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ARITHMETIC FUNCTIONS EVALUATED AT POLYNOMIAL VALUES

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This paper is dedicated to Professor János Galambos on the occasion of his seventieth anniversary

1. Introduction

Let $f:\mathbb{N}\to\mathbb{Z}\setminus\{0\}$ be a multiplicative function such that $f(p^a)$ depends only on a for all prime powers p^a, and let $F_1, F_2, \dots, F_t$ be polynomials with integer coefficients. We establish minimal conditions on the polynomials F_i's which guaranty that

$$\lim_{x\to\infty}\frac{1}{x}\#\{n\le x : f(F_j(n))|f(F_{j+1}(n)) \text{ for } i=1,2,\dots,t-1\}\quad\text{exists.}$$

Given $g\in\mathbb{Z}[x]$, we let $\rho_g(m)=\#\{u \mod m : g(u)\equiv 0 \pmod m\}$ and we write $\mathrm{Discr}(g)$ to denote the discriminant of g. Given $Q_1, Q_2\in\mathbb{Z}[x]$, we let $\mathrm{Res}(Q_1,Q_2)$ stand for their resultant.

Given a positive integer n, we let $\tau(n)$ stand for the number of divisors of n and, for any fixed integer $k\ge 1$, we let $\tau_k(n)$ stand for the number of ways one can write n as the product of k positive integers taking into account the order in which they are written. For each $n\ge 2$, let $\beta(n)$ stand for the product of the exponents in the prime factorization of n, with $\beta(1)=1$. Let $\omega(n)$ stand for the number of distinct prime factors of $n\ge 2$, with $\omega(1)=0$.

Let $\pi(x;k,\ell)$ stand for the number of primes $p\le x$ such that $p\equiv\ell \pmod k$.

We denote by $\mathrm{LCM}(a_1,\dots,a_k)$ the least common multiple of the positive integers $a_1,\dots,a_k$. In what follows, $c_1, c_2,\dots$ stand for absolute positive constants, while c and C also denote constants, but not necessarily the same at

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each occurrence. Finally, p and q, with or without subscripts, always stand for prime numbers.

At times, we shall also write x_1 for $\log x$, x_2 for $\log\log x$, and so on.

2. Main results

Theorem 1. *Let $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be a multiplicative function such that $f(p^a)$ depends only on a for all prime powers p^a. Let $Q_1, Q_2, \dots, Q_h$ be distinct irreducible primitive monic polynomials each of degree no larger than 3. For each $\nu = 1, 2, \dots, t$, let $c_1^{(\nu)}, c_2^{(\nu)}, \dots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^{h} Q_j(x + c_j^{(\nu)})$ $(\nu = 1, 2, \dots, t)$. Let us assume that $(F_\nu(x), F_\mu(x)) = 1$ if $\nu \neq \mu$. Then, there exists a non negative constant d_0 such that*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \le x : f(F_\ell(n)) \text{ divides } f(F_{\ell+1}(n)) \text{ for } \ell = 1, 2, \dots, t-1\} = d_0.$$

Remark 1. The condition $(F_\nu(x), F_\mu(x)) = 1$ for $\nu \neq \mu$ holds if the numbers $c_1^{(\nu)}, \dots, c_h^{(\nu)}$ $(\nu = 1, \dots, t)$ are chosen in such a manner that $Q_j(x + c_j^{(\nu)}) \neq Q_i(x + c_i^{(\mu)})$ holds whenever $i \neq j$ for arbitrary values of ν and μ.

Remark 2. Interesting arithmetic functions to which one can apply Theorem 1 are $\tau(n)$, $\tau_k(n)$, $\beta(n)$ and also $a(n)$, the number of finite non isomorphic abelian groups with n elements (studied in particular by Ivić [4]).

Remark 3. From the proof of Theorem 1, the following assertion follows:

If there exists at least one positive integer n_0 such that

$$f(F_\ell(n_0)) \quad \text{divides} \quad f(F_{\ell+1}(n_0)) \qquad (\ell = 1, 2, \dots, t-1),$$

then $d_0 > 0$.

Theorem 2. *Let f be as in Theorem 1 and let $Q_1, Q_2, \dots, Q_h$ be distinct irreducible primitive monic polynomials of degree no larger than 2. Then, define $F_\nu(x)$ $(\nu = 1, 2, \dots, t)$ as in Theorem 1. Then, there exists a non negative constant e_0 such that*

$$\lim_{x\to\infty} \frac{1}{\pi(x)}\#\{p \le x : f(F_\ell(p)) \text{ divides } f(F_{\ell+1}(p)) \text{ for } \ell = 1, 2, \dots, t-1\} = e_0.$$

3. Preliminary lemmas

Lemma 1. *Given $F_1, F_2 \in \mathbb{Z}[x]$, which are relatively prime, then the congruences*

$$F_1(m) \equiv 0 \pmod{a} \quad \text{and} \quad F_2(m) \equiv 0 \pmod{a}$$

have common roots for at most finitely many a's.

Proof. A proof of this result was established by Tanaka [8]. ■

Lemma 2. *Let $F(m)$ be an arbitrary primitive polynomial with integer coefficients and of degree ν. Let D be the discriminant of F and assume that $D \neq 0$. Let $\rho(m)$ be the number of solutions n of $F(n) \equiv 0 \pmod{m}$. Then ρ is a multiplicative function whose values on the prime powers satisfy*

$$\rho(p^\alpha) \qquad \begin{cases} = \rho(p) & \text{if } p \not| D, \\ \leq 2D^2 & \text{if } p | D. \end{cases}$$

Moreover, there exists a positive constant $c = c(f)$ such that $\rho(p^\alpha) \leq c$ for all prime powers p^α.

Proof. This assertion is well known. ■

Lemma 3. *If $g \in \mathbb{Q}[x]$ is an irreducible polynomial and $\rho(m)$ stands for the number of residue classes mod m for which $g(n) \equiv 0 \pmod{m}$, then*

(i) $\displaystyle\sum_{p \leq x} \rho(p) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$;

(ii) $\displaystyle\sum_{p \leq x} \frac{\rho(p)}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right)$.

Proof. This result is due to Landau [6]. ■

Lemma 4. (Brun–Titchmarsh Inequality.) *There exists a positive constant c_3 such that*

$$\pi(x; k, \ell) < c_3 \frac{x}{\varphi(k) \log(x/k)} \quad \text{for all} \quad k < x.$$

Proof. For a proof, see the book of Halberstam and Richert [2]. ■

Lemma 5. (Siegel–Walfisz Theorem.) *There exists a constant $c > 0$ such that for every fixed number $A > 0$, the estimate*

$$\pi(x; k, \ell) - \frac{li(x)}{\varphi(k)} = O\left(xe^{-c\sqrt{\log x}}\right)$$

holds uniformly, as $(\ell, k) = 1$, for $k \le \log^A x$.

Proof. For a proof, see the book of Prachar [7]. ∎

Lemma 6. (Bombieri–Vinogradov Theorem.) *Given any fixed number $A > 0$, there exists a number $B = B(A) > 0$ such that*

$$\sum_{k \le \sqrt{x}/(\log^B x)} \max_{(k,\ell)=1} \max_{y \le x} \left| \pi(x; k, \ell) - \frac{li(x)}{\varphi(k)} \right| = O\left(\frac{x}{\log^A x}\right).$$

Moreover, an appropriate choice for $B(A)$ is $2A + 6$.

Proof. For a proof, see the book of Iwaniec and Kowalski [5]. ∎

Lemma 7. *Let F be a square-free polynomial with integer coefficients and of positive degree such that the degree of each of its irreducible factors is of degree no larger than 3. Let $Y(x)$ be a function which tends to $+\infty$ as $x \to +\infty$. Then*

$$\lim_{x \to \infty} \frac{1}{x} \{n \le x : p^2 | F(n) \text{ for some } p > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [3] (pp. 62–69). ∎

Lemma 8. *Let F and Y be as in Lemma 7. Assume that each of the irreducible factors of F is of degree no larger than 2 and that $F(0) \ne 0$. Then*

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \{p \le x : q^2 | F(p) \text{ for some } q > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [3] (pp. 69–72). ∎

Lemma 9. *Let $f(n)$ be a real valued non negative arithmetic function. Let a_n, $n = 1, \dots, N$, be a sequence of integers. Let r be a positive real number, and let $p_1 < p_2 < \dots < p_s \le r$ be prime numbers. Set $Q = p_1 \cdots p_s$. If $d|Q$, then let*

$$\sum_{\substack{n=1 \\ a_n \equiv 0 \pmod{d}}}^{N} f(n) = \kappa(d)X + R(N, d), \tag{3.1}$$

where X and R are real numbers, $X \geq 0$, and $\kappa(d_1 d_2) = \kappa(d_1)\kappa(d_2)$ whenever d_1 and d_2 are co-prime divisors of Q.

Assume that for each prime p, $0 \leq \kappa(p) < 1$. Setting

$$I(N,Q) := \sum_{\substack{n=1 \\ (a_n,Q)=1}}^{N} f(n),$$

then the estimate

$$I(N,Q) = \{1 + 2\theta_1 H\} X \prod_{p|Q} (1-\kappa(p)) + 2\theta_2 \sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)} |R(N,d)|$$

holds uniformly for $r \geq 2$, $\max(\log r, S) \leq \frac{1}{8}\log z$, where $|\theta_1| \leq 1$, $|\theta_2| \leq 1$, and

$$H = \exp\left(-\frac{\log z}{\log r}\left\{\log\left(\frac{\log z}{S}\right) - \log\log\left(\frac{\log z}{S}\right) - \frac{2S}{\log z}\right\}\right)$$

and

$$S = \sum_{p|Q} \frac{\kappa(p)}{1-\kappa(p)} \log p.$$

When these conditions are satisfied, there exists an absolute positive constant c such that $2H \leq c < 1$.

Proof. This result is Lemma 2.1 in the book of Elliott [1]. ■

4. The first part of the proof of Theorem 1

Since $\rho_{Q_j(x)}(m) = \rho_{Q_j(x+c)}(m)$ for any constant c, it follows that $\rho_{F_\nu}(m) = \rho_{F_\mu}(m)$. Observe also that $\mathrm{Res}(Q_i, Q_j) \neq 0$ if $i \neq j$. We shall now define four sets of primes, namely $\wp_1, \wp_2, \wp_3, \wp_4$, as follows.

First, as elements of $\wp_1$, we include

1. the prime divisors of $\prod_{1 \leq i < j \leq h} \mathrm{Res}(Q_i, Q_j)$,
2. the prime divisors of $\prod_{1 \leq i \leq h} \mathrm{Discr}(Q_i)$,
3. those primes p for which $\rho_F(p) \geq p$,
4. and no other primes.

Then, let $\mathcal{N}(\wp_1)$ be the semigroup generated by the set of primes $\wp_1$.

Observe that:

(a) If $(m, \mathcal{N}(\wp_1)) = 1$, then $\rho_F(m) = \rho_{Q_1}(m) + \cdots + \rho_{Q_h}(m)$.

(b) If $(m_1 m_2, \mathcal{N}(\wp_1)) = 1$ with $(m_1, m_2) = 1$, then $\rho_F(m_1 m_2) = \rho_F(m_1) + +\rho_F(m_2)$.

(c) If $p \notin \wp_1$, then $\rho(p^a) = \rho(p)$ for each $a \in \mathbb{N}$.

Let $Y = Y(x)$ be a large number. Moreover, let A_x and $\varepsilon(x)$ be such that $\varepsilon(x)A_x \to 0$ as $x \to \infty$, and define $r = r_x = x^{\varepsilon(x)}$.

We now define the other sets of primes $\wp_2$, $\wp_3$ and $\wp_4$ (which depend on x) as follows:

$$\begin{aligned} \wp_2 &= \{p : p \le Y, p \notin \wp_1\}, \\ \wp_3 &= \{p : Y < p \le r\}, \\ \wp_4 &= \{p : p > r\}. \end{aligned}$$

Now consider the sets $\mathcal{N}(\wp_2)$, $\mathcal{N}(\wp_3)$, $\mathcal{N}(\wp_4)$, that is the semigroups generated respectively by the sets of primes $\wp_2, \wp_3, \wp_4$.

For each positive integer ν, we now define $A(\nu)$, $B(\nu)$, $C(\nu)$ and $D(\nu)$ by

$$\nu = A(\nu)B(\nu)C(\nu)D(\nu),$$

where

$$A(\nu) \in \mathcal{N}(\wp_1), \quad B(\nu) \in \mathcal{N}(\wp_2), \quad C(\nu) \in \mathcal{N}(\wp_3), \quad D(\nu) \in \mathcal{N}(\wp_4).$$

Finally, let $T(u) := \prod_{p \le u} p$.

We now choose $\xi_1, \xi_2, \ldots, \xi_t \in \mathcal{N}(\wp_1)$ in such a way that there exists at least one solution $n = n_0$ of

$$\text{(4.1)} \qquad F_\nu(n) \equiv 0 \pmod{\xi_\nu}, \quad \left(\frac{F_\nu(n)}{\xi_\nu}, \mathcal{N}(\wp_1)\right) = 1 \quad (\nu = 1, \ldots, t).$$

Further define

$$\begin{aligned} \xi^* &= \mathrm{LCM}(\xi_1, \ldots, \xi_t), \\ \xi^{**} &= \xi^* \prod_{p \in \wp_1} p. \end{aligned}$$

Clearly, (4.1) holds for all those positive integers n for which $n \equiv n_0 \pmod{\xi^{**}}$.

Now let $\kappa=\kappa(\xi_1,\dots,\xi_t)$ be the number of those residue classes $r \pmod{\xi^{**}}$ for which

$$F_\nu(r)\equiv 0 \pmod{\xi_\nu},\qquad \left(\frac{F_\nu(r)}{\xi_\nu},\mathcal{N}(\wp_1)\right)=1 \quad (\nu=1,\dots,t) \tag{4.2}$$

holds. Note that, in the case where (4.1) has no solutions, we simply set $\kappa(\xi_1,\dots,\xi_t)=0$.

We now choose

$$\begin{aligned} m_1,\dots,m_t\in\mathcal{N}(\wp_2),&\qquad (m_i,m_j)=1 \text{ if } i\neq j,\\ d_1,\dots,d_t\in\mathcal{N}(\wp_3),&\qquad (d_i,d_j)=1 \text{ if } i\neq j. \end{aligned}$$

With these notations in mind, we introduce the set

$$\mathcal{M}_x=\mathcal{M}_x(\xi_1,\dots,\xi_t;m_1,\dots,m_t;d_1,\dots,d_t)= \tag{4.3}$$

$$=\{n\le x: A(F_\ell(n))=\xi_\ell, B(F_\ell(n))=m_\ell, C(F_\ell(n))=d_\ell \text{ for } \ell=1,\dots,t\}.$$

Observe that if $(m_i,m_j)>1$ or $(d_i,d_j)>1$ for some $i\neq j$, then

$$\mathcal{M}_x(\xi_1,\dots,\xi_t;m_1,\dots,m_t;d_1,\dots,d_t)=\emptyset.$$

One can easily see that $\mathcal{M}_x$ is the set of those integers $n\le x$ for which

$$\xi_j m_j d_j | F_j(n),\qquad \left(\frac{F_j(n)}{\xi_j m_j d_j},T(r)\right)=1 \qquad \text{for } j=1,\dots,t. \tag{4.4}$$

We now let $\mathcal{E}(\xi_1,\dots,\xi_t)$ be the set of those integers n for which (4.4) holds for $j=1,\dots,t$ for an appropriate choice of $m_1,\dots,m_t,d_1,\dots,d_t$.

Let $\nu_1,\dots,\nu_t$ be those residues mod ξ^{**} for which $\mathcal{E}(\xi_1,\dots,\xi_t)$ is covered exactly by

$$\bigcup_{u=1}^{\kappa}\{n\le x: n\equiv\nu_u \pmod{\xi^{**}}\},$$

that is, if $n\in\mathcal{E}(\xi_1,\dots,\xi_t)$, then $n\equiv\nu_u \pmod{\xi^{**}}$ for some $u\in\{1,\dots,\kappa\}$, and

$$\{n\le x: n\equiv\nu_u \pmod{\xi^{**}}\}\cap\mathcal{E}(\xi_1,\dots,\xi_t)\neq\emptyset.$$

Now let $\xi_1,\dots,\xi_t$, $\nu\in\{\nu_1,\dots,\nu_\kappa\}$, where $\kappa=\kappa(\xi_1,\dots,\xi_t)$ is fixed. Then the fact that $n\equiv\nu \pmod{\xi^{**}}$ guarantees that

$$\left(\frac{F_\ell(n)}{\xi_\ell},\xi^{**}\right)=1 \qquad (\ell=1,\dots,t)$$

holds.

We further define

$$\begin{aligned}\underline{m} &= (m_1, \dots, m_t),\\ \underline{d} &= (d_1, \dots, d_t),\end{aligned}$$

and

$$\widetilde{\mathcal{M}}_x(\nu \pmod{\xi^{**}}; \underline{m}, \underline{d}) =$$
$$= \{n \le x : n \equiv \nu \pmod{\xi^{**}}, B(F_j(n)) = m_j, C(F_j(n)) = d_j \text{ for } j = 1, \dots, t\}$$

Observe that, for each $j = 1, \dots, t$, the number of solutions of $F_j(\nu + s\xi^{**}) \equiv \equiv 0 \pmod{m_j d_j} \bmod m_1 \dots m_t d_1 \dots d_t$ is equal to $\rho(m_1 \dots m_t)\rho(d_1 \dots d_t)$.

Let μ_0 be one of these solutions, that is let $0 \le \mu_0 < m_1 \dots m_t d_1 \dots d_t$, $F_j(\nu + \mu_0 \xi^{**}) \equiv 0 \pmod{m_j d_j}$ $(j = 1, \dots, t)$, and set

$$R = \xi^{**} m_1 \dots m_t d_1 \dots d_t. \tag{4.5}$$

We would like to estimate the size of the number of those integers $k \le x/R$ for which

$$\varphi_j(k) := \frac{F_j(\nu + \mu_0 \xi^{**} + kR)}{\xi_j m_j d_j}$$

is coprime to $T(r)$ for every $j = 1, \dots, t$.

We shall only consider those $k \le x/R$ for which m_j, d_j are both not very large, that is when $m_j \le Y^{A_x}$ and $d_j \le r^{A_x}$. Indeed, one can easily prove, in light of Lemma 7, that

$$\lim_{x \to \infty} \frac{1}{x} \#\{n \le x : \max m_j > Y^{A_x} \text{ or } \max d_j > r^{A_x}\} = 0, \tag{4.6}$$

and we will therefore skip the proof. Now, define

$$\Phi(k) := \varphi_1(k) \cdots \varphi_t(k). \tag{4.7}$$

Since, if $p \in \wp_1$, then $\rho_{\varphi_j}(p^a) = \rho_{\varphi_j}(p) = 0$, it follows that $\rho_\Phi(p^a) = \rho_\Phi(p) = 0$.

Furthermore, if $p \in \wp_2 \cup \wp_3$, then $\rho_{\varphi_j}(p^a) = \rho_{\varphi_j}(p)$ and we shall prove that

(P1) if $p | d_j m_j$, then $\rho_{\varphi_j}(p) = 1$ and $\rho_{\varphi_\ell}(p) = 0$ for all $\ell \ne j$;

(P2) if $(p, d_1 m_1 \cdots d_t m_t) = 1$, then $\rho_{\varphi_j}(p) = \rho(p)$ for $j = 1, \dots, t$.

Consequently, assuming that (P1) and (P2) are true, and letting $\eta(M)$ stand for the number of those $k \pmod M$ for which $\Phi(k) \equiv 0 \pmod M$, we then have

$$\eta(p^a) = \begin{cases} 0 & \text{if} \quad p \in \wp_1, \\ \rho_{\varphi_j}(p) = 1 & \text{if} \quad p \in \wp_2 \cup \wp_3 \text{ and } p | d_j m_j, \\ t\rho(p) & \text{if} \quad p \in \wp_2 \cup \wp_3 \text{ and } (p, d_1 m_1 \cdots d_t m_t) = 1. \end{cases} \tag{4.8}$$

We now prove (P1). We prove it only in the case $j=1$, the general case being similar. Assume that $p|d_1$ (the same reasoning will work if one assumes that $p|m_1$). Let a be the positive integer defined by $p^a\|d_1$. Then,

$$\varphi_1(k)\equiv 0 \pmod p \Longleftrightarrow F_1(\nu+\mu_0\xi^{**}+kR)\equiv 0 \pmod{p^{a+1}},$$

meaning that, since $\rho_{\varphi_1}(p)$ stands for the number of solutions k of $\varphi_1(k)\equiv 0 \pmod p$, while $\rho_{F_1}(p^{a+1})$ stands for the number of solutions of $F_1(\nu+\mu_0\xi^{**}+ +kR)\equiv 0 \pmod{p^{a+1}}$, it follows that $\rho_{\varphi_1}(p)=\rho_{F_1}(p)=1$. It remains to prove that $\rho_{\varphi_\ell}(p)=0$ if $\ell\neq 1$. To do so, we assume that $\rho_{\varphi_\ell}(p)\neq 0$. In this case, we have that $p|\varphi_1(k_1)$ and $p|\varphi_\ell(k_2)$, in which case

$$\begin{aligned} F_1(\nu+\mu_0\xi^{**}+k_1R) &\equiv 0 \pmod p,\\ F_\ell(\nu+\mu_0\xi^{**}+k_2R) &\equiv 0 \pmod p. \end{aligned}$$

Now, in light of (4.5), we have that $p|R$, implying that

$$\begin{aligned} F_1(\nu+\mu_0\xi^{**}) &\equiv 0 \pmod p,\\ F_\ell(\nu+\mu_0\xi^{**}) &\equiv 0 \pmod p, \end{aligned}$$

which is an impossible situation in light of Lemma 1, because $F_1(a)=0$ and $F_2(a)$ cannot occur simultaneously due to the fact that $p\notin\wp_1$. This completes the proof of (P1).

The proof of (P2) is almost obvious. Indeed,

$$\varphi_1(k)\equiv 0 \pmod p \Longleftrightarrow F_1(\nu+\mu_0\xi^{**}+kR)\equiv 0 \pmod p.$$

Thus, $F_1(u)\equiv 0 \pmod p$ holds for $u=u_1,\dots,u_{\rho(p)} \bmod p$, and therefore, $\nu+\mu_0\xi^{**}+kR\equiv u_j \pmod p$ $(j=1,\dots,\rho(p))$ can be solved in k.

We now move on to estimate $\#\mathcal{M}$ and $\#\widetilde{\mathcal{M}}$ using Lemma 9. We choose $Q=T(r)$, $f(k)=1$, $a_k=\Phi(k)$ (which was defined in (4.7)), $X=x/R$ (where R is as in (4.5)) and $\kappa(d)=\eta(d)/d$ with η as defined in (4.8). We thus obtain

$$\sum_{\substack{k\le X\\ a_k\equiv 0 \pmod d}} 1=\frac{\eta(d)}{d}X+R(X,d)$$

with

$$|R(X,d)|\le t\rho(d). \tag{4.9}$$

With $I(X,Q):=\#\{k\le X:(a_k,Q)=1\}$, we obtain from Lemma 9 that

$$I(X,Q)=(1+O(H))\frac{x}{R}\prod_{p|Q}\left(1-\frac{\eta(p)}{p}\right)+O\left(\sum_{\substack{d|Q\\ d\le z^3}} 3^{\omega(d)}|R(X,d)|\right). \tag{4.10}$$

In light of (4.9), we have that

$$\sum_{\substack{d\mid Q\\ d\le z^3}} 3^{\omega(d)}|R(X,d)| \le t \sum_{\substack{d\mid Q\\ d\le z^3}} 3^{\omega(d)}\eta(d). \tag{4.11}$$

We shall prove that for $z \ge 2$,

$$\sum_{\substack{d\mid Q\\ d\le z^3}} 3^{\omega(d)}\eta(d) \le cz^3(\log z)^K, \tag{4.12}$$

for a suitable large constant K.

$$\begin{aligned} \sum_{d\le Y} 3^{\omega(d)}\eta(d)|\mu(d)|\log d &\le \sum_{pu\le Y} 3^{\omega(pu)}(\log p)\eta(p)\eta(u)|\mu(u)| \le \\ &\le 3\sum_{u\le Y} 3^{\omega(u)}\eta(u)|\mu(u)| \sum_{p\le Y/u} \eta(p)\log p. \end{aligned} \tag{4.13}$$

Since $\sum_{p\le Y/u} \eta(p)\log p \le c\frac{Y}{u}$, (4.13) becomes

$$\begin{aligned} \sum_{d\le Y} 3^{\omega(d)}\eta(d)|\mu(d)|\log d &\le cY\sum_{u\le Y} \frac{3^{\omega(u)}\eta(u)}{u}|\mu(u)| \le \\ &\le cY\prod_{p\le Y}\left(1+\frac{3\eta(p)}{p}\right) \le cY\exp\left\{3\sum_{p\le Y}\frac{\eta(p)}{p}\right\} \le \\ &\le cY\exp(3th\log\log Y) = cY(\log Y)^{3th}. \end{aligned} \tag{4.14}$$

Let us write

$$\sum_{d\le Y} 3^{\omega(d)}\eta(d)|\mu(d)| = \sum_{d\le\sqrt{Y}} + \sum_{\sqrt{Y}<d\le Y} = S_1 + S_2, \tag{4.15}$$

say. Clearly we have

$$S_1 \ll \sqrt{Y}\cdot Y^{\varepsilon}, \tag{4.16}$$

where $\varepsilon > 0$ can be taken arbitrarily small. On the other hand, in light of (4.14), we have

$$S_2 \le \frac{2}{\log Y}\cdot cY(\log Y)^{3th} \ll Y(\log Y)^{3th-1}. \tag{4.17}$$

Setting $Y = z^3$ and using (4.16) and (4.17) in (4.14) proves (4.12).

We now move to obtain the size of S and to find an upper bound for H. First of all, since

$$0 < \frac{c_1}{p} < \frac{\eta(p)}{p-\eta(p)} < \frac{c_2}{p} \quad \text{if} \quad p \in \wp_2 \cup \wp_2,$$
$$\text{while} \quad \eta(p) = 0 \quad \text{if} \quad p \in \wp_1,$$

it follows that

$$S \asymp \log r \asymp \varepsilon(x) \log x. \tag{4.18}$$

Let $B(x)$ be a real valued function satisfying $B(x) \to 0$ and $B(x)/\varepsilon(x) \to \to +\infty$ as $x \to \infty$, and set $z = x^{B(x)}$. Note that, in our context, the condition $\max(\log r, S) \geq \frac{1}{8} \log z$ (of Lemma 9) clearly holds for every large x.

Then, we have

$$H = \exp\left\{ -\frac{\log z}{\varepsilon(x)\log x} \left[\log\left(\frac{\log z}{\varepsilon(x)\log x}\right) - \right.\right.$$
$$\left.\left. - \log\log\left(\frac{\log z}{\varepsilon(x)\log x}\right) - \frac{2\varepsilon(x)\log x}{\log z} + O(1) \right] \right\}.$$

From this representation, it follows that

$$0 \leq H \leq C \exp\left\{ -\frac{1}{2}\frac{B(x)}{\varepsilon(x)} \log\left(\frac{B(x)}{\varepsilon(x)}\right) \right\} =: H_1,$$

for an appropriate constant C.

Hence, applying Lemma 9, we obtain

$$I(X,Q) = \{1 + O(H_1)\} \frac{x}{R} \prod_{p|Q} \left(1 - \frac{\eta(p)}{p}\right) + O\left(x^{3B(x)} \left[(\log x) B(x)\right]^K\right). \tag{4.19}$$

Now observe that

$$\prod_{p|Q}\left(1 - \frac{\eta(p)}{p}\right) = \prod_{\substack{p \in \wp_2 \\ (p, m_1 \cdot m_t)=1}} \left(1 - \frac{t\rho_F(p)}{p}\right) \cdot \prod_{p | m_1 \cdots m_t} \left(1 - \frac{1}{p}\right) \cdot$$
$$\cdot \prod_{\substack{p \in \wp_3 \\ (p, d_1 \cdots d_t)=1}} \left(1 - \frac{t\rho_F(p)}{p}\right) \cdot \prod_{p | d_1 \cdots d_t} \left(1 - \frac{1}{p}\right). \tag{4.20}$$

Summing up over all the $\kappa = \kappa(\xi_1, \ldots, \xi_t)$ residue classes mod ξ^{**}, we obtain that

$$\begin{aligned}(4.21)\quad &\#\mathcal{M}_x(\xi_1, \ldots, \xi_t; m_1, \ldots, m_t; d_1, \ldots, d_t) = \\ &= \frac{\kappa(\xi_1, \ldots, \xi_t) x \rho_F(m_1 \cdots m_t d_1 \cdot d_t)}{\xi^{**} m_1 \cdots m_t d_1 \cdot d_t} \cdot \frac{\varphi(m_1 \ldots m_t)}{m_1 \cdots m_t} \cdot \frac{\varphi(d_1 \ldots d_t)}{d_1 \cdots d_t} \cdot \\ &\cdot \prod_{\substack{p \in \wp_2 \cup \wp_3 \\ (p, m_1 \cdots m_t d_1 \cdots d_t) = 1}} \left(1 - \frac{t \rho_F(p)}{p}\right) \cdot (1 + \theta_1 \kappa(\xi_1, \ldots, \xi_t) H_1) + \\ &+ O\left(x^{3B(x)} \left[(\log x) B(x)\right]^K\right).\end{aligned}$$

Using the fact that

$$\sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_F(p)}{p} = \sum_{j=1}^{h} \sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_{Q_j}(p)}{p} = h \log\log x^{\varepsilon(x)} + c_2 + O\left(\frac{1}{\varepsilon(x) \log x}\right),$$

it follows that

$$\begin{aligned}(4.22)\quad &\prod_{p \in \wp_2 \cup \wp_3} \left(1 - \frac{t \rho_F(p)}{p}\right) = \exp\left\{-t \sum_{p \in \wp_2 \cup \wp_3} \frac{\rho_F(p)}{p}\right\} + c_1 + O\left(\frac{1}{\varepsilon(x) \log x}\right) = \\ &= \exp\{-th \log\log x^{\varepsilon(x)} + c_1 - t c_2\} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right) = \\ &= \left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} e^{c_3} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right).\end{aligned}$$

Thus, by defining the strongly multiplicative function λ on $\mathcal{N}(\wp_1 \cup \wp_2 \cup \wp_3)$ by

$$\lambda(p) = \begin{cases} 1, & \text{if } p \in \wp_1 \\ \left(1 - \frac{t \rho_F(p)}{p}\right)^{-1} \cdot \left(1 - \frac{1}{p}\right), & \text{if } p \mid m_1 \cdots m_t d_1 \cdots d_t \\ 1, & \text{if } p \in \wp_2 \cup \wp_3 \text{ and } (p, m_1 \cdots m_t d_1 \cdots d_t) = 1 \end{cases}$$

then, in light in the above, (4.21) becomes

$$\begin{aligned}(4.23)\quad &\#\mathcal{M}_x(\xi_1, \ldots, \xi_t; m_1, \ldots, m_t; d_1, \ldots, d_t) = \\ &= \frac{x}{R} \frac{\kappa(\xi_1, \ldots, \xi_t)}{\xi^{**}} \cdot \frac{\varphi(m_1 \ldots m_t)}{m_1 \cdots m_t} \cdot \frac{\varphi(d_1 \ldots d_t)}{d_1 \cdots d_t} \cdot \\ &\cdot \lambda(m_1 \cdots m_t) \lambda(d_1 \cdots d_t) \rho_F(m_1 \cdots m_t d_1 \cdots d_t) \cdot \\ &\cdot \left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} e^{c_3} \left(1 + O\left(\frac{1}{\varepsilon(x) \log x}\right)\right) \cdot \\ &\cdot (1 + \theta_1 \kappa(\xi_1, \ldots, \xi_t) H_1) + O\left(x^{3B(x)} \left[(\log x) B(x)\right]^K\right).\end{aligned}$$

Now, from the estimates (4.21) and (4.23), we can formulate the following straightforward and important assertions:

Proposition 1. *Let $m'_1,\dots,m'_t$ and $d'_1,\dots,d'_t$ be arbitrary permutations of $m_1,\dots,m_t$ and $d_1,\dots,d_t$ respectively. Then, for some constant $c>0$,*

$$|\#\mathcal{M}_x(\xi_1,\dots,\xi_t;m_1,\dots,m_t;d_1,\dots,d_t)-$$
$$-\#\mathcal{M}_x(\xi_1,\dots,\xi_t;m'_1,\dots,m'_t;d'_1,\dots,d'_t)|\le$$
$$\le\frac{cx\kappa(\xi_1,\dots,\xi_t)}{\xi^{**}}\rho_F(m_1\cdots m_t d_1\cdots d_t)\cdot$$
$$\cdot\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th}\left(\frac{1}{\varepsilon(x)\log x}+\kappa(\xi_1,\dots,\xi_t)H_1\right)+$$
$$+O\left(x^{3B(x)}\left[(\log x)B(x)\right]^K\right).$$

Proposition 2. *Let $M\in\mathcal{N}(\wp_1)$ and $D\in\mathcal{N}(\wp_2)$ be two square-free integers satisfying $M\le Y^{A_x}$ and $D\le r^{A_x}$. Assume that $M=m_1\cdots m_t$ and $D=d_1\cdots d_t$. Let $m'_1,\dots,m'_t$ and $d'_1,\dots,d'_t$ be permutations of $m_1,\dots,m_t$ and $d_1,\dots,d_t$ respectively. Then, for some constant $c>0$,*

$$|\#\mathcal{M}_x(\xi_1,\dots,\xi_t;m_1,\dots,m_t;d_1,\dots,d_t)-$$
$$-\#\mathcal{M}_x(\xi_1,\dots,\xi_t;m'_1,\dots,m'_t;d'_1,\dots,d'_t)|\le$$
$$\le\frac{cx\kappa(\xi_1,\dots,\xi_t)}{\xi^{**}}\rho_F(MD)\cdot$$
$$\cdot\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th}\left(\frac{1}{\varepsilon(x)\log x}+\kappa(\xi_1,\dots,\xi_t)H_1\right)+$$
$$+O\left(x^{3B(x)}\left[(\log x)B(x)\right]^K\right).$$

5. The second part of the proof of Theorem 1

Given a positive integer n, we write it as $n=M(n)S(n)$, where $M(n)$ is the squarefree part of n and $S(n)$ the squarefull part of n. Then we clearly have $f(n)=U^{\omega(S(n))}f(M(n))$, where $U=q_1^{\beta_1}\cdots q_v^{\beta_v}$, say. With this set up, we may write $f(m)=f_1(m)f_2(m)$, where $f_2(m)\in\mathcal{N}(\{q_1,\dots,q_v\})$ and $f_1(m)=$

$= f(m)/f_2(m)$ satisfies $(f_1(m), U) = 1$. Of course, f_1 and f_2 are easily seen to be multiplicative functions.

Writing $m = M(m) \cdot S(m)$, we have that

$$f(n)|f(m) \iff \begin{cases} (1) & f_1(S(n))|f_1(S(m)) \\ \text{and} & \\ (2) & f_1(S(n)) \cdot U^{\omega(M(n))}\Big| f_1(S(m)) \cdot U^{\omega(M(m))} \end{cases}$$

Define $L(S(n))$ as the smallest (nonnegative) integer for which $f_2(S(n))$ divides $U^{L(S(n))}$. Then, in order for the condition $f(n)|f(m)$ to be satisfied, it is sufficient that the conditions (1) and

$$(2)' \qquad L(S(n)) + \omega(M(n)) \le \omega(M(m))$$

be satisfied, while it is necessary that conditions (1) and

$$(2)'' \qquad \omega(M(n)) \le \omega(M(m)) + L(S(m))$$

hold.

From this, it follows that in order to have

$$f(F(_\ell(n)))|f(F(_{\ell+1}(n))) \qquad (\ell = 1, \dots, t-1),$$

the conditions

$$(5.1) \qquad f_1(S(F(_\ell(n))))|f_1(S(F(_{\ell+1}(n)))) \qquad (\ell = 1, \dots, t-1)$$

and

$$(5.2) \qquad L(S(f_\ell(n))) + \omega(M(F_\ell(n))) \le \omega(M(F_\ell(n)))$$

are sufficient, while the conditions (5.1) and

$$(5.3) \qquad \omega(M(F_\ell(n))) \le \omega(M(F_{\ell+1}(n))) + L(S(f_{\ell+1})) \quad (\ell = 1, \dots, t-1)$$

are necessary.

Now, let $S_1, \dots, S_t$ be squarefull numbers. By using a method developed by Hooley (see [3], Chapter 4) and using also the Eratosthenian sieve, one can prove that

(5.4)
$$\frac{1}{x}\#\{n \le x : S(F_\ell(n)) = S_\ell,\ \ell = 1, \dots, t\} = d(S_1, \dots, S_t) + O\left(\frac{x}{\log\log x}\right),$$

for some nonnegative constant $d(S_1, \dots, S_t)$ which satisfy

$$\sum_{S_1, \dots, S_t} d(S_1, \dots, S_t) = 1,$$

and where the constant implied in the error term is absolute.

Let $\mathcal{B}$ be the set of all those vectors $(S_1, \dots, S_t)$ for which $S_1, \dots, S_t$ are squarefull numbers and

$$f_1(S_\ell(n)) | f_1(S_{\ell+1}(n)) \qquad (\ell = 1, \dots, t-1). \tag{5.5}$$

We will prove that

$$d_0 = \frac{1}{t!} \sum_{(S_1,\dots,S_t)\in\mathcal{B}} d(S_1, \dots, S_t). \tag{5.6}$$

Since

$$\sum_{\max(S_1,\dots,S_t)\geq Y} d(S_1, \dots, S_t) \to 0 \text{ as } Y \to \infty,$$

it is sufficient to prove that, for each fixed $(S_1, \dots, S_t) \in \mathcal{B}$,

$$\begin{aligned} &\frac{1}{x}\#\{n \leq x : S(F_\ell(n)) = S_\ell,\ f(F_\ell(n))|f(F_{\ell+1}(n)) \text{ for } \ell = 1, \dots, t\} = \\ &\qquad = \frac{1}{t!} d(S_1, \dots, S_t) + o(1) \qquad (x \to \infty). \end{aligned} \tag{5.7}$$

Let Y be large enough so that $\max(S_1, \dots, S_t) \leq Y$. We now move on to count the number of those integers $n \leq x$ for which both

$$S(F_\ell(n)) = S_\ell \qquad (\ell = 1, \dots, t) \tag{5.8}$$

and

$$f(F_\ell(n)) | f(F_{\ell+1}(n)) \qquad (\ell = 1, \dots, t) \tag{5.9}$$

hold. We must compute the number of those integers $n \leq x$ appearing in the set displayed in equation (4.3), with the additional condition $S(\xi_\ell m_\ell) = S_\ell$ and with also d_ℓ and $D(F_\ell(n))$ both squarefree for $\ell = 1, \dots, t$. But it is clear that, choosing $\varepsilon(x) = 1/\log\log\log\log x$, we have

$$\omega(A(F_\ell(n))B(F_\ell(n))D(F_\ell(n))) \leq \frac{c_4}{\varepsilon(x)} + Y \leq c_5 \log\log\log\log x.$$

In light of (4.6), we only need to consider those $n \leq x$ for which

$$\max_{i=1,\dots,t} m_j \leq Y^{A_x} \text{ and } \max_{i=1,\dots,t} d_j \leq r^{A_x}. \tag{5.10}$$

For now, fix $\xi_1, \dots, \xi_t, m_1, \dots, m_t, d_1, \dots, d_t$ and assume that $\omega(d_i) \neq \omega(d_j)$ when $i \neq j$. Under these conditions, there exists one and only one permutation $d_1^*, \dots, d_t^*$ of $d_1, \dots, d_t$ for which $\omega(d_1^*) < \dots < \omega(d_t^*)$.

In the event that $|\omega(d_i)-\omega(d_j)|\geq 2c_5\log\log\log\log x$ whenever $i\neq j$, then (5.9) will hold for the corresponding number n.

Hence, it remains to prove that the sum of

$$\#\mathcal{M}_x(\xi_1,\ldots,\xi_t;m_1,\ldots,m_t;d_1,\ldots,d_t)$$

running over all those $\xi_1,\ldots,\xi_t,m_1,\ldots,m_t,d_1,\ldots,d_t$ for which $|\omega(d_i)-\omega(d_j)|< < 2c_5\log\log\log\log x$ holds for some $i\neq j$ is $o(x)$.

In order to prove this, observe that, in light of Proposition 1,

$$\#\mathcal{M}_x(\xi_1,\ldots,\xi_t;m_1,\ldots,m_t;d_1,\ldots,d_t)\leq \leq \frac{cx\kappa(\xi_1,\ldots,\xi_t)}{\xi^{**}}\rho_F(m_1\cdots m_t d_1\cdots d_t)\cdot\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th}.$$

For short, let us write $d_1\Delta d_2$ to express the condition $\omega(d_1)\leq\omega(d_2)\leq\omega(d_1)+ +c_5\log\log\log\log x$.

We then have

$$\begin{aligned}(5.11)\qquad &\sum_{\substack{d_1,d_2\\ d_1\Delta d_2}}\ \sum_{\substack{\xi_1,\ldots,\xi_t\\ m_1,\ldots,m_t\\ d_1,\ldots,d_t}}\#\mathcal{M}_x(\xi_1,\ldots,\xi_t;m_1,\ldots,m_t;d_1,\ldots,d_t)\leq\\
&\leq c(Y)\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} x\sum_{\substack{d_1,d_2\\ d_1\Delta d_2}}\frac{\rho_F(d_1d_2)}{d_1d_2}\sum_{d_3,\ldots,d_t}\frac{\rho_f(d_3\ldots d_t)}{d_3\ldots d_t}\leq\\
&\leq c(Y)\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} x\left(\sum_{d\in\mathcal{N}(\wp_3)}\frac{\rho_F(d)}{d}\right)^{t-2}\sum_{\substack{d_1,d_2\\ d_1\Delta d_2}}\frac{\rho_F(d_1d_2)}{d_1d_2}\leq\\
&\leq c(Y)\left(\frac{1}{\log x^{\varepsilon(x)}}\right)^{th} x\left(\log x^{\varepsilon(x)}\right)^{(t-2)h}\sum_{\substack{d_1,d_2\\ d_1\Delta d_2}}\frac{\rho_F(d_1d_2)}{d_1d_2}.\end{aligned}$$

Now, because

$$\sum_{\omega(d)=r}\frac{\rho_F(d)}{d}\leq\frac{1}{r!}\left(\sum_{p\in\wp_3}\frac{\rho_F(p)}{p}\right)^r\leq\frac{(h\log\log x+O(1))^r}{r!},$$

it follows, in light of Lemma 3, that, as $x\to\infty$,

(5.12)

$$\sum_{\substack{d_1,d_2\\ d_1\Delta d_2}}\frac{\rho_F(d_1d_2)}{d_1d_2}\leq\sum_{r=1}^{\infty}\frac{(h\log\log x+O(1))^r}{r!}\sum_{t=0}^{\lfloor c_5x_4\rfloor}\frac{(hx_2+O(1))^{t+r}}{(t+r)!}=o(\log^{2h}x).$$

Using (5.12) in (5.11), it follows that

$$\sum_{\substack{d_1,d_2 \\ d_1 \Delta d_2}} \sum_{\substack{\xi_1,\dots,\xi_t \\ m_1,\dots,m_t \\ d_1,\dots,d_t}} \#\mathcal{M}_x(\xi_1,\dots,\xi_t; m_1,\dots,m_t; d_1,\dots,d_t) = o(x),$$

thus proving our claim and thereby completing the proof of Theorem 1.

6. The proof of Theorem 2

The proof of Theorem 2 can be obtained along the same lines as that of Theorem 1. We only provide here the main ideas. Indeed, Hooley [3] proved that

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \text{for some } \ell,\ C(F_\ell(p))D(F_\ell(p)) \neq \text{squarefree}\} = 0.$$

Using this result, the Siegel-Walfisz form of the Prime Number Theorem (Lemma 5) as well as the Bombieri-Vinogradov Inequality (Lemma 6), one can proceed as earlier and prove the analogues of Propositions 1 and 2, thereby easily completing the proof of Theorem 2.

7. Further applications

It is interesting to observe that the following two results are consequences of Proposition 1. Here, $\phi(y)$ is the standard cumulative distribution function defined by $\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$.

Theorem 3. *Let $F_1,\dots,F_t$ be as in Theorem 1. Then,*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \frac{\omega(F_\ell(n)) - h\log\log x}{\sqrt{h\log\log x}} < y_\ell,\ \ell = 1,\dots,t\} = \phi(y_1)\dots\phi(y_t).$$

Theorem 4. *Let $F_1,\dots,F_t$ be as in Theorem 2. Then,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \frac{\omega(F_\ell(p)) - h\log\log x}{\sqrt{h\log\log x}} < y_\ell,\ \ell = 1,\dots,t\} = \phi(y_1)\dots\phi(y_t).$$

8. Final remarks

We now state a few remarks shedding some light on the value of d_0 and whether it is strictly positive.

The main idea is that the value of d_0 as well as the fact that it is positive or zero depends on the values taken by f on squarefull numbers.

Remark 4. Let $u(n)$ stand for the squarefull part of n, and $v(n)$ for the part of n which is coprime to $f(p)$, that is

$$v(n) = \prod_{q^a \| n, (q, f(p))=1} q^a.$$

Under the additional assumption that $f(p) > 1$, then

$$d_0 = \frac{1}{t!} \lim_{x\to\infty} \frac{1}{x} \# \{n \le x : v(f(u(F_j(n)))|v(f(u(F_{j+1}(n))), j = 1, \dots, t-1\}.$$

Indeed, let $\omega_k(n)$ be defined as

$$\omega_k(n) := \sum_{p^k \| n} 1.$$

Then, except for a set of density zero,

$$\min_{1\le j<r\le t} |\omega_1(F_j(n)) - \omega_1(F_r(n))| > \left(\log\log(\max_{1\le j\le t} F_j(n)\right)^{1/3}.$$

Furthermore for any function $g(n)$ tending to infinity with n, if $u(n)$ stands for the squarefull part of n,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \max_{1\le j\le t} f(u(F_j(n))) > g(n) \right\} = 0.$$

From the above equations, it follows that except on a set of density zero,

$$f(F_j(n))|f(F_{j+1}(n)) \iff v(f(u(F_j(n))))|v(f(u(F_{j+1}(n)))),$$

thus completing the proof of our claim.

Remark 5. The constant d_0 will be positive regardless of the polynomials F if and only if for any values of A and n, there exists a number m such that $p|m \Rightarrow p > A$ and $v(f(u(n)))|v(f(u(m)))$.

In order to prove this, first assume that the assumption does not hold. Let n_0 and A_0 be such that if $m > n_0$ and $v(f(u(n_0)))|v(f(u(m)))$, then there is prime $p < A_0$ such that $p|m$. Set $t = (u(n_0)\prod_{p<A_0} p)^2$ and $F_j(n) = n + j$. Then for at least one j, $f(u(n_0))|f(n+j)$ while $(n+j+1, \prod_{p<A} p) = 1$. It follows that $f(n+j)$ does not divide $f(n+j+1)$. Assume now that the assumption holds. Let $F_j(n)$ be a suitable family of polynomials. Choose Y large enough so that for any t-uple $m_1, \ldots, m_t$ of integers, and for any prime $p > Y$, there exists n such that $F_j(n) \equiv m_j \pmod p$ for $j = 1, \ldots, t$. It follows that n can be chosen so that

$$v(f(u(F_j(n)))) \text{ divides } v(f(u(F_{j+1}(n)))) \qquad (j = 1, \ldots, t),$$

thus completing the proof.

Remark 6. Assume that f is such that on prime powers p^a, we have $f(p^a) = g(a)$ for a certain function g. Then, for any value of t and any family of polynomials $F_1, \ldots, F_t$, we have that d_0 is strictly positive.

Indeed, this is an easy corollary of Remark 5.

The following remark provides perhaps the simplest instance for which $d_0 = 0$.

Remark 7. Let f be a multiplicative function such that $f(p) = 1$ and $f(p^a) = p^a$ if $a \geq 2$. Then there exists no integer n such that

$$f(n)|f(n+1)|f(n+2)|f(n+3)|f(n+4).$$

Indeed, for exactly one value of $j = 0, 1, 2, 3$, we have that $n + j$ is divisible by 4. It follows that $f(n+j)$ is even while $f(n+j+1)$ is odd, a non sense.

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