}$ is absolutely convergent, therefore such a K exists. (See (2.5).)

Let

$$F_\Delta^{(K)}(u+iv) = \sum_{\substack{|m|\le K\\ |n|\le K}} b_{m,n} e(mu+nv). \tag{2.11}$$

Since

$$|F_\Delta(u+iv) - F_\Delta^{(K)}(u+iv)| \le \Delta,$$

from (2.9) we have

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = \sum_{\substack{m_1,\dots,m_k\\ n_1,\dots,n_k}}^{*} b_{m_1,n_1}\dots b_{m_k,n_k} \sum_{\beta\in xW} g(\beta)\chi_{m_1,\dots,n_k}(\beta).$$

The star indicates that we sum over those m_j, n_j for which $|m_j| \le K$, $|n_j| \le$ $\le K$ $(j = 1,\dots,k)$, where $\chi_{m_1,\dots,n_k}(\beta) = e(\lambda \mathrm{Re}\,\beta + \mu \mathrm{Im}\,\beta)$,

$$\lambda = \sum_{j=1}^{k}(m_j\xi_j + n_j\eta_j), \quad \mu = \sum_{j=1}^{k}(n_j\xi_j - m_j\eta_j).$$

From the assumption of the theorem we have that either λ or μ is irrational, consequently, by Theorem D we have that

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = a_{0,0}^k \sum_{\beta\in xW} g(\beta) + o_x(|xW|) + \mathcal{O}(\Delta|xW|).$$

Hence we obtain that

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) - \frac{a_{0,0}^k}{|xW|} \sum_{\beta\in xW} g(\beta) \right| \le c\Delta.$$

Since Δ is arbitrary, therefore our theorem is true. ■

The proof of Theorem 5 is similar. We omit it.

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