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ON CHARACTERIZATIONS OF PARETO DISTRIBUTION BY HAZARD RATE OF RECORD VALUE

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Dedicated to Janos Galambos on the occasion of his 70th birthday

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non negative random variables which has absolutely continuous distribution function $F(x)$ with $F(1) = 0$ and probability density function $f(x)$. Let $F(x) < 1$ for all $x > 1$ and let F belong to class C_2^+. Then $X_k \in PAR(\alpha)$ if and only if for some fixed n, $n \geq 1$, the hazard rate r of X_k is the same as the hazard rate r_1 of $W_{n,n+1}$ or the hazard rate r_2 of $Z_{n,n+1}$, where $W_{n,n+1} = \frac{X_{U(n+1)}}{X_{U(n)}}$, $Z_{n,n+1} = X_{U(n)} \cdot X_{U(n+1)}$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative density function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max(min)\{\ X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say that X_j is an upper(lower) record value of this sequence that if $Y_j > (<)Y_{j-1}$ for $j > 1$.

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By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables. If $F(x)$ has density $f(x)$, the ratio $r(x) = \frac{f(x)}{\bar{F}(x)}$, for $0 < F(x) < 1$, is called the hazard rate. We will say that F belongs to the class C_2, if $r(x)$ is either monotonically increasing or monotonically decreasing and F belongs to the class C_2^+ if $r(x)$ is monotonically increasing.

By definition the random variable $X \in PAR(\alpha)$ if the corresponding probability cdf $F(x)$ of X is of the form

$$F(x) = \begin{cases} 1 - x^{-\alpha}, & x > 1,\ \alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Ahsanullah(1995) characterized those X_k's that belong to the class C_2. Then $X_k \in E(x, \sigma)$ if and only if for some fixed n, $n \geq 1$, the hazard rate r_1 of $X_{U(n+1)} - -X_{U(n)}$ is the same as the hazard rate r of X_k.

In this paper we show characterizations of Pareto distribution by hazard rate of record value. Namely $X_k \in PAR(\alpha)$ if and only if for some fixed n, $n \geq 1$, the hazard rate r of X_k is the same as the hazard rate r_1 of $W_{n,n+1}$ or the hazard rate r_2 of $Z_{n,n+1}$ where $W_{n,n+1} = \frac{X_{U(n+1)}}{X_{U(n)}}$, $Z_{n,n+1} = X_{U(n)} \cdot X_{U(n+1)}$.

2. Results

We prove the following theorems.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d non negative random variables which has absolutely continuous cdf $F(x)$ with $F(1) = 0$ and pdf $f(x)$. Let $F(x) < 1$ for all $x > 1$ and let F belong to class C_2^+. Then X_k has the Pareto distribution if and only if for some fixed n, $n \geq 1$, the hazard rate r of X_k is the same as the hazard rate r_1 of $W_{n,n+1}$ where $W_{n,n+1} = \frac{X_{U(n+1)}}{X_{U(n)}}$.*

Proof. If $X_k \in PAR(\alpha)$, then it can easily be shown that r is the same as r_1. We need to prove sufficiency only. Suppose $r = r_1$. Now we can write the joint pdf of $X_{U(n+1)}$ and $X_{U(n)}$ as

$$f_{n,n+1}(x, y) = \begin{cases} \dfrac{\{R(x)\}^{n-1}}{\Gamma(n)} r(x) f(y), & 1 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $W_{n,n+1} = \frac{X_{U(n+1)}}{X_{U(n)}}$ and $U_n = X_{U(n)}$, we get the joint pdf of $W_{n,n+1}$ and U_n as

$$f_1(u,w) = \frac{\{R(u)\}^{n-1}}{\Gamma(n)} r(u) f(uw) u \tag{2.1}$$

Thus by (2.1), we can write

$$r_1(w) = \frac{\int_1^\infty \{R(u)\}^{n-1} r(u) f(uw) u du}{\int_1^\infty \{R(u)\}^{n-1} r(u) \bar{F}(uw) du}$$

for all $w > 1$.

Since $r_1(w) = r(w)$ for all w, we have

$$\frac{\int_1^\infty \{R(u)\}^{n-1} r(u) f(uw) u du}{\int_1^\infty \{R(u)\}^{n-1} r(u) \bar{F}(uw) du} = \frac{f(w)}{\bar{F}(w)} \tag{2.2}$$

for all $w > 1$. By simplifying (2.2) we obtain

$$\int_1^\infty R(u)^{n-1} r(u) \bar{F}(w) \bar{F}(uw) \{r(uw)u - r(w)\} du = 0$$

for all $w > 1$.

Since F belongs to class C_2^+, the following equation

$$r(uw)u = r(w) \tag{2.3}$$

holds for almost all u and for any fixed $w > 1$. Integrating (2.3) with respect to w from 1 to w_1, we get

$$\bar{F}(uw_1) = \bar{F}(u)\bar{F}(w_1), \ \textit{for all } w_1 > 1. \tag{2.4}$$

By the theory of functional equations[see Aczél (1966)], the only continuous solution of (2.4) with the boundary condition $F(1) = 0$ is

$$\bar{F}(x) = x^{-\alpha}$$

for all $x > 1$ and $\alpha > 0$. Consequently, $F(x) = 1 - x^{-\alpha}$.

This completes the proof. ■

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d non negative random variables which has absolutely continuous cdf $F(x)$ with $F(1) = 0$ and pdf $f(x)$. Let $F(x) < 1$ for all $x > 1$ and let F belong to class C_2^+. Then X_k has the Pareto distribution if and only if for some fixed n, $n \geq 1$, the hazard rate r of X_k is the same as the hazard rate r_2 of $Z_{n,n+1}$ where $Z_{n,n+1} = X_{U(n)} \cdot X_{U(n+1)}$.*

Proof. The necessary condition is easy to establish. We will prove here the sufficiency of the condition. Suppose that r is the same as r_2.

Now we can write the joint pdf of $X_{U(n+1)}$ and $X_{U(n)}$ as

$$f_{n,n+1}(x,y) = \begin{cases} \frac{\{R(x)\}^{n-1}}{\Gamma(n)} r(x) f(y), & 1 < x < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $Z_{n,n+1} = X_{U(n)} \cdot X_{U(n+1)}$ and $U_n = X_{U(n)}$, we get the pdf of $Z_{n,n+1}$ and U_n as

$$f_2(u,z) = \frac{\{R(u)\}^{n-1}}{\Gamma(n)} r(u) f(\frac{z}{u}) u^{-1}. \tag{2.5}$$

Thus by (2.5), we can write

$$r_2(z) = \frac{\int_1^\infty \{R(u)\}^{n-1} r(u) f(\frac{z}{u}) u^{-1} du}{\int_1^\infty \{R(u)\}^{n-1} r(u) \bar{F}(\frac{z}{u}) du}$$

for all $z > 1$.

Since $r_2(z) = r(z)$ for all z, we have

$$\frac{\int_1^\infty \{R(u)\}^{n-1} r(u) f(\frac{z}{u}) u^{-1} du}{\int_1^\infty \{R(u)\}^{n-1} r(u) \bar{F}(\frac{z}{u}) du} = \frac{f(z)}{\bar{F}(z)} \tag{2.6}$$

for all $z > 1$. By simplifying (2.6) we have

$$\int_1^\infty R(u)^{n-1} r(u) \bar{F}(z) \bar{F}(\frac{z}{u}) \{r(\frac{z}{u}) u^{-1} - r(z)\} du = 0$$

Thus if $F \in C_2^+$, then above equation holds if

$$r(\frac{z}{u}) u^{-1} = r(z) \tag{2.7}$$

for almost all u and for any fixed $z > 1$. Integrating (2.7) with respect to z from 1 to z_1 we get

$$\bar{F}(\frac{z_1}{u}) = \bar{F}(\frac{1}{u}) \bar{F}(z_1), \ for \ all \ z_1 > 1. \tag{2.8}$$

By the theory of functional equations[see Aczel(1966)], the only continuous solution of (2.8) with the boundary condition $F(1) = 0$ is

$$\bar{F}(x) = x^{-\alpha}$$

for all $x > 1$ and $\alpha > 0$. Hence $F(x) = 1 - x^{-\alpha}$.

This completes the proof. ∎

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