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SOME FURTHER REMARKS ON A PAPER OF K. RAMACHANDRA

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Dedicated to Professor János Galambos on his seventieth anniversary

1. Introduction

Let $\mathcal{P}$ be the whole set of primes. Let $D > 1$ be an integer, $l_1, \dots, l_k$ be distinct residues $\mod D$ coprime to D, $\quad k < \varphi(D)$. Let $\tilde{\mathcal{P}}$ be the set of the primes $p \equiv l_1, \dots, l_k \pmod{D}$, and $\tilde{\mathcal{N}} = \mathcal{N}\left(\tilde{\mathcal{P}}\right)$ be the semigroup generated by $\tilde{\mathcal{P}}$. Let $\tilde{\mathcal{P}}_k := \{n | n \in \mathcal{N}\left(\tilde{\mathcal{P}}\right), \omega(n) = k\}$, $\quad \tilde{\mathcal{N}}_k = \{n | n \in \mathcal{N}\left(\tilde{\mathcal{P}}\right), \Omega(n) = k\}$, where $\omega(n), \Omega(n)$ are additive arithmetical functions defined for prime power p^α by $\omega(p^\alpha) = 1, \quad \Omega(p^\alpha) = \alpha$.

Let

$$\begin{aligned} \pi_k(x) &:= \#\{n | n \le x\}, \\ \tilde{\Pi}_k(x) &:= \#\{n \le x | n \in \tilde{\mathcal{P}}_k\}, \\ \tilde{N}_k(x) &:= \#\{n \le x | n \in \tilde{\mathcal{N}}_k\}. \end{aligned}$$

Our purpose in this paper is to give the asymptotic of $\tilde{\Pi}_k(x+y) - \tilde{\Pi}_k(x)$, and that of $\tilde{N}_k(x+y) - \tilde{N}_k(x)$, where $y \asymp x^\theta, \quad \theta < 1$.
This can be done by combining the method of Sathe-Selberg, and that of K. Ramachandra ([1], [6], [7]).

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2. The method of Ramachandra

2.1. Ramachandra [1] proved the following assertion:

Let S_1, S_2 and S_3 be the sets of L-series, the derivatives, and the logarithms of L-series, respectively. $\log L(s,\chi)$ is defined by analytic continuation from the halfplane $\sigma = \mathrm{Re} s > 1$; for some complex z, we define

$$L(s,\chi)^z = \exp\left(z \log L(s,\chi)\right).$$

Let $P_1(s)$ be any finite power product (with complex exponents) of functions of S_1. Let $P_2(s)$ be any finite power product (with nonnegative integral exponents) of functions of S_2. Let also $P_3(s)$ denote any finite power product with nonnegative integral exponents of functions of S_3. Let c_n be a sequence of complex numbers such that $|c_n| \ll n^{\varepsilon}$ for every $\varepsilon > 0$ and

$$\sum \frac{|c_n|}{n^{\sigma}} < \infty \qquad \text{for} \quad \sigma > 1/2.$$

Let $F_0(s) = \sum\limits_n \frac{c_n}{n^s}$. Furthermore, let

$$F_1(s) = P_1(s)\, P_2(s)\, P_3(s)\, F_o(s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s}$$

and

$$E(x) = \sum_{n \le x} g_n.$$

Let $r\ (\le 1/2)$ be a positive number. We define the contour $C(r)$ by starting from the circle $\{s \big| |s-1| = r\}$, removing the point $1-r$, and proceeding on the remaining portion of the circle in the anticlockwise direction. Let $C_0 = C(r)$.

Assume that r is so small that $F_1(s)$ has no singularities on the boundary and in interior of it, except, possibly, the places $s = 1$.

Let $C_1 = C\left(\frac{1}{\log x}\right)$, and let L^-, L^+ be defined as the intervals on straightlines

$$L^- = \left[\left(1 - \frac{1}{r}\right) e^{-i\pi}, \left(1 - \frac{1}{\log x}\right)^{-i\pi}\right],$$

$$L^+ = \left[\left(1 - \frac{1}{\log x}\right) e^{i\pi}, \left(1 - \frac{1}{r}\right)^{i\pi}\right].$$

Let C^* be the contour going along L^- starting from $\left(1-\frac{1}{2}\right)e^{-i\pi}$, then on C_1, and, finally, on L^+.

Let B be the constant occurring in the density result

$$N_\chi(\alpha,T)=\mathcal{O}\left(T^{B(1-\alpha)}(\log T)^2\right),$$

which is valid for all characters occurring in P_1, P_2 and P_3. Let $\varphi = 1-1/B+\varepsilon$ with arbitrary $\varepsilon>0$.

Remark. According to Huxley's result, φ can be any constant greater than $7/12$.

Theorem of Ramachandra. *Let x be sufficiently large and $1\le h\le x$. Let*

$$I(x,h)=\frac{1}{2\pi i}\int_0^h\left(\int_{C_0}F_1(s)(v+x)^{s-1}\,ds\right)dv. \tag{2.1}$$

Then

$$E(x+h)-E(x)=I(x,h)+\mathcal{O}_\varepsilon\left(h\cdot\exp\left(-(\log x)^{1/6}\right)+x^\varphi\right). \tag{2.2}$$

Ramachandra used the Hooley-Huxley contour for proving his very general theorem. Kátai [2] applied Ramachandra's theorem to obtain the uniform result

$$\frac{1}{h}\sum_{\substack{\omega(n)=k\\ x\le n\le x+h}}1=(1+o(1))\frac{\pi_k(x)}{x},$$

uniformly for any $k\le\log\log x+c_x\sqrt{\log\log x}$, where $c_x\to\infty$ sufficiently slowly, and $x\ge h\ge x^{\varphi+\varepsilon}$.

Sankaranarayanan and Srinivas [3] gave a version of Ramachandra's result in which the function $F_1(s)$ may depend on a parameter.

2.2. Some consequence proved in [4]

Integrating on the same contour as Ramachandra did, we have

$$E(x)=J(x)+\mathcal{O}\left(x\cdot\exp\left(-(\log x)^{1/6}\right)\right), \tag{2.3}$$

where

$$J(x)=\frac{1}{2\pi i}\int_{C_0}F_1(s)\frac{x^s}{s}ds. \tag{2.4}$$

Furthermore, $I(x,h)$ can be written as

$$I(x,h)=\frac{1}{2\pi i}\int_{C_0}F_1(s)\frac{(x+h)^s-x^s}{s}ds. \tag{2.5}$$

Let

$$D(x,h,s):=\frac{1}{s}\left(\frac{(x+h)^s-x^s}{h}-x^{s-1}\right). \tag{2.6}$$

Assume that $\frac{1}{2}\le|s|\le 2$ and that $h=x^{\eta},\eta<\frac{2}{3}-\frac{2r}{3}$ with small r. Then

$$\frac{(x+h)^s-x^s}{sh}=x^{s-1}+\frac{hx^{s-2}(1-s)}{2}+\mathcal{O}\left(h^3x^{\sigma-3}\right)$$

and, thus,

$$D(x,h,s)=x^{s-1}\left(1-\frac{1}{s}\right)+\mathcal{O}\left(h^3\cdot x^{\sigma-3}\right),$$

which by $h^3\cdot x^{\sigma-3}\ll x^{2-2r+r-2}\ll x^{-r}$ and $hx^{\sigma-2}\ll x^{-r}$ implies that

$$D(x,h,s)=x^{s-1}\frac{(s-1)}{s}+\mathcal{O}\left(x^{-r}\right).$$

Hence, we obtain that

$$\begin{aligned}\frac{E(x+h)-E(x)}{h}-\frac{E(x)}{x}=&\frac{1}{2\pi i}\int_{C_0}F_1(s)\frac{x^{s-1}}{s}(s-1)\,ds+\\&+\mathcal{O}\left(x^{-r}\right)+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right)\end{aligned} \tag{2.7}$$

and, thus, by (2.3) and (2.4) we have

$$\begin{aligned}\frac{E(x+h)-E(x)}{h}=&\frac{1}{2\pi i}\int_{C_0}F_1(s)\,x^{s-1}ds+\\&+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right)+\\&+\mathcal{O}\left(x^{-r}\right).\end{aligned} \tag{2.8}$$

Since $F_1(s)$ is analytic on the domain with boundary $C_o\cup C^*$, we can transform the integration line on the right side of (2.8) to the contour C^*.

We have proved the following:

Theorem A. *Assume that $F_1(s)$ satisfies the conditions stated in Ramachandra's theorem. Let $r>0$ and $\varepsilon>0$ be sufficiently small constants, and let $x^{7/12+\varepsilon}\le h\le$ $\le x^{\frac{2}{3}-\frac{2r}{3}}$. Then*

$$\frac{E(x+h)-E(x)}{h}=\frac{1}{2\pi i}\int_{C^*}F_1(s)\,x^{s-1}dx+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right). \tag{2.9}$$

Let us assume that

$$F_1(s) = \frac{U(s)}{(s-1)^z}, \tag{2.10}$$

where the function $U(s)$ is analytic in the disc $|s-1| \le r$. Then, for each fixed k,

$$U(s) = A_0 + A_1(s-1) + \ldots + A_k(s-1)^k + (s-1)^{k+1} V(s),$$

where $V(s)$ is bounded in $|s-1| \le r$.

Furthermore, since

$$\frac{1}{2\pi i}\int_{C^*} x^{s-1}(s-1)^{a-z}\,ds = \frac{\Gamma(a-z)}{(\log x)^{a-z+1}}\frac{\sin\pi(a-z)}{\pi} + \mathcal{O}\left(x^{-r/2}\right) \tag{2.11}$$

(for the proof, see Lemma 8 in [10]), we deduce the following:

Theorem B. *Under the conditions stated above, we have*

$$\begin{aligned}\frac{1}{2\pi i}\int_{C^*}\frac{U(s)}{(s-1)^z}x^{s-1}ds = &\sum_{l=0}^{k} A_l\frac{\Gamma(l-z)}{(\log x)^{l-z+1}}\frac{(-1)^{l+1}\sin\pi z}{\pi} + \\ &+ \mathcal{O}\left(\frac{1}{(\log x)^{k+2-\mathrm{Re}z}}\right),\end{aligned} \tag{2.12}$$

whenever Re$z \le k+1$.

Proof. By (2.11), we have only to prove that

$$\frac{1}{2\pi i}\int_{C^*} V(s)(s-1)^{k+1-z}\,ds \tag{2.13}$$

can be majorated by the error term on the right-hand side of (2.12). The integral (2.13) extended to the contour $C(1/\log x)$ is obviously less than the error term of (2.12).

To estimate the integral on L^+ and L^-, let us write $s = 1-\tau$. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{L^\pm}|V(s)||(s-1)|^{k+1-\mathrm{Re}z}x^{-\tau}ds \le &\frac{K}{2\pi}\int_{1/\log x}^{r} x^{-\tau}\tau^{k+1-\mathrm{Re}z}d\tau \ll \\ &\ll \frac{1}{(\log x)^{k+2-\mathrm{Re}z}},\end{aligned}$$

and the proof is completed. ■

3. Short interval version theorems for $\tilde{\mathbf{N}}_{\mathbf{k}}(\mathbf{x}), \tilde{\mathbf{\Pi}}_{\mathbf{k}}(\mathbf{x})$

Let χ run over the Dirichlet characters $\mod D$, $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, χ_0 be the principal character $\mod D$, $L(s,\chi_0) = \zeta(s) \prod_{p|D} \left(1 - \frac{1}{p^s}\right)$.

Let

$$c(\chi) := \frac{1}{\varphi(D)} \sum_{j=1}^{k} \bar{\chi}(l_j), \tag{3.1}$$

especially

$$c(\chi_0) = \frac{k}{\varphi(D)}. \tag{3.2}$$

Let $z \in \mathbb{C}$

$$F(s,z) := \sum_{n \in \tilde{\mathcal{N}}} \frac{z^{\Omega(n)}}{n^s} = \prod_{p \in \tilde{\mathcal{P}}} \frac{1}{1 - \frac{z}{p^s}}, \tag{3.3}$$

$$G(s,z) := \sum_{n \in \tilde{\mathcal{N}}} \frac{z^{\omega(n)}}{n^s} = \prod_{p \in \tilde{\mathcal{P}}} \left(1 + \frac{z}{p^s - 1}\right), \tag{3.4}$$

$$H(s,z) := \sum_{n \in \tilde{\mathcal{N}}} \frac{z^{\omega(n)} |\mu(n)|}{n^s} = \prod_{p \in \tilde{\mathcal{P}}} \left(1 + \frac{z}{p^s}\right). \tag{3.5}$$

Let p^* be the smallest element of $\tilde{\mathcal{P}}$.

We can write

$$F(s,z) = F(s,1)^z Q(s,z), \tag{3.6}$$

where

$$Q(s,z) = \prod_{p \in \tilde{\mathcal{P}}} \frac{\left(1 - \frac{1}{p^s}\right)^z}{\left(1 - \frac{z}{p^s}\right)}. \tag{3.7}$$

The product on the right-hand side of (3.7) is absolutely and uniformly convergent in $\mathrm{Re}s > \frac{1}{2}+\delta, \quad |z| \le p^{*\frac{1}{2}+\delta} - \varepsilon$, if δ, ε are arbitrary positive constants.

Let

(3.8) $$T(s) := \prod_{\chi} L(s,\chi)^{c(\chi)}, \quad A(s) = T(s)\cdot(s-1)^{c(\chi_0)}.$$

Thus

(3.9) $$A(s) := \zeta(s)(s-1)^{c(\chi_0)} \prod_{p|D}\left(1-\frac{1}{p^s}\right)^{c(\chi_0)} \prod_{\chi\neq\chi_0} L(s,\chi)^{c(\chi)}.$$

Let

(3.10) $$K(s) := \frac{F(s,1)}{T(s)},$$

(3.11) $$U(s,z) := (A(s)K(s))^z Q(s,z),$$

(3.12) $$F(s,z) := \frac{U(s,z)}{(s-1)^{(\chi_0)z}}.$$

$F(s,z)$ satisfies the conditions stated for $F_1(s)$ in 2.1. We can use Theorem A and B.

Let

(3.13) $$U(s,z) = B_0(z) + B_1(z)(s-1) + B_2(z)(s-1)^2 + \cdots.$$

It is easy to prove that there exists $r > 0$ and a constant c such that

(3.14) $$\sup_{n\ge 0}\max_{|z|\le 2-\varepsilon} |B_n(z)|\cdot r^n \le c.$$

We have

(3.15) $$B_0(z) = U(1,z) = (A(1)K(1))^z Q(1,z),$$

(3.16) $$A(1) = \left(\frac{\varphi(D)}{D}\right)^{c(\chi_0)} \prod_{\chi=\chi_0} L(1,\chi)^{c(\chi)}.$$

Let

(3.17) $$u(s) := \sum_{p\in\tilde{\mathcal{P}}} \frac{1}{p^s}; \quad t(s,\chi) := \sum_{p\in\mathcal{P}} \frac{\chi(p)}{p^s}.$$

Then

$$\log K(s) = \log F(s,1) - \log T(s) = \\ = \sum_{l\ge 2} \frac{1}{l}\left\{u(ls) - \sum_{\chi} c(\chi)\, t\left(ls, \chi^l\right)\right\},$$

and so

$$\log K(1) = \sum_{l\ge 2} \frac{1}{l}\left\{u(l) - \sum_{\chi} c(\chi)\, t\left(l, \chi^l\right)\right\}, \tag{3.18}$$

the right-hand side is absolute convergent.

Since

$$\log Q(s,z) = \sum_{l\ge 2} \frac{1}{l}\left(z - z^l\right) u(ls),$$

therefore $\log Q(1,z) = \left(\sum_{l=2}^{\infty} \frac{u(l)}{l}\right) z - \sum_{l=2}^{\infty} \frac{u(l)}{l} z^l$. Let

$$C = \sum_{l=2}^{\infty} \frac{u(l)}{l}, \tag{3.19}$$

$$Q^*(1,z) = \exp\left(-\sum \frac{u(l)}{l} z^l\right) \\ = Q_0 + Q_1 z + Q_2 z^2 + \cdots. \tag{3.20}$$

Then $Q_0 = 1$,

$$\sum_{\nu} |Q_\nu| \cdot |z|^\nu \le \exp\left(\sum \frac{u(l)}{l} |z|^l\right), \tag{3.21}$$

the right-hand side is finite if $|z| < p^* - \varepsilon$.

We can write

$$B_0(z) = \left(A(1)\, K(1)\, e^C\right)^z \cdot Q^*(1,z).$$

Let $y = x^{7/12+\varepsilon}$. From Theorem A and B we have that

$$\begin{aligned}\mathcal{L}(z) :=& \frac{1}{y}\left(\sum_{\substack{x\le n\le x+y\\ n\in\tilde{\mathcal{N}}}} z^{\Omega(n)}\right) = \frac{1}{2\pi i}\int_{C^*} F(s,z)\,x^{s-1}ds+\\ &+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right) =\\ =&\frac{1}{2\pi i}\int_{C^*}\frac{U(s,z)}{(s-1)^{c(\chi_0)z}}x^{s-1}dx+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right) =\\ =&B_0(z)\,\frac{\Gamma(-c(\chi_0)z)(-1)\sin\pi c(\chi_0)z}{\pi(\log x)^{1-c(\chi_o)z}}+\mathcal{O}\left(\frac{1}{(\log x)^{2-\mathrm{Re}z}}\right)+\\ &+\mathcal{O}\left(\exp\left(-(\log x)^{1/6}\right)\right).\end{aligned}$$

From $\frac{1}{\Gamma(w-k)} = \frac{\sin\pi w}{\pi}\cdot(-1)^k\,\Gamma(k+1-w)$, applied for $k=-1,\quad w=c(\chi_0)$, we have

$$\begin{aligned}\frac{\Gamma(-c(\chi_0)z)(-1)\sin\pi c(\chi_0)z}{\pi} =& \frac{1}{\Gamma(1+c(\chi_0)z)} =\\ =&\frac{1}{c(\chi_0)z}\cdot\frac{1}{\Gamma c(\chi_0)z}.\end{aligned}$$

It is wellknown that $\frac{1}{\Gamma(w)}$ is an entire function. Let

$$R(z) := \frac{Q^*(1,z)}{\Gamma(c(\chi_0)z)} = T_0+T_1z+T_2z^2+\cdots. \tag{3.22}$$

It is clear that

$$\sum|T_\nu||z|^\nu$$

is convergent for $|z|\le p^*-\varepsilon$.

Since

$$\begin{aligned}\frac{\tilde{N}_k(x+y)-\tilde{N}_k(x)}{y} =& \int_0^1\mathcal{L}\left(e^{2\pi i\theta}\right)e^{-2\pi ik\theta}d\theta =\\ =&\operatorname*{coeff}_{z^{k-1}}\frac{\left(A(1)\,K(1)\,e^C(\log x)^{c(\chi_0)z}\right)}{c(\chi_0)(\log x)}R(z)+\\ &+\mathcal{O}\left(\frac{1}{\log x}\right) = \frac{S_k}{\log x}+\mathcal{O}\left(\frac{1}{\log x}\right).\end{aligned}$$

Let

$$l(x) := c(\chi_0)\log\log x + C + \log A(1)\,K(1).$$

We have

$$S_k = \operatorname*{coeff}_{z^{k-1}} e^{zl(x)} \cdot \frac{R(z)}{c(\chi_0)},$$

and so

$$\begin{aligned} S_k = \sum_{l+m=k-1} \frac{T_l}{m!} l^m(x) &= \frac{l(x)^{k-1}}{(k-1)!c(\chi_0)} \sum_{l \le (k-1)} \frac{(k-1)!}{(k-l)!} T_l \cdot l(x)^{-l} \\ &= \frac{l(x)^{k-1}}{c(\chi_0)(k-1)!} U_k, \end{aligned}$$

where

$$U_k = \sum_{l \le k-1} \frac{(k-1)!}{(k-1-l)!} T_l \cdot l(x)^{-l}.$$

Since

$$\frac{(k-1)!}{(k-1-l)!} = (k-1)^l + \mathcal{O}\left(l^2 \cdot k^{l-1}\right),$$

we have

$$U_k = \sum_{l=0}^{k-1} T_l \cdot \left(\frac{k-1}{l(x)}\right)^l + \mathcal{O}\left(\frac{1}{k} \sum_{l=0}^{\infty} l^2 T_l \left(\frac{k-1}{l(x)}\right)^l\right).$$

Collecting our inequalities we obtain the following assertion.

Theorem 1. *Let $\varepsilon > 0$ be fixed. Then, uniformly as*

$$1 \le k \le (p^* - \varepsilon)\, c(\chi_0) \log\log x,$$

for $y = x^{7/12+\varepsilon}$ we have

$$\begin{aligned} \frac{\tilde{N}_k(x+y) - \tilde{N}_k(x)}{y} = &\frac{(c(\chi_0)\log\log x + C + \log A(1)\, K(1))^{k-1}}{c(\chi_0)(k-1)!} \times \\ &\times R\left(\frac{k}{c(\chi_0)\log\log x + C + \log A(1)\, K(1)}\right) \times \\ &\times \left(1 + \mathcal{O}\left(\frac{1}{\log\log x}\right)\right). \end{aligned}$$

$A(1), K(1)$ are defined in (3.9), (3.11), and C in (3.19), (3.17), R in (3.22).

Arguing similarly, as above, we can prove Theorem 2.

Let

$$M(1,z) := \prod_{p\in\tilde{\mathcal{P}}} \frac{\left(1-\frac{1}{p}\right)^z}{1+\frac{z}{p-1}},$$

$$S(z) := \frac{Q^*(1,z)\,M(1,z)}{\Gamma(c(\chi_0)\,z)}.$$

Theorem 2. *Let* $0 < B < \infty$, $\left(\frac{1}{2}\right) > \varepsilon > 0$ *be fixed constants. Then, uniformly as* $1 \le k \le B \log\log x$, *we have*

$$\frac{\tilde{\Pi}_k(x+y) - \tilde{\Pi}_k(x)}{y} = \frac{1}{c(\chi_0)\log x} \cdot \frac{l(x)^{k-1}}{(k-1)!} S\left(\frac{k-1}{l(x)}\right) \times \\ \times \left(1 + \mathcal{O}\left(\frac{1}{\log\log x}\right)\right),$$

where $y = x^{7/12+\varepsilon}$.

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