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RENEWAL THEOREMS FOR SOME WEIGHTED RENEWAL FUNCTIONS

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Dedicated to János Galambos on his seventieth anniversary

1. Introduction

Let $X, X_1, X_2, \ldots$ be a family of integer valued, independent and identically distributed random variables with positive mean μ and finite (positive) variance σ. Let $S_n = X_1 + \ldots + X_n$. The asymptotic behaviour of the weighted sum

$$R(k) = \sum_{n=1}^{\infty} a_n P(S_n = k) \tag{1.1}$$

has been investigated in a paper of Galambos, Indlekofer and Kátai [1]. In the special case $a_n = \tau_r(n)$, the number of solutions of the equation $n = n_1 n_2 \ldots n_r$ in positive integers n_j, $1 \leq j \leq r$, $R(k)$ becomes the renewal function $Q(k)$ for a random walk in r dimensional time whose terms are distributed as X. This special (important!) case has been investigated earlier by Maejima and Mori [2], Ney and Wainger [3] and Galambos and Kátai [4].

The main results proved in [1] are the following.

Assume that $a_n \geq 0, \quad a_n = \mathcal{O}(n^{\varepsilon})$, for every $\varepsilon > 0$. Let us assume that, with some positive constants c_1, c_2, c_3, c_4 the inequalities

$$c_1 h L(x) \leq A(x+h) - A(x) \leq c_2 h L(x) \tag{1.2}$$

and

$$c_3 \leq \frac{L(h)}{L(x)} \leq c_4 \tag{1.3}$$

hold with a positive function $L(x)$ for all $x \geq 1$ and $\sqrt{x} \leq h \leq x$.

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Let

$$R_1(k) = \sum_{n=1}^{\infty} a_n \varphi_n(k), \tag{1.4}$$

where

$$\varphi_n(k) = \varphi_n(k; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{1}{2}\xi_{n,k}^2\right) \tag{1.5}$$

with

$$\xi_{n,k} = \frac{n\mu - k}{\sigma\sqrt{n}}.$$

Theorem A. *Assume that X and a_n satisfy the conditions related above hold true. Then*

$$R(k) = R_1(k) + o(R_1(k)) \qquad \textit{as} \quad k \to \infty. \tag{1.6}$$

Furthermore, with suitbale positive constants c_5 and c_6,

$$c_5 \le \frac{R_1(k)}{L(k)} \le c_6. \tag{1.7}$$

Theorem B. *Let $a_n \ge 0$, $\quad a_n = \mathcal{O}(n^{\varepsilon})$ for every $\varepsilon > 0$. Let $L(x)$ be a positive function for which* (1.3) *holds. Furthermore assume that $A(x) < cxL(x)$ with some positive constants c and that the lower inequality of* (1.2) *is valid. Let X satisfy the conditions of Theorem A as well as the condition $\int\limits_{|x| \ge z} x^2 dF(x) = \mathcal{O}(z^{-a})$ with a suitable constant $0 < a < 1$, where $F(x)$ is the distribution function of X. Then* (1.6) *holds.*

Theorem C. *Let a_n be as in Theorem B, furthermore assume that* (1.2) *and* (1.3) *hold. Furthermore assume that there exists a positive function $\varrho(x)$, tending to zero monotonically, such that*

$$(A(x+h) - A(x))/hL^*(x) \to 1 \qquad (\textit{as} \quad x \to \infty) \tag{1.8}$$

uniformly in $h \in (\varrho(x)\sqrt{x}, \sqrt{x})$, where $L^*(x)$ is a very slowly varying function in the sense that, as $x \to \infty$,

$$L^*(Y(x))/L^*(x) \to 1, \qquad \textit{whenever} \quad \frac{\log Y(x)}{\log x} \to 1. \tag{1.9}$$

Then, as $k \to \infty$,

$$\frac{R_1(k)}{\frac{1}{\mu} L^*\left(\frac{k}{\mu}\right)} \to 1 \qquad (k \to \infty). \tag{1.10}$$

In the next section we shall give some examples of a_n originated by some functions defined on the lattice points for which the above conditions assumed in Theorem A, B, C hold.

2.

2.1.

Let

$$\varepsilon(x) = (\log\log x)^{-\frac{1}{5}}(\log x)^{\frac{3}{5}}. \tag{2.1}$$

Theorem 1. *Let* $\alpha(m) \quad (m = 1, 2, \ldots)$ *be such a sequence of real numbers for which* $\alpha(m) = \mathcal{O}(\tau_3(m))$*, and*

$$E(x) = \sum_{m\le x} \alpha(m) \ll x\exp(-c\varepsilon(x)) \tag{2.2}$$

holds with a positive constant c.

Let

$$\begin{cases} a_N^{(1)} := \displaystyle\sum_{N=nm^2} \tau(n)\alpha(m), \\ a_N^{(2)} := \displaystyle\sum_{N=nm^2} \tau_3(n)\alpha(m). \end{cases} \tag{2.3}$$

Let

$$A^{(1)}(x) = \sum_{N\le x} a_N^{(1)}, \quad A^{(2)}(x) = \sum_{N\le x} a_N^{(2)}. \tag{2.4}$$

Then

$$A^{(1)}(x) = A_1 x\log x + A_2 x + \mathcal{O}(\sqrt{x}\exp(-c\varepsilon(x))) \tag{2.5}$$

and

$$A^{(2)}(x) = A_3 x(\log x)^2 + A_4 x\log x + A_5 x + \mathcal{O}(\sqrt{x}\exp(-c\varepsilon(x))) \tag{2.6}$$

where A_1, A_2, A_3, A_4, A_5 *are constants.*

Proof of Theorem 1.

Lemma 1. *From* (2.2) *we have*

$$\sum_{m\ge z} \frac{\alpha(m)}{m^2} \ll \frac{1}{z}\exp(-c_1\varepsilon(z)), \tag{2.7}$$

$$\sum_{m \geq z} \frac{\alpha(m)(\log m)^l}{m^2} \ll \frac{1}{z} \exp\left(\frac{-c_1}{2}\varepsilon(z)\right) \quad l = 1, 2 \dots \tag{2.8}$$

where $c_1 > 0$ is a suitable constant.

Proof of Lemma 1. The left hand side of (2.7) is

$$\int_z^\infty \frac{dE(u)}{u^2} = \frac{E(u)}{u^2}\bigg|_z^\infty + 2\int_z^\infty \frac{E(u)}{u^3} du \ll \frac{1}{z} \exp(-c_1 \varepsilon(z))$$

i.e. (2.7) is true. The proof of (2.8) is similar, we omit it.

We shall prove (2.6).

Let $A^{(2)}(x) = \Sigma_1 + \Sigma_2 - \Sigma_3$, where

$$\Sigma_1 = \sum_{m \leq \sqrt{Y}} \alpha(m) \left\{ \sum_{\nu \leq \frac{x}{m^2}} \tau_3(\nu) \right\} = \sum_{m \leq \sqrt{Y}} \alpha(m) T_3\left(\frac{x}{m^2}\right), \tag{2.9}$$

$$\Sigma_2 = \sum_{\nu \leq \frac{x}{Y}} \tau_3(\nu) E\left(\sqrt{\frac{x}{\nu}}\right), \tag{2.10}$$

$$\Sigma_3 = E(\sqrt{Y}) T_3\left(\frac{x}{Y}\right), \tag{2.11}$$

$$Y = x \exp(-c_1 \varepsilon(x)) \tag{2.12}$$

and

$$T_3(x) = \sum_{n \leq x} \tau_3(n). \tag{2.13}$$

Let

$$P(t) = \frac{1}{2}t^2 + (3\gamma - 1)t + (3\gamma^2 - 3\gamma + 3\gamma_1 + 1) = a_2 t^2 + a_1 t + a_0,$$

where γ and γ_1 are given from the Laurent expansion of $\zeta(s)$ around $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^\infty \gamma_k (s-1)^k.$$

According to the result of G. Kolesnik [5], (see A. Ivič [6])

$$|T_3(x) - xP(\log x)| \ll x^{\frac{43}{96}+\varepsilon} \tag{2.14}$$

for every $\varepsilon > 0$. Let $\theta = \frac{43}{96} + \varepsilon (< \frac{1}{2})$.

We have

(2.15)

$$\Sigma_1 = \sum_{m\le\sqrt{Y}} \frac{\alpha(m)x}{m^2} P\left(\log\frac{x}{m^2}\right) + \mathcal{O}\left(x^\theta \cdot \sum_{m\le\sqrt{Y}} \frac{|\alpha(m)|}{m^{2\theta}}\right) = \Sigma_{1,1} + \Sigma_{1,2}.$$

Since

$$\sum_{m\le\sqrt{Y}} \frac{|\alpha(m)|}{m^{2\theta}} \le c \sum_{m\le\sqrt{Y}} \frac{\tau_3(m)}{m^{2\theta}} \ll Y^{\frac{1-2\theta}{2}} (\log x)^2,$$

therefore the error term $\Sigma_{1,2}$ in (2.15) is $\mathcal{O}(\sqrt{x}\exp(-c_1\varepsilon(x)))$.

Let us write

$$\begin{aligned}\Sigma_{1,1} =& x\sum_{m\le\sqrt{Y}} \frac{\alpha(m)}{m^2}\left\{a_2\left(\log\frac{x}{m^2}\right)^2 + a_1\left(\log\frac{x}{m^2}\right) + a_0\right\} = \\ =& x\sum_{m\le\sqrt{Y}} \frac{\alpha(m)}{m^2}\{P(\log x) + (-2a_2\log m + a_1)\log x + \\ &+ \{2(\log m)^2 - 2a_1\log m\}\}.\end{aligned}$$

From the conditions on $\alpha(m)$, and from Lemma 1 we obtain that

$$\Sigma_{1,1} = a_4 x(\log x)^2 + a_5 x(\log x) + a_6 x + \mathcal{O}(\sqrt{x}\exp(-c_1\varepsilon(x)))$$

holds with suitable constants a_4, a_5, a_6, furthermore $a_4 = \frac{1}{4}$.

Now we estimate the sum Σ_2.

We have

$$\sum_{U\le\nu\le 2U} \frac{\tau_3(\nu)}{\sqrt{\nu}} \ll \sqrt{U}\log U.$$

From (2.2),

$$\begin{aligned}&\Sigma_2 \le \sum_{\nu\le\frac{x}{Y}} \tau_3(\nu)\left(\frac{x}{\nu}\right)^{\frac{1}{2}} \exp\left(-c\varepsilon\left(\frac{x}{\nu}\right)^{\frac{1}{2}}\right) = c\sum_{t\ge 0}\sum_{\nu\in J_t},\\ &J_t = \left[\frac{x}{Y}\cdot 2^{-t-1}, \frac{x}{Y}\cdot 2^{-t}\right].\end{aligned}$$

Since

$$\sqrt{x}\cdot\sum_{\nu\in J_t} \frac{\tau_3(\nu)}{\sqrt{\nu}} \exp(-c\varepsilon(2^tY)) \ll \sqrt{x}\cdot\left(\frac{x}{Y}2^{-t}\right)^{\frac{1}{2}} (\log x)\exp(-c\varepsilon(2^tY)),$$

therefore

$$\Sigma_2 \ll \frac{x \log x}{\sqrt{Y}} \sum_t \exp(-c\varepsilon(2^t Y)) \ll \sqrt{x} \exp(-c_2 \varepsilon(x)),$$

if $0 < c_2$ is small enough.

Finally,

$$\Sigma_3 \ll \sqrt{Y} \exp(-c_1 \varepsilon(x)) \cdot \left(\frac{x}{Y} \log^2 \frac{x}{Y}\right) \ll \sqrt{x} \exp(-c_3 \varepsilon(x)),$$

with some $c_3 > 0$.
The proof of (2.6) is completed. ■

Remark. The proof of (2.5) is similar. Instead of $T_3(x)$ we have to take

$$T_2(x) = \sum_{n \le x} \tau(n),$$

and instead of $\tau_3(n)$ the function $\tau(n)$.
It is known that

$$|T_2(x) - x(\log x + (2\gamma - 1))| \le c x^{\frac{35}{108}} \cdot \log^2 x.$$

We omit the details.

2.2.

Let $\zeta(s)$ be the Riemann zeta function. Then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{2.16}$$

where μ is the Möbius function. μ is multiplicative $\mu(p) = -1, \quad \mu(p^a) = 0 \quad (a = 2, 3, \ldots)$ for primes p.

Lemma 2. *Let*

$$\frac{1}{\zeta(s)^l} = \sum_{n=1}^{\infty} \frac{\nu_l(n)}{n^s} \quad (l = 1, 2, \ldots), \tag{2.17}$$

$$N_l(x) = \sum_{\nu \le x} \nu_l(n). \tag{2.18}$$

For every fixed $l \in \mathbb{N}$

$$N_l(x) \ll x \exp(-c_l \varepsilon(x)), \tag{2.19}$$

where c_l is a positive, suitable constant.

Proof. We can follow the argument used by A. Ivič. According to Lemma 12.3 in [6] (page 310) there is an absolute constant $C > 0$ such that

$$\frac{1}{\zeta(s)} = \mathcal{O}\left((\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}} \right) \tag{2.20}$$

in the region $(s = \sigma + it)$

$$\sigma \geq 1 - \frac{c}{(\log t)^{\frac{2}{3}}} \frac{1}{(\log\log t)^{\frac{1}{3}}} \quad T_0 \leq t \leq T \tag{2.21}$$

and $\zeta(s) \neq 0$ in (2.21). We note that this assertion is a very deep result due to N.M. Korobov and I.M. Vinogradov (see [6]).

By using the Perron-formula,

$$N_l(x) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x} - iT}^{1+\frac{1}{\log x} + iT} \frac{x^s}{s\zeta^l(s)} ds + \mathcal{O}(\sqrt{x}),$$

if $T \geq x^2, \quad x = [x] + \frac{1}{2}$.

See e.g. in K. Prachar [7], Appendix §3. Transforming the integration line as Ivič did (see page 314), we obtain Lemma 2. ■

2.3.

Let D be the set of those lattice points n_1, n_2, n_3 for which $n_1, n_2, n_3 \in \mathbb{N}$ and n_1, n_2, n_3 are square-free numbers.

Let $A_D(x) := \#\{(n_1, n_2, n_3) \in D | n_1 n_2 n_3 \leq x\}$. Since $\frac{\zeta(s)}{\zeta(2s)} = \sum_{n \geq 1} \frac{|\mu(n)|}{n^s}$, therefore

$$\left(\frac{\zeta(s)}{\zeta(2s)} \right)^3 = \zeta(s)^3 \cdot \frac{1}{\zeta(2s)^3} = \left\{ \sum \frac{\tau_3(n)}{n^s} \right\} \left\{ \sum \frac{\nu_3(n)}{n^{2s}} \right\}.$$

Since the conditions for $\alpha(m) = \nu_3(m)$ in Theorem 1 hold, therefore for $A_D(x)$ $(= A^{(2)}(x))$ the relation (2.6) is true. A_3, A_4, A_5 are suitable constants.

Corollary 1. *Let*

$$R_D(k) = \sum_{(n_1, n_2, n_3) \in D} P(S_{n_1 n_2 n_3} = k).$$

Assume that the conditions stated for X in Theorem A *hold true. Then*

$$\frac{Q(k)}{\frac{1}{\mu} P\left(\log \frac{k}{\mu}\right)} \to 1 \qquad (k \to \infty)$$

where $P(u) = A_3u^2 + A_4u + A_5$, $\quad A_3, A_4, A_5$ *are computed from* $A^{(2)}(x)$ *in the case* $A^{(2)}(x) = A_D(x)$.

Remarks. We can prove similar theorems

(i) for the subset of lattice points (n_1, n_2, n_3) such that n_1, n_2 run over the square-free positive numbers, and n_3 over all positive integers;

(ii) for the subset of lattice points (n_1, n_2, n_3) such that n_1 runs over the positive square-free numbers, $n_2, n_3 \in \mathbb{N}$;

(iii) for $D = \{(n_1, n_2) | n_1, n_2$ are square-free$\}$;

(iv) for $D = \{(n_1, n_2) | n_1$ square-free, $n_2 \in \mathbb{N}\}$.

3.

Let $M_1, M_2 \in \mathbb{N}$, $\quad (0 \le) l_1 < \ldots < l_T (\le M_1)$, $\quad (0 \le) k_1 < \ldots < k_R (\le \le M_2)$, $\quad GCD(l_j, M_1) = 1$ $(j = 1, \ldots, T)$, $\quad GCD(k_l, M_2) = 1$ $(l = 1, \ldots, R)$.

Let $D = \left\{ (n,m) \middle| \begin{array}{ll} n \equiv l_\nu \pmod{M_1}, & \nu = 1, \ldots, T \\ m \equiv k_\mu \pmod{M_2}, & \mu = 1, \ldots, R \end{array} \right\}$,

$A_D(x) = \sum_{\substack{nm \le x \\ (n,m) \in D}} 1.$

Theorem 2. *We have*

$$A_D(x) = A_5 x \log x + A_6 x + \mathcal{O}\left(x^{\frac{1}{3}} (\log x)^A\right). \tag{3.1}$$

Consequently, if

$$Q_D(k) = \sum_{(n_1,n_2) \in D} P(S_{n_1,n_2} = k), \tag{3.2}$$

and the conditions of Theorem A *hold true, then*

$$\frac{Q_D(k)}{\frac{1}{\mu} \log \frac{k}{\mu}} \to 1 \qquad (k \to \infty). \tag{3.3}$$

The proof of (3.1) can be done by the standard method used for proving that $\sum_{n \le x} \tau(n) - x(\log x + (2\gamma - 1)) = \mathcal{O}\left(x^{\frac{1}{3}}\right)$. See e.g. E. Krätzel [8]. (3.3) is a consequence of (3.1) of Theorem A.

4.

4.1.

Let $0 \leq \alpha < 1, \quad 0 \leq \beta < 1,$

$$D(x) = \# \left\{ (n,m) \in \mathbb{N}^2, \quad (n+\alpha)(m+\beta) \leq x \right\}.$$

Theorem 3. *We have*

$$D(x) = x \log x + c(\alpha,\beta)x + R_x(\alpha,\beta)\sqrt{x} + \mathcal{O}\left(x^{\frac{1}{3}} (\log x)^2 \right),$$

where $c(\alpha,\beta)$ and $R_x(\alpha,\beta)$ are defined in the end of the proof. $R_x(\alpha,\beta)$ is bounded.

Proof. Let $\psi(u) = \{u\} - \frac{1}{2}$. Then $[u] = \left(u - \frac{1}{2}\right) - \left(\{u\} - \frac{1}{2}\right) = \left(u - \frac{1}{2}\right) - \psi(u)$. Let us write $D(x) = \Sigma_1 + \Sigma_2 - \Sigma_3$, where

$$\begin{aligned}
\Sigma_1 &= \sum_{n \leq \sqrt{x} - \alpha} [\varphi(n) - \beta], \qquad \varphi(n) = \frac{x}{n+\alpha}, \\
\Sigma_2 &= \sum_{m \leq \sqrt{x} - \beta} [\varphi^*(m) - \alpha], \qquad \varphi^*(m) = \frac{x}{m+\beta}, \\
\Sigma_3 &= \#\{n | n \leq \sqrt{x} - \alpha\} \cdot \#\{m | m \leq \sqrt{x} - \beta\}. \qquad \blacksquare
\end{aligned}$$

Lemma 3 (Theorem 2.2 in E. Krätzel [8]). *Let $f(t)$ be a real function in $[a,b]$, twice continuously differentiable, and let $|f''(t)| \geq \lambda_2 > 0$. Then*

$$\sum_{a < n \leq b} \psi(f(n)) \ll \frac{|f'(b) - f'(a)|}{\lambda_2^{\frac{3}{2}}} + \frac{1}{\sqrt{\lambda_2}}.$$

Lemma 4 (Theorem 2.3 in E. Krätzel [8]). *Let $f(t)$ be a real function in $[a,b]$, twice continuously differentiable. Let $f''(t)$ be monotonic and b either positive or negative throughout. Then*

$$\sum_{a < n \leq b} \psi(f(n)) \ll \int_a^b |f''(t)|^{\frac{1}{3}} dt + \frac{1}{\sqrt{|f''(a)|}} + \frac{1}{\sqrt{|f''(b)|}}.$$

Estimation of Σ_1.

We shall write $\Sigma_1 = \Sigma_A - \Sigma_B$, where

$$\Sigma_A = \sum_{n \le \sqrt{x}-\alpha} \left(\varphi(n) - \beta - \frac{1}{2}\right), \qquad \Sigma_B = \sum_{n \le \sqrt{x}-\alpha} \psi(\varphi(n) - \beta)$$

$$\Sigma_A = \sum_{n \le \sqrt{x}-\alpha} \varphi(n) - (\beta+1)\left[\sqrt{x}-\alpha\right].$$

We have

$$\sum_{n \le \sqrt{x}-\alpha} \frac{1}{n+\alpha} = \int_{1-0}^{\sqrt{x}-\alpha} \frac{d([u]-\frac{1}{2})}{u+\alpha} = \int_{1-0}^{\sqrt{x}-\alpha} \frac{d\left(u-\frac{1}{2}\right)}{u+\alpha} - \int_{1-0}^{\sqrt{x}-\alpha} \frac{d\psi(u)}{u+\alpha}$$

$$= \log\sqrt{x} - \log(1+\alpha) - \frac{\psi(u)}{u+\alpha}\bigg|_1^{\sqrt{x}-\alpha} + \int_{1-0}^{\sqrt{x}-\alpha} \frac{\psi(u)}{(u+\alpha)^2} du.$$

Thus

$$\sum_{n \le \sqrt{x}-\alpha} \frac{1}{n+\alpha} = \frac{1}{2}\log x + C_0(\alpha) - \frac{\psi(\sqrt{x}-\alpha)}{\sqrt{x}} - \int_{\sqrt{x}-\alpha}^{\infty} \frac{\psi(u)}{(u+\alpha)^2} du$$

where

$$C_0(\alpha) = -\log(1+\alpha) + \int_{1-0}^{\infty} \frac{\psi(u)}{(u+\alpha)^2} du.$$

Let

$$\sigma(x|a) = \frac{\psi(\sqrt{x}-\alpha)}{\sqrt{x}} + \int_{\sqrt{x}-\alpha}^{\infty} \frac{\psi(u)}{(u+\alpha)^2} du.$$

Observe that $\sigma(x|\alpha)\sqrt{x} = \mathcal{O}(1)$.
Then

$$\Sigma_A = \frac{x}{2}\log x + xC_0(\alpha) - x\sigma(x|\alpha).$$

To estimate Σ_B we shall use Lemma 4.

We have $\varphi'(u) = \frac{-x}{(u+\alpha)^2}$, $\varphi''(u) = \frac{2x}{(u+\alpha)^3}$. Let us apply Lemma 4 with $[a,b] = [U, 2U]$, where $2U \le \sqrt{x}-\alpha$. Then

$$\sum_{U \le n \le 2U} \psi(\varphi(n)) \ll \int_U^{2U} \left(\frac{x}{u^3}\right)^{\frac{1}{3}} du + \frac{1}{\sqrt{\frac{x}{U^3}}} \ll x^{\frac{1}{3}} \log U + \frac{U^{\frac{3}{2}}}{\sqrt{x}}.$$

Doing this with $U = (\sqrt{x}-\alpha)\cdot 2^{-l}$ $(l = 1, 2, \ldots, l_0)$ where l_0 is the smallest integer for which $(\sqrt{x}-\alpha)\cdot 2^{-l_0} \le x^{\frac{1}{3}}$, we have

$$\sum_{n \le \sqrt{x}-\alpha} \psi(\varphi(n)) \ll x^{\frac{1}{3}} (\log x)^2.$$

Thus we have

$$\Sigma_1 = \frac{x}{2}\log x + xC_0(\alpha) - x\sigma(x|\alpha) - \left(\beta + \frac{1}{2}\right)x^{\frac{1}{2}} + \mathcal{O}\left(x^{\frac{1}{3}}(\log x)^2\right).$$

Estimation of Σ_2. Completely analogously, we have

$$\Sigma_2 = \frac{x}{2}\log x + xC_0(\beta) - x\sigma(x|\beta) - \left(\alpha + \frac{1}{2}\right)x^{\frac{1}{2}} + \mathcal{O}\left(x^{\frac{1}{3}}(\log x)^2\right).$$

Estimation of Σ_3.

$$\begin{aligned}\Sigma_3 = &\left(\sqrt{x} - \alpha - \frac{1}{2} - \psi(\sqrt{x} - \alpha)\right)\left(\sqrt{x} - \beta - \frac{1}{2} - \psi(\sqrt{x} - \beta)\right) = \\ =& x - \sqrt{x}\left\{\alpha + \beta + 1 + \psi\left(\sqrt{x} - \alpha\right) + \psi\left(\sqrt{x} - \beta\right)\right\} + \mathcal{O}(1).\end{aligned}$$

Collecting our inequalities we have

$$\begin{aligned}D(x) =& x\log x + x\{C_0(\alpha) + C_0(\beta) - 1\} + \\ &+ \sqrt{x}\big\{\alpha + \beta + 1 + \psi\left(\sqrt{x} - \alpha\right) + \psi\left(\sqrt{x} - \beta\right) - \\ &- \alpha - \beta - 1 - \sqrt{x}(\sigma(x|\alpha) + \sigma(x|\beta))\big\} + \mathcal{O}(1).\end{aligned}$$

Thus our theorem holds with

$$\begin{aligned}&c(\alpha, \beta) = C_0(\alpha) + C_0(\beta) - 1, \\ &R_x(\alpha, \beta) = \psi\left(\sqrt{x} - \alpha\right) + \psi\left(\sqrt{x} - \beta\right) + \sqrt{x}\sigma(x|\alpha) + \sqrt{x}\sigma(x|\beta)),\end{aligned}$$

where

$$C_0(\alpha) = -\log(1 + \alpha) + \int_1^\infty \frac{\psi(u)}{(u + \alpha)^2}du,$$

$$\begin{aligned}R_x(\alpha, \beta) =& -\sqrt{x}\left\{\int_{\sqrt{x}-\alpha}^\infty \frac{\psi(u)}{(u + \alpha)^2}du + \int_{\sqrt{x}-\beta}^\infty \frac{\psi(u)}{(u + \beta)^2}du\right\} = \\ =& -\sqrt{x}\int_{\sqrt{x}}^\infty \frac{\psi(u - \alpha) + \psi(u - \beta)}{u^2}du.\end{aligned}$$

The theorem is proved. ■

Remark. $R_x(\alpha, \beta)$ is not constant, $\limsup |R_x(\alpha, \beta)| > 0$, $R_x(\alpha, \beta)$ is bounded in x.

4.2.

Let $E=\{e_1<e_2<\ldots\}$, $F=\{f_1<f_2<\ldots\}$, $E(x):=\#\{e\in E|e\leq \leq x\}$, $F(x)=\#\{f\in F|f\leq x\}$. Let $D=\{(e,f)|e\in E, f\in F\}$, $A_D(x)= =\#\{(e,f)\in D|ef\leq x\}$.

Theorem 4. *Assume that*

$$E(x)=c_1x+\mathcal{O}(x^\alpha),\quad F(x)=c_2x+\mathcal{O}(x^\beta),$$

where c_1, c_2 *are positive constants,* $0\leq\alpha\leq 1$, $0\leq\beta\leq 1$. *Then*

$$\begin{aligned}A_D(x)=&c_3x\log x+c_4x+\mathcal{O}(x^\gamma),\\ c_3=c_1c_2,\quad \gamma=&\max\left\{\frac{\alpha+1}{2},\frac{\beta+1}{2}\right\},\end{aligned}$$

c_4 *is a calculable constant.*

Proof. We shall start from the formula

$$\begin{aligned}A_D(x)=&\sum_{f_\mu\leq\sqrt{x}}E\left(\frac{x}{f_\mu}\right)+\sum_{e_\nu\leq\sqrt{x}}F\left(\frac{x}{e_\nu}\right)-E\left(\sqrt{x}\right)F\left(\sqrt{x}\right)+\mathcal{O}(1)\\ =&\Sigma_1+\Sigma_2-\Sigma_3+\mathcal{O}(1).\end{aligned}$$

We have

$$\Sigma_1=c_1x\sum_{f_\mu\leq\sqrt{x}}\frac{1}{f_\mu}+\mathcal{O}(x^\alpha)\sum_{f_\mu\leq\sqrt{x}}\frac{1}{f_\mu^\alpha}.$$

Let $\Delta(u):=F(u)-c_2u$.

$$\begin{aligned}T:=&\sum_{f_\mu\leq\sqrt{x}}\frac{1}{f_\mu}=\int_1^{\sqrt{x}}\frac{dF(u)}{u}=c_2\int_1^{\sqrt{x}}\frac{du}{u}+\int_1^{\sqrt{x}}\frac{\Delta(u)}{u}=\\ =&c_2\log x+\left.\frac{\Delta(u)}{u}\right|_1^{\sqrt{x}}+\int_1^{\sqrt{x}}\frac{\Delta(u)}{u^2}du.\end{aligned}$$

Let $c_4=-\Delta(1)+\int_1^\infty\frac{\Delta(u)}{u^2}du$.
We have

$$\begin{aligned}\frac{\Delta(\sqrt{x})}{\sqrt{x}}\ll&(\sqrt{x})^{\alpha-1},\\ \int_{\sqrt{x}}^\infty\frac{\Delta(u)}{u^2}du\ll&\int_{\sqrt{x}}^\infty u^{\alpha-2}du\ll(\sqrt{x})^{\alpha-1},\end{aligned}$$

thus

$$Tx = \frac{c_1 c_2}{2} x \log x + c_4 x + \mathcal{O}\left(x^{\frac{\alpha+1}{2}}\right).$$

Furthermore

$$\sum_{f_\mu \le \sqrt{x}} \frac{1}{f_\mu^\alpha} \le \sum_{n \le \sqrt{x}} \frac{1}{n^\alpha} \le (\sqrt{x})^{1-\alpha} = x^{\frac{1}{2}-\frac{\alpha}{2}}.$$

Hence we obtain that

$$\Sigma_1 = \frac{c_1}{2} x \log x + c_4 x + \mathcal{O}\left(x^{\frac{\alpha+1}{2}}\right).$$

Similarly, we can prove that

$$\Sigma_2 = \frac{c_2}{2} x \log x + c_5 x + \mathcal{O}\left(x^{\frac{\beta+1}{2}}\right).$$

with a numerically calculable constant.

Finally

$$\begin{aligned}\Sigma_3 = &\left(c_1\sqrt{x} + \mathcal{O}\left(x^{\frac{\alpha}{2}}\right)\right)\left(c_2\sqrt{x} + \mathcal{O}\left(x^{\frac{\beta}{2}}\right)\right) = \\ =&c_1 c_2 x + \mathcal{O}\left(x^{\frac{(1+\alpha)}{2}}\right) + \mathcal{O}\left(x^{\frac{(1+\beta)}{2}}\right).\end{aligned}$$

Collecting our estimations, we obtain our theorem. ■

We can prove similarly

Theorem 5. *Let*

$$E(x) = c_1 x + \mathcal{O}(\varepsilon(x)x), \quad F(x) = c_2 x + \mathcal{O}(\varepsilon(x)x),$$

$c_1, c_2 > 0, \quad \varepsilon(x) \downarrow 0$. *Then*

$$A_D(x) = c_1 c_2 x \log x + \mathcal{O}(\varepsilon(\sqrt{x}) x \log x)$$

where c_5 *is a calculable constant.*

5.

Let $\{y\}$ = fractional part of y, $\quad \| \; y \; \| = \min_{n \in \mathbb{Z}} |x - n|$. Let $x_1, \dots, x_N$ be real numbers, $S(I) = \sum_{\substack{\{x_i\} \in I \\ i=1}}^{N} 1$, where $I \subseteq [0, 1)$ is an interval.

Let

$$D(x_1, \ldots, x_N) = \sup_{I \subseteq [0,1)} \frac{1}{N} |S(I) - \lambda(I)N|,$$

$\lambda(I) =$ length of I.

Let

$$\psi_k = \sum_{j=1}^{N} e(x_j) \qquad (k = 1, 2, \ldots), \quad e(x) := e^{2\pi i x}.$$

According to a wellknown theorem due to P. Erdős and P. Turán [9]

$$ND(x_1, \ldots, x_N) \le C \left(\sum_{1 \le k \le Y} \frac{|\psi_k|}{k} + \frac{N}{Y} \right), \tag{5.1}$$

where C is an absolute constant, $Y \ge 1$ is an arbitrary number.

Let α be an irrational number, I an interval in $[0, 1)$. Let $\mathcal{A} = \{n | \{n\alpha\} \in \in I\}$, $\quad A(x) = \#\{n \le x | n \in \mathcal{A}\}$. From (5.1) we obtain that

$$|A(x) - \lambda(I)x| \le C \left(2 \sum_{1 \le k \le Y} \frac{1}{k} \cdot \frac{1}{||k\alpha||} + \frac{x}{Y} \right), \tag{5.2}$$

since in this case

$$\psi_k = \sum_{1 \le n \le x} e(kn\alpha),$$

and so

$$|\psi_k| = \left| \frac{e([x]k\alpha) - 1}{e(k\alpha) - 1} \right| \le \frac{2}{||k\alpha||}.$$

Let $\tau = \sqrt{x}$, and $\frac{A}{Q}$, $(A, Q) = 1$ be such a rational number for which $\left| \alpha - \frac{A}{Q} \right| \le \le \frac{1}{Q\tau}$, $\quad Q < \tau$ holds. Choose $Y = Q - 1$. Since $\left| k\alpha - \frac{kA}{Q} \right| < \frac{k}{Q\tau}$, therefore $||k\alpha|| > \frac{1}{2Q}$, and so $\frac{1}{||k\alpha||} \le \frac{2Q}{Q}$, thus

$$|A(x) - \lambda(I)x| \le C \left(\frac{x}{Q} + 4Q \sum_{1 \le k \le Y} \frac{1}{k} \right) \le C \left(\frac{x}{Q} + 4Q \log Q \right).$$

Lemma 5. *Let $\alpha \in (0, 1)$ be an irrational number, such that $||k\alpha|| > \frac{1}{k^{1+\kappa}}$ $\quad (k \in \in \mathbb{N})$, where κ is a fixed positive number. Let I be a subinterval in $[0, 1)$, $\mathcal{A}$ and $A(x)$ be as above. Then*

$$|A(x) - \lambda(I)x| \le Cx^{1 - \frac{1}{2(1+\kappa)}}. \tag{5.3}$$

Proof. Since Q, defined earlier satisfies $\frac{1}{Q^{1+\kappa}} < |Q\alpha - A| \leq \frac{1}{\tau} = \frac{1}{\sqrt{x}}$, we have $Q > x^{\frac{1}{2(1+\kappa)}}$. From Theorem 4 and Lemma 5 the following assertion is straightforward. ∎

Theorem 6. *Let α, β be irrational numbers, $||k\alpha||k^{1+\kappa_1} \geq 1, \quad ||k\beta||k^{1+\kappa_2} \geq \geq 1 \quad (k \in \mathbb{N})$. Let I, J be subintervals in $[0, 1)$,*

$$\mathcal{A} = \{n|\{n\alpha\} \in I\}, \quad \mathcal{B} = \{m|\{m\beta\} \in J\},$$

$$A(x) = \#\{n \leq x|n \in \mathcal{A}\}, \quad B(x) = \#\{m \leq x|m \in \mathcal{B}\},$$

$$D(x) = \#\{(n,m)|nm \leq x, n \in \mathcal{A}, m \in \mathcal{B}\}.$$

Then

$$A(x) = \lambda(I)x + \mathcal{O}\left(x^{1-\frac{1}{2(1+\kappa_1)}}\right),$$
$$B(x) = \lambda(J)x + \mathcal{O}\left(x^{1-\frac{1}{2(1+\kappa_2)}}\right)$$

and so

$$D(x) = \lambda(I)\lambda(J)x\log x + cx + \mathcal{O}\left(x^{\gamma}\right),$$

where c is a calculable constant,

$$\gamma = \max\left(1 - \frac{1}{4(1+\kappa_1)}, 1 - \frac{1}{4(1+\kappa_2)}\right).$$

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