Annales Univ. Sci. Budapest., Sect. Comp. 34 (2011) 25–32

HOW LARGE CAN THE COEFFICIENTS OF A POWER SERIES BE?

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Dedicated to Professor János Galambos on the occasion of his 70th birthday

Abstract. The paper is inspired by the problem of estimating the deviation of two discrete probability distributions in terms of the supremum distance between their generating functions over the interval $[0, 1]$. Under certain conditions on the tail it is clarified how large can the terms of a real sequence be if the sup norm of its generating function is known.

1. Introduction

Let $A_1, \ldots, A_n$ be an arbitrary collection of events in an arbitrary probability space. Let N denote the number of events that occur. In many cases we have to determine the distribution of the random variable N, or, at least, to estimate the probability that none of the events occur. Such problems typically arise when stochastic methods are applied in combinatorics, see [1].

The probability $P(N = 0)$ can be estimated in several ways. In messy situtations, where the dependence structure of the events is rather complicated, sieve methods,

Research supported by the Hungarian Scientific Research Fund (OTKA), Grant K67961.

2000 AMS Mathematics Subject Classifications: Primary 41A17, Secondary 26D15.

Key words and phrases: Poisson approximation, Rényi sieve, Bonferroni inequalities, generating function, Chebyshev polynomials.

https://doi.org/10.71352/ac.34.025

like the Rényi sieve, can sometimes help. Those methods provide Bonferroni type lower or upper bounds of the form

$$P(N=0) \le (\ge) \sum_M c(M) P\Big(\bigcap_{i\in M} A_i\Big), \tag{1.1}$$

where in the sum M runs over the subsets of $\{1,2,\dots,n\}$, and the $c(M)$ are real constants. The interested reader is referred to the excellent monograph by Galambos and Simonelli [3].

Such inequalities can easily be transformed into bounds for the probability generating function of N. By [4, Theorem 1], together with (1.1) we also have

$$g_N(x) = E(x^N) \le (\ge) \sum_M c(M) P\Big(\bigcap_{i\in M} A_i\Big)(1-x)^{|M|}, \tag{1.2}$$

for $0 \le x \le 1$. If we want to show the asymptotic Poissonity of N, in the way above we can estimate the difference between $g_N(x)$ and the generating function of the corresponding Poisson distribution (i.e., that with expectation equal to EN). Now the question is: how to estimate the difference of probabilities, if we have bounds for the difference of generating functions?

This problem can be reformulated in the following way. Let $\mathcal{F}$ be the set of real power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ such that $\sum_{k=0}^{\infty} |a_k| \le 2$ and $\sum_{k=0}^{\infty} a_k = 0$. When we have two discrete probability distributions $\mathbf{p} = (p_0, p_1, \dots)$ and $\mathbf{q} = (q_0, q_1, \dots)$ with generating functions $g_{\mathbf{p}}(x)$ and $g_{\mathbf{q}}(x)$, resp., then $f = g_{\mathbf{p}} - g_{\mathbf{q}} \in \mathcal{F}$, and $a_k = p_k - q_k$. We want to estimate a_k in terms of $\Delta = \max_{0\le x\le 1} |f(x)|$.

The main difficulty of the problem is in the restriction that we only know f over the real interval $[0,1]$, not in a whole neighbourhood of the origin on the complex plane.

The first results in this direction appeared in [4]. It is shown there that the coefficients can not be estimated uniformly, and that for every $\ell = 0, 1, \dots$ and $\varepsilon > 0$ there exists a constant C, depending on ℓ and ε, such that

$$|a_\ell| \le C\,\Delta^{1-\varepsilon}.$$

This was improved in [5] to

$$|a_\ell| \le \Delta \exp\bigg(2\Big(\ell \log\frac{1}{\Delta}\Big)^{4/5}\bigg).$$

On the other hand, the limitations of such estimations are illustrated by the following counterexamples, borrowed from [5].

Let $\mathbf{p} = (p_0, p_1, \dots)$ be a fixed discrete probability distribution such that $p_k > 0$ for every $k = 0, 1, \dots$, and

$$\limsup_{k\to\infty} \frac{1}{2k} \log\frac{1}{p_k} < \infty. \tag{1.3}$$

Let ℓ be a positive integer and C a sufficiently small positive constant. Then for every sufficiently small positive Δ there exists a discrete probability distribution $\mathbf{q} = (q_0, q_1, \dots)$, such that $\max\limits_{0\le x\le 1} |g_{\mathbf{p}}(x) - g_{\mathbf{q}}(x)| = \Delta$, and

$$|p_\ell - q_\ell| > C\,\Delta \left(\log \frac{1}{\Delta}\right)^{2\ell}. \tag{1.4}$$

Analogous result holds for probability distributions $\mathbf{p}$ having a tail that is lighter than exponential.

Suppose that, instead of (1.3), we have

$$\limsup_{k\to\infty} \frac{1}{h(k)} \log \frac{1}{p_k} = v, \tag{1.5}$$

where v is positive and finite, h is a positive, continuous, increasing function, regularly varying at infinity with exponent α, and $\lim_{k\to\infty} h(k)/k = \infty$ (hence $\alpha \ge 1$). Let ℓ be a positive integer and C a sufficiently small positive constant. Then for every sufficiently small positive Δ there exists a discrete probability distribution $\mathbf{q}$, such that $\max\limits_{0\le x\le 1} |g_{\mathbf{p}}(x) - g_{\mathbf{q}}(x)| = \Delta$, and

$$|p_\ell - q_\ell| > C\,\Delta \left(h^{-1}\left(\log \frac{1}{\Delta}\right)\right)^{2\ell}. \tag{1.6}$$

These examples inspired our results in Section 2. We are going to drop the condition $\sum_{k=0}^{\infty} |a_k| \le 2$, but in that case (for $\ell > 0$) $|a_\ell|$ can be arbitrary large, no matter how small Δ is, see Theorem 2.

In order to derive upper bounds in the form of the right hand sides of (1.4) and (1.6) we have to impose additional conditions on the sequence (a_k) in consideration.

2. Results

Let us start with a fundamental lemma.

The following theorem is a variant of a result by V. A. Markov, who proved a similar theorem on the extremal properties of Chebyshev polynomials over the interval $[-1,\ 1]$ (see Chapter 2 of [2]).

Theorem 1. *Consider an arbitrary polynomial of the form* $Q_n(x) = \sum_{k=0}^{n} a_k x^k$. *Introduce* $\Delta = \max_{0\le x\le 1} |Q_n(x)|$. *Then*

$$|a_k| \le \frac{n}{k+n} \binom{k+n}{2k} 2^{2k}\,\Delta. \tag{2.1}$$

For $k > 0$ equality holds if and only if $Q_n(x) = \pm\Delta T_n(2x-1)$, where T_n is the degree n Chebyshev polynomial of the first kind, defined as $T_n(\cos\theta) = \cos(n\theta)$.

Let us remark that

$$\frac{n}{k+n}\binom{k+n}{2k}2^{2k} \le \frac{(2n)^{2k}}{(2k)!} \tag{2.2}$$

by the inequality of arithmetic and geometric means, and the ratio of the two sides tends to 1 as k is fixed and $n \to \infty$.

Proof. We may assume that $a_k = 1$. Then $T_n(2x_i - 1) = (-1)^i$ for $x_i = \cos^2\frac{i\pi}{2n}$, $i = 0, 1, \ldots, n$. Let $T_n(2x-1) = \sum_{k=0}^n d_k x^k$. Suppose

$$|d_k|\Delta < 1 = \max_{0\le x\le 1}|T_n(2x-1)|.$$

Let $p(x) = T_n(2x-1) - d_k Q_n(x)$, then $p(x_i)$ is positive or negative, according as i is even or odd. Thus $p(x)$ has n distinct roots in the interval $(0,\ 1)$. Let them be denoted by $y_1, y_2, \ldots, y_n$, then

$$p(x) = (d_n - d_k a_n)\prod_{i=1}^n (x - y_i),$$

hence the coefficient of x^k in $p(x)$ is equal to

$$(-1)^{n-k}(d_n - d_k a_n)\sum_{1\le i_1<\cdots<i_{n-k}\le n} y_{i_1}\ldots y_{i_{n-k}}.$$

It should be 0, which is a contradiction. Consequently, $|a_k| = 1 \le |d_k|\Delta$.

Suppose $k > 0$ and $a_k = \Delta d_k$. Let $p(x) = \Delta T_n(2x-1) - Q_n(x)$, and suppose that p is not identically equal to 0. Then $p(x_i) \ge 0$ for even values of i, and $p(x_i) \le 0$ for odd i. Hence p has a zero in every closed interval $[x_i, x_{i-1}]$, $i = 1, 2, \ldots, n$. If $p(x_i) = 0$ for some $1 < i < n$, it must be a multiple root, since p does not change sign at x_i. This shows that there are exactly n zeros in $[0,\ 1]$, if each root is counted up to its multiplicity. In addition, if $x_n = 0$ is a root, it must be single. Again, let $y_1, y_2, \ldots, y_n$ be the roots, and consider the coefficient of x^k in $p(x)$,

$$0 = (-1)^{n-k}(\Delta d_n - a_n)\sum_{1\le i_1<\cdots<i_{n-k}\le n} y_{i_1}\ldots y_{i_{n-k}}.$$

Since $k > 0$, the sum on the right-hand side must have at least one positive term, leading to a contradiction.

Finally, it is known [5] that

$$d_k = (-1)^{n-k}\frac{n}{k+n}\binom{k+n}{2k}2^{2k}.$$

This can be shown by induction on n, basing on the recursion formula $T_n(2x-1) = = 2(2x-1)T_{n-1}(2x-1) - T_{n-2}(2x-1)$. ■

Let us add that for $k = 0$ equality can also hold for polynomials other than $Q_n(x) = \pm\Delta T_n(2x-1)$. For example, every polynomial of the form $Q_n(x) = = \pm\Delta\big(1 - xP_{n-1}(x)\big)$ will do, where $\deg P_{n-1} \le n-1$, and $0 \le P_{n-1}(x) \le 2$ on $[0,1]$.

In what follows we consider real sequences (a_k) such that the series $\sum_{k=0}^{\infty} a_k$ is absolutely convergent. As in Section 1, we introduce the generating function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad 0 \le x \le 1, \tag{2.3}$$

and the notation

$$\Delta = \sup_{0\le x\le 1} |f(x)|. \tag{2.4}$$

First we point out that a_ℓ cannot be estimated without any further restriction.

Theorem 2. *For arbitrary $\ell > 0$ and $\Delta > 0$ we have*

$$\sup\left\{ |a_\ell| : \max_{0\le x\le 1} |f(x)| = \Delta \right\} = \infty.$$

Proof. Let $f(x) = \Delta\, T_n(2x-1)$, then $|a_\ell| = \Delta \cdot 2^{2\ell} \dfrac{n}{\ell+n} \dbinom{\ell+n}{2\ell}$, which tends to infinity with n. ■

This is the reason why we set additional conditions. We formulate them in the flavor of (1.4) and (1.5).

Theorem 3. *Let h be a positive function defined on $[0,\infty)$, such that $\dfrac{h(x)}{x}$ tends nondecreasingly to a limit ϱ as $x \to \infty$; $0 < \varrho \le \infty$. Suppose $\Delta \le e^{-h(\ell)}$, and*

$$\sum_{k=n+1}^{\infty} |a_k| \le K\, e^{-h(n)}, \quad n \in \mathbb{N}, \tag{2.5}$$

with some positive constant K. Then

$$|a_\ell| \le (K+1)\, C_\ell\, \Delta \left[h^{-1}\left(\log \frac{1}{\Delta}\right)\right]^{2\ell}, \tag{2.6}$$

where $C_\ell = \dfrac{2^{2\ell}}{(2\ell)!}$. *On the other hand, for every positive* $K' < \Big(\dfrac{\varrho}{2+\varrho}\Big)^{2\ell}$ *and every sufficiently small* $\Delta > 0$ *there exists a sequence* (a_k) *such that* (2.4) *and* (2.5) *hold, and*

$$|a_\ell| \geq K'\, C_\ell\, \Delta \Big\lceil h^{-1}\Big(\log\frac{1}{\Delta}\Big)\Big\rceil^{2\ell}. \tag{2.7}$$

Theorem 4. *Suppose the conditions of Theorem 3 are met except that*

$$\sum_{k=0}^{\infty} |a_k|\, e^{h(k)} \leq K < \infty \tag{2.5$'$}$$

is satisfied instead of (2.5). *Then* (2.6) *follows. On the other hand, for every positive* $K' < \Big(\dfrac{\varrho}{2+\varrho}\Big)^{2\ell}$ *and every sufficiently small* $\Delta > 0$ *there exists a sequence* (a_k) *such that* (2.4), $(2.5')$, *and* (2.7) *hold.*

Proof of Theorems 3 and 4. Note that condition $(2.5')$ implies (2.5), for

$$\sum_{k=n+1}^{\infty} |a_k| \leq e^{-h(n)} \sum_{k=0}^{\infty} |a_k|\, e^{h(k)}.$$

Suppose (2.5) holds. Choose

$$n = \Big\lceil h^{-1}\Big(\log\frac{1}{\Delta}\Big)\Big\rceil,$$

then $e^{-h(n)} \leq \Delta$. Cutting the power series (2.3) into two at the nth term we obtain that

$$\sup_{[0,1]} \Big|\sum_{k=0}^{n} a_k x^k\Big| \leq \Delta + \sum_{k>n} |a_k| \leq \Delta + K\, e^{-h(n)} \leq (K+1)\Delta.$$

Hence Theorem 1 and inequality (2.2) immediately imply (2.6).

For the other direction choose n so that

$$2n + h(n) \leq \log\frac{K}{\Delta} < 2(n+1) + h(n+1),$$

and let $f = Q_n\Delta$, where $Q_n(x) = T_n(2x-1)$. Then (2.4) is fulfilled. For the coefficient of x^ℓ in $f(x)$ we have

$$|a_\ell| = \Delta \cdot 2^{2\ell}\, \frac{n}{\ell+n} \binom{\ell+n}{2\ell} \sim C_\ell\, \Delta\, n^{2\ell}.$$

Firstly, suppose that $\varrho < \infty$. Then $h^{-1}(x) \sim x/\varrho$ as $x \to \infty$, and

$$(2+\varrho)n \sim 2n + h(n) \sim \log\frac{1}{\Delta}$$

as $\Delta \to 0$, that is, as $n \to \infty$. Hence

$$n \sim \frac{1}{2+\varrho}\log\frac{1}{\Delta} \sim \frac{\varrho}{2+\varrho}\, h^{-1}\Big(\log\frac{1}{\Delta}\Big) \sim \frac{\varrho}{2+\varrho}\left\lceil h^{-1}\Big(\log\frac{1}{\Delta}\Big)\right\rceil .$$

Secondly, let $\varrho = \infty$. Then $h^{-1}(x)/x$ converges nonincreasingly to 0. It follows that $h^{-1}\big(h(n)+c\big) \sim n$ if $c = c(n) = o\big(h(n)\big)$. Indeed, for $c \geq 0$ we have

$$n = h^{-1}\big(h(n)\big) \leq h^{-1}\big(h(n)+c\big) \leq \frac{h^{-1}\big(h(n)\big)}{h(n)}\big(h(n)+c\big) = n\Big(1+\frac{c}{h(n)}\Big) \sim n.$$

Similarly, for $c < 0$ all inequalities hold reversed. Since

$$h(n) + 2n - \log K \leq \log\frac{1}{\Delta} \leq h(n+1) + 2(n+1) - \log K,$$

we obtain that

$$h^{-1}\big(h(n)+2n-\log K\big) \leq h^{-1}\Big(\log\frac{1}{\Delta}\Big) \leq h^{-1}\big(h(n+1)+2(n+1)-\log K\big),$$

thus

$$n \sim h^{-1}\Big(\log\frac{1}{\Delta}\Big) \sim \left\lceil h^{-1}\Big(\log\frac{1}{\Delta}\Big)\right\rceil .$$

All we have left is to show that condition $(2.5')$ is satisfied.

$$\sum_{k=m+1}^{n} |a_k|\, e^{h(k)} \leq \sum_{k=m+1}^{n} \frac{(2n)^{2k}}{(2k)!}\,\Delta\, e^{h(k)} \leq \Delta\, e^{h(n)+2n} \leq K. \quad \blacksquare$$

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