ON QUANTITATIVE MEAN VALUE ESTIMATIONS FOR MULTIPLICATIVE FUNCTIONS

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Dedicated to the 70th Birthday of Professor János Galambos

Abstract. In this paper we use the convolution identity of Indlekofer to derive quantitative mean value estimations for a class of multiplicative functions f the values of which at primes satisfy $|f(p)-\kappa| \leq \eta < \kappa$ where $\kappa > 1/2$. This generalizes earlier results by Halász and Elliott which are valid only for completely multiplicative functions and for the case $\kappa = 1$

1. Introduction

In [3, 4, 5] a method was established to prove quantitative mean-value estimations for multiplicative functions f of modulus ≤ 1. The underlying idea was to estimate the difference of means of two arithmetic functions if the behaviour of one of them is known. To be specific, we started with an estimation of

$$M(f - A_x g, x) := \sum_{n \leq x} (f(n) - A_x g(n)) \tag{1.1}$$

where $A_x \in \mathbb{C}$ and g is multiplicative with $g(1) = 1$.

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If $|f| \le 1$ is multiplicative, we could prove, by choosing $g(n) = n^{ia}$, $a \in \mathbb{R}$, quantitative version of results by Wirsing [6] and Halász [2]. In this paper we show that the idea works perfectly in the situation where we compare f with the function $g = \tau_\kappa$ where τ_κ is defined by $(s = \sigma + it)$

$$\sum_{n=1}^{\infty} \frac{\tau_\kappa(n)}{n^s} = (\zeta(s))^\kappa \quad (\sigma > 1). \tag{1.2}$$

Here $\zeta(s)$ denotes Riemann's zeta function and $\kappa \in \mathbb{R}$. We assume that $f \neq 0$ is multiplicative and that the generating function

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p (1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^{\alpha s}}) \tag{1.3}$$

is absolutely convergent for $\sigma > 1$ and can be written in the form

$$F(s) = \exp(\sum_{n=2}^{\infty} \frac{\tilde{f}(n)\Lambda(n)}{\log n} n^{-s}) \quad \text{for } \sigma > 1 \tag{1.4}$$

where Λ denotes von Mangoldt's function. Obviously $F(s) \neq 0$ if $\sigma > 1$.

Remark 1. The connection between f and $\tilde{f}$ is given by Dirichlet's convolution

$$f(n) \log n = (\Lambda \tilde{f} * f)(n) \quad n \in \mathbb{N}$$

which holds since $f \neq 0$. From this we conclude

$$\begin{aligned} \tilde{f}(p) &= f(p) \\ \tilde{f}(p^\alpha) &= \alpha f(p^\alpha) - \sum_{\beta=1}^{\alpha-1} \tilde{f}(p^\beta) f(p^{\alpha-\beta}) \quad (\alpha \ge 2). \end{aligned} \tag{1.5}$$

To give an estimate of $M(f - A_x g, x)$ with some $A_x \in \mathbb{C}$ we assume that f is „near" to $g = \tau_\kappa$. The essential condition for this will be

$$|f(p) - \kappa| \le \eta < \kappa.$$

We prove

Theorem 1.1. *Let $f \neq 0$ be multiplicative and let $x \ge 2$. Let $\kappa > 1/2$ and $0 \le \eta_0 < \kappa$, $0 < \lambda_0 \le 2$. Let $\tilde{f}$ be defined by (1.5). Assume that*

$$|\tilde{f}(p^\alpha) - \kappa| \le \eta\alpha(2 - \lambda)^{\alpha-1} \text{ for all primes } p \text{ and all } \alpha \in \mathbb{N} \tag{1.6}$$

with $p^\alpha \le x$*, where* $0 \le \eta \le \eta_0$*, and* $\lambda_0 \le \lambda \le 2$*. Put*

$$A_x = \exp(\sum_{p \le x} \frac{f(p) - \kappa}{p}). \tag{1.7}$$

Then, if τ_κ *is given by* (1.2)*, there exist positive constants* c_1, c_2 *which depend at most on* $\kappa, \lambda_0, \eta_0$ *such that,*

$$\begin{aligned} |\sum_{n \le x} f(n) - A_x \sum_{n \le x} \tau_\kappa(n)| \le c_1 \eta x \log^{\kappa-1} x |A_x| + \\ + c_1 x \log^{\kappa-1} x \exp(\sum_{p \le x} \frac{|f(p)| - \kappa}{p}) \{\exp(-\frac{c_2}{\eta}) + \log^{-c_2} x\}. \end{aligned} \tag{1.8}$$

Remark 2. It is easy to show that the conditions of Theorem 1.1 imply that $F(s)$ converges absolutely for $\sigma > 1$ and that the estimate

$$\sum_{n \le x} f(n) \ll \frac{x}{\log x} \sum_{n \le x} \frac{|f(n)|}{n} \tag{1.9}$$

holds uniformly for $x \ge 2$. Since

$$\sum_{\substack{p^\alpha \\ \alpha \ge 2}} \frac{f(p^\alpha)}{p^\alpha} \ll \frac{1}{p^2}$$

the inequality

$$\sum_{n \le x} \frac{|f(n)|}{n} \ll \exp(\sum_{p \le x} \frac{|f(p)|}{p}).$$

is obvious.

Example 1.1.

1. Let λ be a positive real number smaller than one. Choose, for every prime p, $\lambda \le c(p) \le 2 - \lambda$. Define the multiplicative function f by

$$f(p^\alpha) = \begin{cases} \dfrac{c(p)^{\alpha+1} - 1}{c(p) - 1} & \text{if } c(p) \ne 1 \\ \alpha + 1 & \text{otherwise.} \end{cases}$$

Then f satisifies the conditions of Theorem 1.1 with $\kappa = 1$.

2. Assume that $\kappa > 1/2$ and $0 \le \eta \le \eta_0 < 1$. Let g be a completely multiplicative function with

$$|g(p) - 1| \le \eta.$$

Let the multiplicative function defined by

$$f = \tau_{\kappa-1} * g. \tag{1.10}$$

Then f fulfills the conditions of Theorem 1.1.

The case $\kappa = 1$ has been proved by Elliott and Halász (see [1], Theorem 19.2).

3. Assume that $\kappa > 1/2$ and $0 \le \eta \le \eta_0 < 1$. Let g be multiplicative with

$$|g(p) + 1| \le \eta$$

and $g(p^\alpha) = 0$ for all primes p and $\alpha \ge 2$. Then Theorem 1.1 holds for

$$f = \tau_{\kappa+1} * g. \tag{1.11}$$

Theorem 1.1 will follow from

Theorem 1.2. *Let $f \ne 0$ be multiplicative and let $x \ge 2$. Let $\kappa > 1/2$ and $0 \le \eta_0 < \kappa$, $0 < \lambda_0 \le 2$. Let $\tilde{f}$ be defined by (1.5). Assume that*

$$|\tilde{f}(p^\alpha) - \kappa| \le \eta\alpha(2-\lambda)^{\alpha-1} \text{ for all primes } p \text{ and all } \alpha \in \mathbb{N} \tag{1.12}$$

with $p^\alpha \le x$, where $0 \le \eta \le \eta_0$, and $\lambda_0 \le \lambda \le 2$. Put,

$$M(x) = \sum_{n \le x} (f - A\tau_\kappa)(n).$$

Then the estimate

$$\log^2 x |M(x)| \ll x \log x \int_1^x \frac{|M(u)|}{u^2} du + x \sum_{n \le x} \frac{|f(n)|}{n} + (\eta + \log^{-1} x)|A| x \log^{\kappa+1} x, \tag{1.13}$$

holds uniformly for all $A \in \mathbb{C}$. The implied constant depends at most on $\kappa, \eta_0, \lambda_0$.

For $0 < u \le 1$ we define the functions $H_0(u)$ and $H_1(u)$ by

$$H_0^2(u) := \int_{-\infty}^{\infty} \left| \frac{F(1+u+it) - A\zeta^\kappa(1+u+it)}{1+u+it} \right|^2 dt$$

and

$$H_1^2\left(u\right) := \int\limits_{-\infty}^{\infty} \left| \frac{F'\left(1+u+it\right) - A(\zeta^{\kappa}(1+u+it))'}{1+u+it} \right|^2 dt,$$

respectively.

The integral appearing in Theorem 1.2 can be estimated by using the following result proved in [4].

Theorem 1.3. *Let* $M(x) = \sum_{n\le x} \left(f - A\tau_{\kappa}\right)(n)$ *as above. Then*

(i) $$\int\limits_{1}^{x} \frac{|M(u)|}{u^2} du \ll H_0\left(\frac{1}{\log x}\right)(\log x)^{\frac{1}{2}},$$

(ii) $$\int\limits_{1}^{x} \frac{|M(u)|}{u^2} du \ll H_1\left(\frac{1}{\log x}\right),$$

(iii) $$\int\limits_{1}^{x} \frac{|M(u)|}{u^2} du \ll \int\limits_{\frac{1}{\log x}}^{1} \frac{H_1(y)}{y^{1/2}} dy.$$

2. A convolutional identity

As usual c will denote a constant not necessarily having the same value at different occurrences. p, q will denote prime numbers.

We recall some well known properties of the Dirichlet convolution of arithmetical functions.

In the following we use the convolution arithmetic for functions from

$$\mathcal{S} := \left\{f : \mathbb{R} \to \mathbb{C},\ f(x) = 0 \text{ for } x < 1\right\},$$

which coincides with the Dirichlet convolution for the class

$$\mathcal{A} := \left\{f \in \mathcal{S} : f(x) = 0 \text{ for } x \notin \mathbb{N}\right\}.$$

of arithmetical functions.

So, for $f, g \in \mathcal{S}$, the convolution $f * g$ in $\mathcal{S}$ is defined by

$$(f * g)(x) = \sum_{1 \leq n \leq x} f\left(\frac{x}{n}\right) g(n)$$

and

$$(f \cdot g)(x) = (fg)(x) = f(x)g(x).$$

The "action" of this definition on functions of $\mathcal{A}$ is given by the following: if $f \in \mathcal{A}$, $g \in \mathcal{S}$ then $fg \in \mathcal{A}$ and for $n \in \mathbb{N}$,

$$(f * g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d).$$

In general the operation $*$ is not commutative in $\mathcal{S}$, but if $f, g \in \mathcal{A}$ then $f * g = g * f$.

Consider the function ε defined by

$$\varepsilon(x) = \begin{cases} 1 \text{ for } x = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly $\varepsilon \in \mathcal{A}$, and

$$f * \varepsilon = f \text{ for } f \in \mathcal{S}$$

and

$$(\varepsilon * f)(x) = \begin{cases} f(x) \text{ if } \quad x \in \mathbb{N}, \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } f \in \mathcal{S}. \tag{2.1}$$

Thus ε serves as a right identity under convolution for all of $\mathcal{S}$, but it is a left identity only in $\mathcal{A}$.

The relation (2.1) suggests that for each $f \in \mathcal{S}$ we define an image $f_0 \in \mathcal{A}$ by

$$f_0 = \varepsilon * f \text{ for } f \in \mathcal{S}.$$

The Möbius function μ is defined by

$$\mathbf{1}_0 * \mu = \varepsilon,$$

where $\mathbf{1}_0 = \varepsilon * \mathbf{1}$ and $\mathbf{1} \in \mathcal{S}$ with

$$\mathbf{1}(x) = \begin{cases} 1 \text{ for } x \geq 1, \\ 0 \text{ otherwise.} \end{cases}$$

The well known Möbius inversion formula says that if $f, g \in \mathcal{S}$ then $f = g * \mathbf{1}_0$ if and only if $g = f * \mu$.

Let $L \in \mathcal{S}$ denote the logarithm function. Then obviously L acts as a derivation on $\mathcal{S}$, that is

$$L \cdot (f * g) = (L \cdot f) * g + f * (L \cdot g) \text{ for all } f, g \in \mathcal{S}.$$

Further, we introduce the von Mangoldt function $\Lambda \in \mathcal{A}$ by

$$\varepsilon * L = L_0 = \Lambda * \mathbf{1}_0,$$

i.e.

$$\Lambda = L_0 * \mu,$$

and for $f \in \mathcal{S}$ the corresponding von Mangoldt function is defined by

$$f_0 * \Lambda_f = L_0 f.$$

The quantitative estimations are described in [3]. For the sake of completeness we give the proof of

Theorem 2.1. *Let the arithmetical function $f, g \in \mathcal{A}$ satisfy $f(1) \neq 0$ and $g(1) \neq 0$. Put $M = \mathbf{1} * (f - g)$. Then*

$$L^2 M = M * (\Lambda_g * \Lambda_g + L_0 \Lambda_g) + (R_1 + R_2) * \Lambda_g + L(R_1 + R_2),$$

where

$$\begin{aligned} R_1 &= L * (f - g) \\ R_2 &= \mathbf{1} * f * (\Lambda_f - \Lambda_g). \end{aligned}$$

Proof. Obviously

$$LM = L * (f - g) + \mathbf{1} * L_0 (f - g).$$

Then

$$\begin{aligned} LM &= \mathbf{1} * (f * \Lambda_f) - \mathbf{1} * (g * \Lambda_g) + R_1 = \\ &= \mathbf{1} * (f - g) * \Lambda_g + \mathbf{1} * f * (\Lambda_f - \Lambda_g) + R_1 = \\ &= M * \Lambda_g + R_1 + R_2, \end{aligned} \tag{2.2}$$

where

$$R_1 = L * (f - g)$$

and

$$R_2 = \mathbf{1} * f * (\Lambda_f - \Lambda_g).$$

We multiply (2.2) with L and obtain

$$L^2M = LM * \Lambda_g + M * L_0\Lambda_g + LR_1 + LR_2. \tag{2.3}$$

Then, substituting (2.2) in (2.3) we arrive at

$$L^2M = M * (\Lambda_g * \Lambda_g + L_0\Lambda_g) + (R_1 + R_2) * \Lambda_g + L(R_1 + R_2)$$

which leads immediately to Theorem 2.1. ■

We prove

Lemma 2.1. *Let $f, h \in \mathcal{A}$ such that*

$$\sum_{n \le x} h(n) = cx \log x + O(x). \tag{2.4}$$

Then

$$(|M_f| * h)(x) = \sum_{n \le x} |M_f(\frac{x}{n})| h(n) =$$
$$= c \int_1^x |M_f(\frac{x}{t})| (\log t) dt + \mathcal{O}\left(x \sum_{n \le x} \frac{|f(n)|}{n} \right).$$

Proof. Let

$$H(t) := \sum_{n \le t} |f(n)|.$$

Then $H(t)$ is increasing function of t furthermore for $1 \le t < t'$

$$||\mathcal{M}_f(t)| - |\mathcal{M}_f(t')|| \le |\mathcal{M}_f(t) - \mathcal{M}_f(t')| \le H(t') - H(t).$$

The assertion follows by partial summamtion in the same way as in the proof of Lemma 3.1 in [5]. ■

3. Proof of the theorems

Proof of Theorem 1.2. We apply Theorem 2.1 and show first that

$$R_1(x) = \mathcal{O}(\frac{x}{\log x}(\sum_{n \le x} \frac{|f(n)|}{n} + |A| \log^{\kappa} x)) \tag{3.1}$$

and

(3.2)

$$R_2(x) \ll \frac{1}{\log x}\int_1^x |M(\frac{x}{t})|\log t dt + (\eta + \log^{-1} x)|A|x\log^{\kappa} x + \frac{x}{\log x}\sum_{n\le x}\frac{|f(n)|}{n}.$$

Then we deduce

$$(R_1 * \Lambda_{\tau_\kappa})(x) = \mathcal{O}\left(x(\sum_{n\le x}\frac{|f(n)|}{n} + |A|\log^{\kappa} x)\right)$$

and

$$(R_2 * \Lambda_{\tau_\kappa})(x) =$$

$$= \mathcal{O}\left(\int_1^x |M(\frac{x}{t})|\log t dt + (\eta + \log^{-1} x)|A|x\log^{\kappa+1} x + x\sum_{n\le x}\frac{|f(n)|}{n}\right).$$

Now using (1.9) we deduce

$$R_1(x) = (L * (f - A\tau_\kappa))(x) = \int_1^x \frac{M(u)}{u}du \int_1^x \frac{M_f(u) - AM_{\tau_\kappa}(u)}{u}du \ll$$

$$\ll \sum_{n\le x}\frac{|f(n)|}{n}\int_2^x \frac{1}{\log u}du + |A|\log^{\kappa} x\int_2^x \frac{1}{\log u}du.$$

This proves (3.1) since the estimates

$$\sum_{n\le x}\frac{\tau_\kappa(n)}{n} \le e^{-1}\sum_{n\le x}\frac{\tau_\kappa(n)}{n^{1+\frac{1}{\log x}}} \ll \zeta^{\kappa}(1 + \frac{1}{\log x}) \ll \log^{\kappa} x$$

hold. Now

(3.3)
$$\begin{aligned} LR_2(x) &= L(\mathbf{1} * f * (\Lambda_f - \Lambda_{\tau_\kappa}))(x) = \\ &= L * f * (\Lambda_f - \Lambda_{\tau_\kappa})(x) + \mathbf{1} * L_0 f * (\Lambda_f - \Lambda_{\tau_\kappa})(x) + \\ &+ \mathbf{1} * f * L_0(\Lambda_f - \Lambda_{\tau_\kappa})(x). \end{aligned}$$

Since

(3.4)
$$\begin{aligned} &\sum_{\substack{p^\alpha \le u \\ \alpha\ge 2}} \alpha(2-\lambda)^{\alpha-1}\log p \ll \sum_{p\le\sqrt{u}}\log p \sum_{\alpha\le\frac{\log u}{\log p}} \alpha\exp(\alpha\log(2-\lambda_0)) \ll \\ &\ll \log u\sum_{p\le 5}\exp(\frac{\log(2-\lambda_0)}{\log p}\log u) + u^{1/2-\epsilon}\log^2 u\sum_{p\le\sqrt{u}} 1 \ll u^{1-\epsilon} \end{aligned}$$

holds for some appropriate fixed $1/2 > \epsilon > 0$, we conclude

$$\sum_{n\le u} |\Lambda_f(n) - \Lambda_{\tau_\kappa}(n)| \le \eta \sum_{p^\alpha \le u} \log p + c \sum_{\substack{p^\alpha \le u \\ \alpha \ge 2}} \alpha(2-\lambda)^{\alpha-1} \log p \le \tag{3.5}$$
$$\le \eta u + c u^{1-\epsilon},$$

which implies

$$L * (\Lambda_f - \Lambda_{\tau_\kappa})(y) = \int_1^y \frac{\sum_{n\le u}(\Lambda_f - \Lambda_{\tau_\kappa})(n)}{u} du \ll y. \tag{3.6}$$

Thus

$$L * f * (\Lambda_f - \Lambda_{\tau_\kappa})(x) \ll x \sum_{n\le x} \frac{|f(n)|}{n}.$$

Observing $Lf = \Lambda_f * f$ we get

$$|\Lambda_f - \Lambda_{\tau_\kappa}| \le \eta\Lambda + c\tilde{\Lambda},$$

where

$$\tilde{\Lambda}(n) = \begin{cases} \alpha(2-\lambda_0)^\alpha \log p & \text{if } n = p^\alpha, \alpha > 1 \\ 0 & \text{otherwise .} \end{cases}$$

This leads to

$$\begin{aligned} LR_2(x) &= \mathbf{1} * f * (\Lambda_f * (\Lambda_f - \Lambda_{\tau_\kappa}) + L(\Lambda_f - \Lambda_{\tau_\kappa}))(x) + \\ &+ \mathcal{O}(x \sum_{n\le x} \frac{|f(n)|}{n}) \\ &\ll |\mathbf{1} * (f - A\tau_k)| * [(\eta\Lambda + \tilde{\Lambda}) * (\Lambda + \tilde{\Lambda}) + L(\eta\Lambda + \tilde{\Lambda})](x) + \\ &+ x \sum_{n\le x} \frac{|f(n)|}{n} + \\ &+ |A| \mathbf{1} * \tau_k * (|\Lambda_f| * |\Lambda_f - \Lambda_{\tau_\kappa}| + L|\Lambda_f - \Lambda_{\tau_\kappa}|)(x). \end{aligned} \tag{3.7}$$

Selberg's Symmetry Formula in the form

$$\sum_{n\le x} \Lambda(n) * \Lambda(n) + \Lambda(n) \log n = 2x \log x + \mathcal{O}(x),$$

can easily be obtained by using convolution techniques. See for example [5]. Note that by (3.4)

$$\mathbf{1} * \tilde{\Lambda} * \tilde{\Lambda}(x) \ll x \quad \text{and} \quad \mathbf{1} * \Lambda * \tilde{\Lambda}(x) \ll x.$$

Thus, by Selberg's formula, Lemma 2.1 is applicable to the first term on the right hand side of (3.7), and we arrive at

$$R_2(x) \ll \frac{1}{\log x}\int_1^x |M(\frac{x}{t})| \log t dt + \frac{x}{\log x}(\sum_{n\le x}\frac{|f(n)|}{n} + |A|\log^\kappa x) + \\ + \eta |A| x \log^\kappa x,$$

which proves (3.2). Here in the last step we used the inequality

$$\mathbf{1} * \tau_k(n) * (|\Lambda_f| * |\Lambda_f - \Lambda_{\tau_\kappa}| + L|\Lambda_f - \Lambda_{\tau_\kappa}|)(x) \ll \sum_{n\le x} |\tau_\kappa(n)| \frac{x}{n}(\eta \log\frac{x}{n} + 1),$$

which is nothing else but

$$\eta x \int_1^x \frac{\sum_{n\le u}\frac{|\tau_\kappa(n)|}{n}}{u} du + x\sum_{n\le x}\frac{\tau_k(n)}{n} \ll (\eta + \frac{1}{\log x}) x \log^{\kappa+1} x.$$

Concerning $R_1 * \Lambda_{\tau_\kappa}(x)$ we obtain in the same way.

$$L * \Lambda_{\tau_\kappa}(y) \ll y,$$

Therefore

$$R_1 * \Lambda_{\tau_\kappa}(x) = L * (f - A\tau_\kappa) * \Lambda_{\tau_\kappa}(x) = (L * \Lambda_{\tau_\kappa}) * (f - A\tau_\kappa)(x) \ll \\ \ll x(\sum_{n\le x}\frac{|f(n)|}{n} + A\log^\kappa x).$$

We estimate $R_2 * \Lambda_{\tau_\kappa}(x) = \mathbf{1} * f * (\Lambda_f - \Lambda_{\tau_\kappa}) * \Lambda_{\tau_\kappa}(x)$ in the same way as above. Then using

$$\int_1^x |M\left(\frac{x}{t}\right)| (\log t) dt \le x \log x \int_1^x \frac{|M(u)|}{u^2} du$$

ends the proof. ∎

Proof of Theorem 1.1. We use the estimate

$$\zeta^\kappa(s) = \mathcal{O}(\frac{1}{|s-1|^\kappa}), \tag{3.8}$$

which holds uniformly for all $|\tau| \ll 1$, $2 > \sigma > 1$.

Lemma 3.1. *Let* $|f(p)-\kappa|\le\eta$. *Then*

$$F'(s)-A(\zeta^{\kappa}(s))' \ll \frac{|A|}{|s-1|^{\kappa}}\{\eta\log(2+|s-1|\log x)+\Sigma_1\}\frac{\exp\{\Sigma_1\}}{\sigma-1} \tag{3.9}$$

uniformly for all $\tau\ll 1$, $1<\sigma\le 2$, $2<x$, *as long as* $\eta\log(2+|s-1|\log x)\ll 1$, *where*

$$\Sigma_1=\sup_{\tau}|\sum_{p>x}\frac{f(p)-\kappa}{p^s}+\sum_{\substack{p^{\alpha}\\ \alpha\ge 2}}\frac{\tilde f(p^{\alpha})-\kappa}{\alpha p^{\alpha s}}|.$$

Proof of Lemma 3.1. Since

$$\zeta^{\kappa}(s)=\exp(\sum_{n>1}\frac{\kappa\Lambda(n)}{n^s\log n}),$$

we have

$$\begin{aligned}F(s)-A\zeta^{\kappa}(s)&=\zeta^{\kappa}(s)(\exp(\sum_{n>1}\frac{\Lambda(n)(\tilde f(n)-\kappa)}{n^s\log n})-A)\ll\\ &\ll|\zeta^{\kappa}(s)A||\exp(\sum_{n>1}\frac{\Lambda(n)(\tilde f(n)-\kappa)}{n^s\log n}-\sum_{p\le x}\frac{f(p)-\kappa}{p})-1|\ll\\ &\ll|\zeta^{\kappa}(s)A||\exp(\sum_{p\le x}(f(p)-\kappa)(\frac{1}{p^s}-\frac{1}{p})+\sum_{p>x}\frac{f(p)-\kappa}{p^s}+\\ &\quad+\sum_{\substack{p^{\alpha}\\ \alpha\ge 2}}\frac{\tilde f(p^{\alpha})-\kappa}{\alpha p^{\alpha s}})-1|.\end{aligned}\tag{3.10}$$

Note that

$$\sum_{p\le x}|\frac{1}{p^s}-\frac{1}{p}|\ll\log(2+|s-1|\log x), \tag{3.11}$$

holds uniformly for $1\le\sigma\le 2$, $2<x$. Then substituting it in the inequality (3.10) we have for all s with $\eta\log(2+|s-1|\log x)\ll 1$ that

$$\begin{aligned}F(s)-A\zeta^{\kappa}(s)&\ll|\zeta^{\kappa}(s)A|\{\eta\log(2+|s-1|\log x)+\Sigma_1\}\times\\ &\quad\times\exp\{\eta\log(2+|s-1|\log x)+\Sigma_1\}\ll\\ &\ll|\zeta^{\kappa}(s)A|\{\eta\log(2+|s-1|\log x)+\Sigma_1\}\exp\{\Sigma_1\}.\end{aligned}\tag{3.12}$$

Let Γ be the circular path surrounding s with radius $(\sigma-1)/2$. It is easy to check that the conditions for the above inequality are satisfied for the points of Γ. Therefore using Cauchy's theorem and (3.8) we obtain

$$F'(s) - A(\zeta^\kappa(s))' = \int_\Gamma \frac{F(z) - A\zeta^\kappa(z)}{(z-s)^2} dz \ll$$

$$\ll \frac{|A|\{\eta \log(2 + |s-1| \log x) + \Sigma_1\}}{(\sigma - 1)^2} \exp\{\Sigma_1\} \int_\Gamma \frac{1}{|z-1|^\kappa} dz \ll \tag{3.13}$$

$$\ll \frac{|A|\{\eta \log(2 + |s-1| \log x) + \Sigma_1\}}{\sigma - 1} \exp\{\Sigma_1\} \frac{1}{|s-1|^\kappa}$$

uniformly for $|\tau| \ll 1$, $1 \leq \sigma \leq 2$, $2 < x$, $\eta \log(2 + |s-1| \log x) \ll 1$. Here we used the inequalities

$$|s-1|/2 \leq |s-1| - |z-s| \leq |s-1+z-s| = |z-1|$$

and

$$|z-1| \leq 3/2|s-1|$$

which hold on Γ. ■

Put

$$F_0(s) = \exp(\sum_p \frac{|f(p)|}{p^s}).$$

Lemma 3.2. *Let f be a multiplicative function the generating function of which is absolutely convergent for $\Re s > 1$. Suppose further that $f(p) = 0$ or $0 < \lambda_1 \leq |f(p)|$ for each prime p. If $f(p)$ is nonzero let $\theta_p = \arg f(p)$ with $-\pi < \arg z \leq \pi$ for all complex numbers z. Assume that there are real numbers θ_0, and $\delta > 0$ such that*

$$|e^{i\theta_0} - e^{i\theta_p}| \geq \delta$$

holds. Then there are positive constants τ_0, K so that the following inequalities are satisfied for $1 < \sigma \leq 2$:

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\Sigma_2 \exp\left(-\frac{\delta^3\lambda}{64\pi} \log \frac{1}{\sigma - 1} + \frac{\delta^2\lambda}{8} \sum_{f(p)=0} \frac{1}{p^\sigma}\right) \tag{3.14}$$

if

$$\tau_0 < |\tau| < -2 + \exp\left((\sigma - 1)^{\frac{-3\delta^3}{64\pi}}\right), \quad 1 < \sigma \leq 2$$

and

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\Sigma_2 \exp\left(-\frac{\delta^3\lambda}{32\pi} \log\left(1 + \frac{|\tau|}{\sigma - 1}\right) + \frac{\delta^2\lambda}{8} \sum_{f(p)=0} \frac{1}{p^\sigma}\right) \tag{3.15}$$

if $|\tau| \le \tau_0$, $1 < \sigma \le 2$, *where*

$$\Sigma_2 = \sup_\tau \exp(|\sum_{\substack{p^\alpha \\ \alpha \ge 2}} \frac{\tilde{f}(p^\alpha)}{\alpha p^{s\alpha}}|).$$

Proof of Lemma 3.2. A variant of this lemma can be found in [1] Lemma 19.6. Therefore we only sketch the proof. The conditions imply

$$\begin{aligned} \frac{|F(s)|}{F_0(\sigma)} &= \exp(\sum_{\substack{p^\alpha \\ \alpha \ge 2}} \frac{\tilde{f}(p^\alpha)}{\alpha p^{s\alpha}}) \exp(\sum_{f(p)\neq 0} \frac{(\Re\{e^{i\theta_p} p^{it}\} - 1)|f(p)|}{p^\sigma}) \\ &\le \Sigma_2 \exp(\sum_{f(p)\neq 0} \frac{(\Re\{e^{i\theta_p} p^{it}\} - 1)|f(p)|}{p^\sigma}). \end{aligned}$$

For $a, b \in \mathbb{R}$ we use the notation

$$|a-b| \pmod{2\pi} := \min_{k\in\mathbb{Z}} |a - b + 2k\pi|.$$

Let $\psi(e^{i\theta}) \in C_{2\pi}(\mathbb{R})$ such that it is zero at $\theta_0 \pm \delta/2$, $\delta^2/8$ at θ_0, and linear on the intervals between these three points, (mod 2π), and zero otherwise. The Fourier series expansion of ψ is given by

$$\psi(e^{i\theta}) = \sum_{l\in\mathbb{Z}} a_l e^{il\theta}, \tag{3.16}$$

where

$$a_l = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(e^{i\theta}) e^{-i\theta l} d\theta.$$

Obviously $a_0 = \frac{\delta^3}{32\pi}$, and

$$|a_l| \le \frac{8}{\pi l^2}$$

for all $l \neq 0$. Further

$$1 - \Re e^{i\theta_p} p^{-i\tau} \ge \begin{cases} \frac{\delta^2}{8} & \text{if } |\theta_0 - \tau \log p| \pmod{2\pi} \le \delta/2 \\ 0 & \text{otherwise} \end{cases}$$

Thus $1 - \Re e^{i\theta_p} p^{-i\tau}$ is at least as large as $\psi(p^{i\tau})$. This implies

$$\sum_{f(p)\neq 0} (1 - \Re\{e^{i\theta_p} p^{-i\tau}\})|f(p)| p^{-\sigma} \ge \sum \lambda \psi(p^{i\tau}) p^{-\sigma} - \frac{\delta^2 \lambda}{8} \sum_{f(p)=0} p^{-\sigma}, \tag{3.17}$$

and by (3.16) we obtain that the first term on the right hand side of (3.17) is

$$\sum_{l\in\mathbb{Z}} \lambda a_l \log\zeta(\sigma - il\tau) + \mathcal{O}(1).$$

Since

$$\sum_{l\in\mathbb{Z}} a_l \log\zeta(\sigma - il\tau) \geq \begin{cases} \frac{a_0}{2}\log\frac{1}{\sigma-1} + 1 & \text{if } |\tau| > \tau_0 \\ a_0 \log\frac{|\tau|}{\sigma-1} + 1 & \text{if } |\tau| \leq \tau_0, \end{cases}$$

the proof is finished. ■

Lemma 3.3. *Under the conditions of Lemma 3.2. the inequality*

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\Sigma_2 \exp\left(-\frac{\delta^3\lambda}{32\pi(A+2)}\log\left(1+\frac{|\tau|}{\sigma-1}\right) + \frac{\delta^2\lambda}{8}\sum_{f(p)=0}\frac{1}{p^\sigma}\right) \tag{3.18}$$

holds uniformly for all $|\tau| \leq (\sigma-1)^{-A}$.

Proof of Lemma 3.3. Using

$$\log(1+\frac{|\tau|}{\sigma-1}) \leq (A+2)\log(\frac{1}{\sigma-1}) + c$$

which holds uniformly for all $|\tau| \leq (\sigma-1)^{-A}$ we obtain by (3.14) that

$$\frac{|F(s)|}{F_0(\sigma)} \leq K\Pi_2 \exp\left(-\frac{\delta^3\lambda}{32\pi(A+2)}\log\left(1+\frac{|\tau|}{\sigma-1}\right) + \frac{\delta^2\lambda}{8}\sum_{f(p)=0}\frac{1}{p^\sigma}\right)$$

holds for all $\tau_0 \leq |\tau| \leq (\sigma-1)^{-A}$. On the other hand by (3.15) the same inequality is valid for $|\tau| \leq \tau_0$, thus the assertion of Lemma 3.3 follows. ■

Define $\beta_y = \exp(r)y$, and $\delta = \exp(r)$ with $2r = \frac{1}{\eta+1/\log\log x}$. Let

$$H^2(1+y) = \int_{-\infty}^{\infty} |\frac{F'(1+y+it) - A(\zeta^\kappa(1+y+it))'}{1+y+it}|^2 dt. \tag{3.19}$$

In the range $1/\log^{-1} x \leq y \leq \delta\log^{-1} x$ we treat the integral on the right side for $|t| \leq \beta_y$, $\beta_y < |t| \leq T$ and $T < |t|$ separately, where $T = y^{-D}$ with an arbitrary large positive constant D. The integral over this three ranges will be denoted by I_{11}, I_{12} and I_{13}, respectively. Concerning I_{11} we conclude that

$$\eta\log(2+|s-1|\log x) \leq \eta\log(2+y\log x+y\delta\log x) \ll \eta\log\delta^2 \ll 1,$$

and

$$y \le \beta_y \le \delta^2/\log x \le 1.$$

The conditions of Theorem 1.1 imply $\Sigma_1 \ll \eta$. Then, by (3.9), it follows that

$$\begin{aligned} I_{11} \ll & \frac{\eta^2|A|^2}{y^2} \int\limits_{|t|\le\beta_y} \frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa}} dt \\ \ll & \frac{\eta^2|A|^2}{y^2} \int\limits_{|t|\le y} \frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa}} dt \\ & + \frac{\eta^2|A|^2}{y^2} \int\limits_{y<|t|\le\beta_y} \frac{\log^2(2+y\log x+t\log x)}{|y+it|^{2\kappa}} dt. \end{aligned}$$

The first term on the right of the last inequality does not exceed

$$2\eta^2|A|^2 \frac{\log^2(2+2y\log x)}{y^{2\kappa+1}},$$

whilst the integral in the second term is at most

$$\begin{aligned} 2\int\limits_y^\infty \frac{\log^2(2+2t\log x)}{t^{2\kappa}} dt \ll & \log^{2\kappa-1} x \int\limits_{y\log x}^\infty \frac{\log^2(2+2u)}{u^{2\kappa}} du \\ & \ll \log^2(2+2y\log x) y^{-2\kappa+1}. \end{aligned}$$

Thus

$$I_{11} \ll \eta^2|A|^2\log^2(2+2y\log x)y^{-2\kappa-1}.$$

Concerning I_{12} we obtain, using the Cauchy-Schwarz inequality,

$$\begin{aligned} I_{12} \ll & \int\limits_{\beta_y\le|t|\le T} \left|\frac{F'(1+y+it)}{1+y+it}\right|^2 dt + \int\limits_{\beta_y\le|t|\le T} \left|\frac{A(\zeta^\kappa(1+y+it))'}{1+y+it}\right|^2 dt \\ = & I_{121} + I_{122}. \end{aligned}$$

Using the representation

$$F'(s) = F(s)\frac{F'(s)}{F(s)},$$

we obtain

$$I_{121} \ll \sup_{\beta_y\le|t|\le T} |F(1+y+it)|^2 \int\limits_{-\infty}^{\infty} \left|\frac{F'(1+y+iu)}{F(1+y+iu)(1+y+iu)}\right|^2 du.$$

Because of (3.5)

$$L(u) := \sum_{n\le u} \tilde f(n)\Lambda(n) \ll u.$$

By an application of Parseval's identity we deduce that

$$\int_{-\infty}^{\infty} \left|\frac{F'(1+y+iu)}{F(1+y+iu)(1+y+iu)}\right|^2 du = 2\pi \int_{0}^{\infty} |L(e^w)|^2 e^{-2yw} dw \ll y^{-1}.$$

Now, the conditions of Lemma 3.2 are fulfilled with $\lambda = \kappa - \eta_0$, and $\eta = \kappa + \eta_0$. Then $f(p)$ is never zero and we can choose $\theta_0 = \pi$, and $\delta \le \frac{1-2\eta_0}{2-\eta_0}$ in Lemma 3.2. Lemma 3.3 yields

$$\sup_{\beta_y \le |t| \le T} |F(1+y+it)|^2 \ll \Sigma_2 F_0^2(1+y)\exp(-2c\log(1+y^{-1}\beta_y))$$

with some appropriate positive constant $(c_{D,\kappa,\eta_0,\lambda_0} =)c$. The conditions of Theorem 1.1 shows that $\Sigma_2 \ll 1$ uniformly for $0 < y \le 1$. Further, with $\epsilon = \kappa - \eta_0$,

$$\begin{aligned} F_0(1+y) &\ll \exp\Big(\sum_{p\le \exp(y^{-1})} \frac{|f(p)|-\epsilon}{p^{1+y}}\Big) y^{-\epsilon} \ll \exp\Big(\sum_{p\le x} \frac{|f(p)|-\epsilon}{p^{1+y}}\Big) y^{-\epsilon} \ll \\ &\ll \exp\Big(\sum_{p\le x} \frac{|f(p)|-\epsilon}{p}\Big) y^{-\epsilon} \end{aligned}$$

uniformly for $1/\log x \le y < 2$. Here we used that

$$\sum_{p>\exp(y^{-1})} \frac{1}{p^{1+y}} \ll 1$$

holds uniformly for $0 < y \le 1$, which is a direct consequence of (3.11) and the asymptotic estimations

$$\begin{aligned} \sum_{p\le u} \frac{1}{p} &= \log\log u + \mathcal{O}(1) \qquad (u > 2), \\ \zeta(1+y+it) &= \frac{1}{y+it} + \mathcal{O}(1) \qquad (0 < y \le 1, |t| \ll 1). \end{aligned}$$

Further using the inequalities

$$\begin{aligned} \zeta(1+y+it) &\ll \mathcal{O}(\log^B(|t|)) \qquad (0 < y \le 1, |t| \mathbb{G} 1), \\ \frac{\zeta'(1+y+it)}{\zeta(1+y+it)} &\ll \mathcal{O}(\log^B(|t|)) \qquad (0 < y \le 1, |t| \mathbb{G} 1) \end{aligned}$$

with appropriate positive B we obtain

$$I_{122} \ll |A|^2 \int\limits_{\beta_y \le |t| \le 1} \frac{1}{|y+it|^{2\kappa+2}} dt + |A|^2 \int\limits_{1 \le |t|} \frac{\log^{2B\kappa}(|t|)}{|y+it|^2} dt \ll |A|^2 \beta_y^{-2\kappa-1}.$$

This implies

$$I_{12} \ll \exp(2 \sum_{p \le x} \frac{|f(p)| - \epsilon}{p}) y^{-2\epsilon-1} \exp(-2c \log(1+y^{-1}\beta_y)) + |A|^2 \delta^{-2\kappa-1} y^{-2\kappa-1}.$$

Concerning I_{13} we observe

$$|F(1+y+it)| \ll y^{-B} \qquad 0 < y \le 1, t \in \mathbb{R}$$

for some $0 < B$, and we obtain using Cauchy's theorem

$$|F'(1+y+it)| \ll y^{-B-1} \qquad 0 < y \le 1, t \in \mathbb{R}.$$

Choosing D large enough gives

$$\sup_t |F'(1+y+it)|^2 \int\limits_{T \le |t|} \frac{1}{|1+y+it|^2} dt \ll y^B.$$

Similar estimations lead to

$$I_{13} \ll (|A|^2 + 1) y^B.$$

It remains to estimate $H^2(1+y)$ for $\delta / \log x < y \le 1$. In this range we split the integral appearing on the right hand side of (3.19) into two parts, denoted by I_{21} and I_{22}, where $|t| \le T$ and $T < |t|$ respectively. A similar computation as above concerning I_{12} and I_{13} shows that

$$I_{21} \ll \exp(2 \sum_{p \le x} \frac{|f(p)| - \epsilon}{p}) y^{-2\epsilon-1} + |A|^2 y^{-2\kappa-1},$$

and

$$I_{22} \ll (|A|^2 + 1) y^B$$

respectively. Putting it all together we deduce that $\int_{1/\log x}^1 H(1+y) y^{-1/2} dy$ is at most

$$\int\limits_{1/\log x}^{1} \eta|A|\log(2+2y\log x)y^{-\kappa-1}+$$
$$+\exp(\sum_{p\le x}\frac{|f(p)|-\epsilon}{p})y^{-\epsilon-1}\exp(-c\log(1+y^{-1}\beta_y))dy+$$
$$+\int\limits_{1/\log x}^{1} |A|\delta^{-\kappa-1}y^{-\kappa-1}+(|A|+1)y^{B}dy+$$
$$+\int\limits_{\delta/\log x}^{1} \exp(\sum_{p\le x}\frac{|f(p)|-\epsilon}{p})y^{-\epsilon-1}+|A|y^{-\kappa-1}dy,$$

which does not exceed

$$\log^{\kappa} x\{\eta|A|+\exp(\sum_{p\le x}\frac{|f(p)|-\kappa}{p})(\exp(-\frac{c}{\eta})+\log^{-c} x)\}.$$

Here we used that

$$\delta^{-1} \ll \begin{cases} \exp(-\frac{1}{2\eta}) & \text{if} 1/\log\log x < \eta \\ \log^{-1/2} x & \text{otherwise.} \end{cases}$$

The proof is finished. ■

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