Annales Univ. Sci. Budapest., Sect. Comp. 34 (2011) 87–94

ON CONJUGATE MEANS OF n VARIABLES

Z. Daróczy and J. Dascăl

(Debrecen, Hungary)

Dedicated to Professor János Galambos
on the occasion of his 70th birthday

Abstract. Let $I \subset \mathbb{R}$ be a nonvoid open interval and let $n \geq 3$ be a fixed natural number. The question is which conjugate means of n variables generated by the n-variable arithmetic mean are weighted quasi-arithmetic means of n variables at the same time? This question is a functional equation problem. We characterize the real valued continuous and strictly monotone functions φ, ψ defined on I and the parameters $p_1, ..., p_n$, $q_1, ..., q_n$ for which the equation

$$\varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) \varphi\left(\frac{x_1 + \ldots + x_n}{n}\right)\right) =$$

$$= \psi^{-1}\left(\sum_{i=1}^{n} q_i \psi(x_i)\right)$$

holds for all $x_1, ..., x_n \in I$, where

$$q_i > 0 \quad (i = 1, ..., n), \quad \sum_{i=1}^{n} q_i = 1,$$

$$p_j > 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i - p_j \leq 1 \ (j = 1, ..., n).$$

This research has been supported by the Hungarian Scientific Research Fund (OTKA) under grants NK 81402, NK 68040.

2000 AMS Mathematics Subject Classifications: 39B22.

https://doi.org/10.71352/ac.34.087

1. Introduction

Let $n \geq 2$ be a natural number. Denote by K_n the n-tuples $(p_1, \ldots, p_n)$ with the property below:

$$\text{If} \quad \min\{x_i\} \leq M \leq \max\{x_i\} \quad (i = 1, \ldots, n)$$

holds for the real numbers $x_1, \ldots, x_n, M$ then

$$\min\{x_i\} \leq \sum_{i=1}^{n} p_i x_i + \left(1 - \sum_{i=1}^{n} p_i\right) M \leq \max\{x_i\}$$

holds, too.

Daróczy–Páles ([2]) gave the necessary and sufficient condition for the n-tuple $(p_1, \ldots, p_n)$ to be an element of K_n:

Theorem 1. $(p_1, p_2, ..., p_n) \in K_n$ *$(n \geq 2)$ if and only if*

$$p_j \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i - p_j \leq 1 \quad (j = 1, 2, \ldots, n).$$

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Denote by $\mathcal{CM}(I)$ the class of continuous and strictly monotone real valued functions defined on the interval I.

Furthermore, let M be a mean of n variables on I. Then for any $\varphi \in$ $\in \mathcal{CM}(I)$

$$\min\{\varphi(x_i)\} \leq \varphi(M(x_1, x_2, \ldots, x_n)) \leq \max\{\varphi(x_i)\}$$

holds for all $x_1, \ldots, x_n \in I$, hence by Theorem 1 we have that

$$\min\{\varphi(x_i)\} \leq \sum_{i=1}^{n} p_i \varphi(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) \varphi(M(x_1, x_2, \ldots, x_n)) \leq \max\{\varphi(x_i)\}$$

holds for all $(p_1, p_2, \ldots, p_n) \in K_n$ and $x_1, \ldots, x_n \in I$.

We get that

$$\varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) \varphi(M(x_1, \ldots, x_n))\right) =:$$

$$=: M_{\varphi}^{p_1, p_2, \ldots, p_n}(x_1, \ldots, x_n)$$

is always between $\min\{x_i\}$ and $\max\{x_i\}$, that is $M_\varphi^{p_1,p_2,\dots,p_n} : I^n \to I$ is a mean. We call this mean *the conjugate mean generated by the mean M with weights $p_1, p_2, \dots, p_n$*. This class of means includes the weighted quasi-arithmetic means and the quasi-arithmetic means. If for instance we consider the case $n = 2$ we get the following mean:

$$\text{(1)} \qquad M_\varphi^{(p_1,p_2)}(x,y) := \varphi^{-1}(p_1\varphi(x) + p_2\varphi(y) + (1 - p_1 - p_2)\varphi(M(x,y))) \quad (x, y \in I),$$

where $(p_1, p_2) \in [0,1]^2$ (see Theorem 1). If we put $p_1 + p_2 = 1$ in (1) we get the weighted quasi-arithmetic mean. If $p_1 = p_2 = \frac{1}{2}$ we get the quasi-arithmetic mean.

Let now M be the n-variable arithmetic mean:

$$M(x_1, \dots, x_n) := \frac{(x_1 + \ldots + x_n)}{n}.$$

The question is which conjugate means generated by the arithmetic mean M are weighted quasi-arithmetic means at the same time?

In this paper we characterize the functions $\varphi, \psi \in \mathcal{CM}(I)$ and the parameters $p_1, p_2, \dots, p_n$, $q_1, q_2, \dots, q_n$ for which the equation

$$\varphi^{-1}\left(\sum_{i=1}^n p_i\varphi(x_i) + \left(1 - \sum_{i=1}^n p_i\right)\varphi(M(x_1, \dots, x_n))\right) = \psi^{-1}\left(\sum_{i=1}^n q_i\psi(x_i)\right)$$

holds for all $x_1, \dots, x_n \in I$, where $(p_1, p_2, \dots, p_n) \in K_n$, with $p_i > 0$ and $\sum_{i=1}^n q_i = 1, \quad q_i > 0 \ (i = 1, \dots, n)$.

This problem has been solved in the special case $n = 2$ by Daróczy–Dascăl [4]. It is surprising that this special equality problem leaded to the original Matkowski–Sutô problem and some of its generalizations.

2. Preliminary results

We need the following definitions.

Definition 1. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that

$$\psi(x) = a\varphi(x) + b \quad \textit{if } x \in I$$

then we say that *φ is equivalent to ψ on I* and denote it by $\varphi(x) \sim \psi(x)$ if $x \in I$ or in short $\varphi \sim \psi$ on I.

We define the following sets:

$$T_+(I) := \{t \in \mathbb{R} \mid I + t \subset \mathbb{R}_+\},$$
$$T_-(I) := \{t \in \mathbb{R} \mid -I + t \subset \mathbb{R}_+\}.$$

The following comparison theorem was given by Maksa–Páles in [3]:

Theorem 2. *Let $\varphi, \psi : I \to \mathbb{R}$ be continuous strictly monotonic functions, $2 \le n \in \mathbb{N}$, and let $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in (0,1)$ with $\sum_{k=1}^{n} \lambda_k = \sum_{k=1}^{n} \mu_k = 1$. Then the inequality*

$$\varphi^{-1}\left(\sum_{k=1}^{n} \lambda_k \varphi(x_k)\right) \le \psi^{-1}\left(\sum_{k=1}^{n} \mu_k \psi(x_k)\right)$$

holds for all $(x_1, \dots, x_n) \in I^n$ if and only if $\psi \circ \varphi^{-1}$ is convex (concave) if ψ is increasing (decreasing) and $\lambda_k = \mu_k$ for all $k \in \{1, \dots, n\}$.

Theorem 3 (J. Jarczyk, [5]). *Let $I \subset \mathbb{R}$ be a non-trivial interval, $\kappa, \lambda \in \in \mathbb{R} \setminus \{0,1\}$ and $\mu, \nu \in (0,1)$, and let $\varphi, \psi : I \to \mathbb{R}$ be continuous strictly monotonic functions such that (φ, ψ) satisfies the following equation:*

$$\kappa x + (1-\kappa) y =$$
$$= \lambda \varphi^{-1}(\mu\varphi(x) + (1-\mu)\varphi(y)) + (1-\lambda)\psi^{-1}(\nu\psi(x) + (1-\nu)\psi(y)).$$

Then

$$\kappa = \lambda(\mu - \nu) + \nu \tag{2}$$

and one of the following conditions holds in I:

(a) $\varphi(x) \sim x$, $\psi(x) \sim x$;

(b) $\kappa = \lambda = \frac{1}{2}$, $\mu + \nu = 1$ *and* $\varphi(x) \sim e^{tx}$, $\psi(x) \sim e^{-tx}$ *with some* $t \in \mathbb{R} \setminus \{0\}$;

(c) $\kappa = \frac{\mu^2}{\mu^2 + (1-\mu)^2}$, $\lambda = \frac{1}{\mu^2 + (1-\mu)^2}$, $\nu = \frac{1}{2}$ *and either* $\varphi(x) \sim \sqrt{x+t}$ *and* $\psi(x) \sim \log(x+t)$ *if* $t \in T_+(I)$ *or*
$\varphi(x) \sim \sqrt{-x+t}$ *and* $\psi(x) \sim \log(-x+t)$ *if* $t \in T_-(I)$;

(d) $\kappa = \frac{\nu^2}{\nu^2 + (1-\nu)^2}$, $\lambda = \frac{-2\nu(1-\nu)}{\nu^2 + (1-\nu)^2}$, $\mu = \frac{1}{2}$ *and either*
$\varphi(x) \sim \log(x+t)$ *and* $\psi(x) \sim \sqrt{x+t}$ *if* $t \in T_+(I)$ *or*
$\varphi(x) \sim \log(-x+t)$ *and* $\psi(x) \sim \sqrt{-x+t}$ *if* $t \in T_-(I)$;

(e) $\mu \neq \nu$, $\mu + \nu \neq 1$, $\kappa = \frac{\mu\nu}{\mu+\nu-1}$, $\lambda = \frac{\kappa-\nu}{\mu-\nu}$ *and either*
$\varphi(x) \sim \sqrt{x+t}$ *and* $\psi(x) \sim \sqrt{x+t}$ *if* $t \in T_+(I)$ *or*
$\varphi(x) \sim \sqrt{-x+t}$ *and* $\psi(x) \sim \sqrt{-x+t}$ *if* $t \in T_-(I)$.

Conversely, all the pairs listed above satisfy the functional equation.

Observe that if $I = \mathbb{R}$ then only cases (a) and (b) are possible.

3. Main result

Theorem 4. *Let $n \geq 3$ be a fixed natural number, $(p_1, p_2, \ldots, p_n) \in K_n$, $p_i > 0$ and $q_1 + \ldots + q_n = 1$, $q_i > 0$ $(i = 1, \ldots, n)$. If the functions $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of the functional equation*

(3)
$$\varphi^{-1}\left(\sum_{i=1}^{n} p_i\varphi(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right)\varphi(M(x_1, \ldots, x_n))\right) = \psi^{-1}\left(\sum_{i=1}^{n} q_i\psi(x_i)\right)$$

$(x_1, \ldots, x_n \in I)$ *then*

$$q_j - p_j = \frac{1}{n}\left(1 - \sum_{i=1}^{n} p_i\right) \qquad (j = 1, \ldots, n)$$

and the following cases are possible

- *if* $\sum\limits_{i=1}^{n} p_i \neq 1$ *then* $\varphi(x) \sim x$, $\psi(x) \sim x$ $(x, y \in I)$;
- *if* $\sum\limits_{i=1}^{n} p_i = 1$ *then* $\varphi(x) \sim \psi(x)$ $(x, y \in I)$.

Conversely, the functions given in the above cases are solutions of equation (3).

Proof. If $\sum\limits_{i=1}^{n} p_i = 1$ then we have the equality of weighted quasi-arithmetic means and by Theorem 2 we get the statement.

If $\sum\limits_{i=1}^{n} p_i \neq 1$ we put $J := \varphi(I)$ and $u_i := \varphi(x_i)$ and we get

$$p_1u_1 + \ldots p_nu_n + \left(1 - \sum_{i=1}^{n} p_i\right)\varphi\left(\frac{\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_n)}{n}\right) =$$

$$= \varphi \circ \psi^{-1}\left(q_1 \psi \circ \varphi^{-1}(u_1) + \ldots + q_n \psi \circ \varphi^{-1}(u_n)\right) \tag{4}$$

for all $u_i \in J$.

Setting $u := u_1$ and $v := u_2 = \ldots = u_n$ in the equation above we get

$$p_1 u + v \sum_{i=2}^{n} p_i = \left(\sum_{i=1}^{n} p_i - 1\right) \varphi\left(\frac{1}{n}\varphi^{-1}(u) + (1 - \frac{1}{n})\varphi^{-1}(v)\right) +$$

$$+\varphi \circ \psi^{-1}\left(q_1 \psi \circ \varphi^{-1}(u) + (1 - q_1)\psi \circ \varphi^{-1}(v)\right) \ (u, v \in J).$$

Dividing by $\sum\limits_{i=1}^{n} p_i$ we get

$$\frac{p_1}{\sum\limits_{i=1}^{n} p_i} u + \left(1 - \frac{p_1}{\sum\limits_{i=1}^{n} p_i}\right) v =$$

$$= \left(1 - \frac{1}{\sum\limits_{i=1}^{n} p_i}\right) \varphi\left(\frac{1}{n}\varphi^{-1}(u) + \left(1 - \frac{1}{n}\right)\varphi^{-1}(v)\right) +$$

$$+\frac{1}{\sum\limits_{i=1}^{n} p_i} \varphi \circ \psi^{-1}\left(q_1 \psi \circ \varphi^{-1}(u) + (1 - q_1)\psi \circ \varphi^{-1}(v)\right) \ (u, v \in J).$$

Let $\frac{p_1}{\sum_{i=1}^{n} p_i} =: \kappa$, $\frac{1}{\sum_{i=1}^{n} p_i} =: \lambda$, $q_1 =: \mu$ and $\frac{1}{n} =: \nu$, then

$$\kappa u + (1 - \kappa)v = \lambda \varphi \circ \psi^{-1}\left(\mu \psi \circ \varphi^{-1}(u) + (1 - \mu)\psi \circ \varphi^{-1}(v)\right) +$$

$$+(1 - \lambda)\varphi\left(\nu \varphi^{-1}(u) + (1 - \nu)\varphi^{-1}(v)\right) \ (u, v \in J). \tag{5}$$

$p_i > 0$ for all $i = 1, \ldots, n$ implies $\kappa \in \mathbb{R} \setminus \{0, 1\}$, hence we can apply Theorem 3. The case $\lambda = 1$ gives $\sum\limits_{i=1}^{n} p_i = 1$ which we have already discussed. $\lambda = 0$ is not possible. Hence by (2) we get that

$$q_1 - p_1 = \frac{1}{n}\left(1 - \sum_{i=1}^{n} p_i\right).$$

Setting $u := u_2$ and $v := u_1 = u_3 = \ldots = u_n$ in (4) and applying Theorem 3 again to the new equation and so on we get that

$$q_j - p_j = \frac{1}{n}\left(1 - \sum_{i=1}^{n} p_i\right)$$

for $j = 1, \ldots, n$.

From the first case of Theorem 3 we have that $\psi \circ \varphi^{-1}$ and φ^{-1} are the identical functions, which means $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x, y \in I$.

In case (b) of Theorem 3 $\kappa = \lambda = \frac{1}{2}$ and $\mu + \nu = 1$, which gives $p_1 = 1$, $q_1 = 1 - \frac{1}{n}$ and $\sum_{i=1}^{n} p_i = 2$. Since $(p_1, p_2, \ldots, p_n) \in K_n$ we know that $p_j \geq 0$ and $\sum_{i=1}^{n} p_i - p_j \leq 1$ $(j = 1, 2, \ldots, n)$. From these relations we have that $p_j \geq 1$ for all $j = 1, 2, \ldots, n$ and $\sum_{i=2}^{n} p_i = 1$, hence $(p_1, p_2, \ldots, p_n) \in \{(1, 1, 0, \ldots, 0), \ldots, (1, 0, \ldots, 0, 1)\}$ which contradicts $p_1, \ldots, p_n > 0$.

In case (c) of Theorem 3 we have $\nu = \frac{1}{2}$, but $\nu = \frac{1}{n}$ and $n \geq 3$, so this case is not possible.

In case (d) of Theorem 3 we have $\mu = \frac{1}{2}$, but $\mu = q_1$ which implies $q_1 = \frac{1}{2}$. Furthermore, $\kappa = \frac{\nu^2}{\nu^2 + (1-\nu)^2}$ and $\lambda = \frac{-2\nu(1-\nu)}{\nu^2 + (1-\nu)^2}$, which give $p_1 = -\frac{1}{2} \cdot \frac{1}{n-1} < 0$ which is a contradiction, so this case is not possible either.

In case (e) from the relations between the parameters we get that $p_1 = \frac{q_1(nq_1 - 1)}{n-1}$, but $p_1 > 0$, hence $q_1 > \frac{1}{n}$. Applying Theorem 3 n times we get that $q_i > \frac{1}{n}$ $(i = 1, \ldots, n)$, but $q_1 + \ldots + q_n = 1$, which is again a contradiction. By easy computation we can see that the functions found in the above cases are solutions of the functional equation (3) and this completes our proof. ■

References

[1] **Daróczy, Z. and Zs. Páles,** Gauss-composition of means and the solution of the Matkowski–Sutô problem, *Publ. Math. Debrecen*, **61** (1-2) (2002), 157–218.

[2] **Daróczy, Z. and Zs. Páles,** On an elementary inequality and conjugate means (manuscript)

[3] **Maksa, Gy. and Zs. Páles,** Remarks on the comparison of weighted quasi-arithmetic means, *Colloq. Math.*, **120** (1) (2010), 77–84.

[4] **Daróczy, Z. and J. Dascăl,** On the equality problem of conjugate means, *Results Math.*, **58** (1-2) (2010), 69–79.

[5] **Jarczyk, J.,** On an equation involving weighted quasi-arithmetic means, *Acta. Math. Hungar.*, **129** (1-2) (2010), 96–111.

Z. Daróczy and J. Dascăl
Institute of Mathematics
University of Debrecen
H-4010 Debrecen, Pf. 12
Hungary
daroczy@math.klte.hu
jdascal@math.klte.hu