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CONVERGENCE OF FILTERED WALSH–FOURIER SERIES

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Dedicated to Prof. Ferenc Schipp on his 70th birthday and to Prof. Péter Simon on his 60th birthday

Abstract. What happens to almost everywhere convergence and L^p boundedness of Walsh-Fourier series when some coefficients are suppressed? To answer this question, we introduce a modified Walsh-Dirichlet kernel that blocks out certain frequencies, and examine the partial sums it generates.

1. Introduction

Let $\mathbf{N} := \{0, 1, 2, \ldots\}$ denote the set of *nonnegative integers* and let $\oplus$ denote addition modulo two, i.e., if $m, n \in \mathbf{N}$, then

$$m \oplus n := \begin{cases} 0 & \text{when } m+n \text{ is even,} \\ 1 & \text{when } m+n \text{ is odd.} \end{cases}$$

Each $n \in \mathbf{N}$ has a unique *binary expansion*, i.e., for each $n \in \mathbf{N}$ there exists an integer $n_k = 0$ or 1 (called the *binary coefficient* of n *of order* k) such

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that $n = \sum_{k=0}^{\infty} n_k 2^k$. The *dyadic sum* of two integers and $n = \sum_{k=0}^{\infty} n_k 2^k$ and $m = \sum_{k=0}^{\infty} m_k 2^k$ is defined by

$$n \dot{+} m = \sum_{k=0}^{\infty} (n_k \oplus m_k) 2^k.$$

This makes $(\mathbf{N}, \dot{+})$ a group.

Similarly, the set $\mathbf{G} := \{(x_0, x_1, \ldots) : x_k = 0 \text{ or } 1\}$ becomes a group (called the *dyadic group*) if for each $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ in $\mathbf{G}$ we define

$$x \dot{+} y := (x_0 \oplus y_0,\ x_1 \oplus y_1, \ldots).$$

It is well known that $(\mathbf{G}, \dot{+})$ is a compact group and that $(\mathbf{N}, \dot{+})$ is its dual group. Let $\mathbf{Q}$ represent the set *dyadic rationals*, i.e.,

$$\mathbf{Q} := \left\{ \frac{k}{2^n} : k = 0, 1, \ldots, 2^n - 1,\ n \in \mathbf{N} \right\},$$

and let $\mathbf{I} := [0, 1]$ denote the *unit interval*. Fine's map, defined by

$$\varphi(x_0, x_1, \ldots) = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}},$$

identifies $\mathbf{G}$ with the interval $[0, 1]$. Although Fine's map is not 1-1 (since dyadic rationals have two binary expansions), it is 1-1 on the subset $\varphi^{-1}(\mathbf{I} \backslash \mathbf{Q})$ of $\mathbf{G}$. (For details of everything mentioned so far, see the original source Fine [1], or the monograph [3].)

The dual group of $\mathbf{G}$ can be identified with the system $(w_n,\ n \in \mathbf{N})$ defined by

$$w_n(x) := \prod_{k=0}^{\infty} (-1)^{n_k x_k}, \quad x = (x_0, x_1, \ldots) \in \mathbf{G},$$

where the n_k's are the binary coefficients of n. In particular, $w_n(x) w_n(y) = w_n(x \dot{+} y)$ and $w_{n \dot{+} m} = w_n w_m$ for all $x, y \in \mathbf{G}$ and all $n, m \in \mathbf{N}$. The characters $\{w_n\}$ are called the *Walsh functions* because under Fines's map, each w_n gets pulled back to the classical Walsh function of order n, also denoted by w_n (e.g., see [3]). The Walsh functions of order 2^n are called *Rademacher functions*, and are denoted by $r_n := w_{2^n}$.

Denote the integral of a function f on $\mathbf{G}$ with respect to Haar measure, if it exists, by

$$\int_{\mathbf{G}} f(x)dx.$$

For each $1 \le p < \infty$, let $L^p(\mathbf{G})$ represent the collection of real-valued functions f on $\mathbf{G}$ whose pth powers, $|f|^p$, are integrable with respect to Haar measure and set

$$\|f\|_p := \left(\int_{\mathbf{G}} |f(x)|^p dx\right)^{1/p}.$$

Let $\mathcal{C}(\mathbf{G})$ represent all real-valued functions which are continuous on $\mathbf{G}$ and set

$$\|f\|_\infty := \sup_{x \in \mathbf{G}} |f(x)|.$$

The *dyadic convolution* of two functions f and g on $\mathbf{G}$ is defined by

$$(f * g)(x) := \int_{\mathbf{G}} f(x \dot{+} y) g(y) dy.$$

It is well known that convolution makes $L^1(\mathbf{G})$ a commutative Banach algebra, and that if $f \in L^p(\mathbf{G})$ for some $1 \le p \le \infty$ and $g \in L^1(\mathbf{G})$, then

$$\|f * g\|_p \le \|f\|_p \|g\|_1. \tag{1}$$

(See, for example, [3], page 24.)

A *Walsh series* with *coefficients* a_k is an infinite series of the form $S := := \sum_{k=0}^{\infty} a_k w_k$. The partial sums of S of order n are defined by $S_n := \sum_{k=0}^{n-1} a_k w_k(x)$. The *Walsh-Fourier* coefficients of an integrable function on $\mathbf{G}$ are defined by

$$\widehat{f}(k) := \int_{\mathbf{G}} f(x) w_k(x) dx, \qquad k \in \mathbf{N}.$$

A Walsh series $S := \sum_{k=0}^{\infty} a_k w_k$ is called the *Walsh-Fourier series* of an integrable function f (notation: $Sf := S$) if $a_k = \widehat{f}(k)$ for all $k \in \mathbf{N}$. It is well known, and easy to see, that the *Walsh-Dirichlet kernel* $D := \sum_{k=0}^{\infty} w_k$ satisfies $(Sf)_n(x) = = (D_n * f)(x)$ for $x \in \mathbf{G}$ and $n \in \mathbf{N}$. Define

$$I_n(k) := \left\{ (x_0, x_1, \dots) \in \mathbf{G} : \sum_{j=0}^{n-1} \frac{x_k}{2^{k+1}} = \frac{k}{2^n} \right\}, \quad n \in \mathbf{N}, \quad 0 \le k < 2^n,$$

and observe that for each $n \in \mathbf{N}$, the values of the Rademacher function r_n are given by

$$r_n(x) := w_{2^n}(x) = \begin{cases} 1 & x \in I_{n+1}(2k), \\ -1 & x \in I_{n+1}(2k+1) \end{cases} \tag{2}$$

for $k = 0, 1, \dots, 2^{n+1} - 1$. Thus $1 + r_n(x) = 2$ for $x \in I_{n+1}(2k)$ and zero elsewhere.

The $I_n(k)$'s are called *intervals* in $\mathbf{G}$ because Fine's map carries them to dyadic intervals in $\mathbf{I}$:

$$\varphi(I_n(k)) = [k2^{-n}, (k+1)2^{-n}).$$

The Haar measure of each interval $I_n(k)$ is exactly 2^{-n}. Intervals in $\mathbf{G}$ play a special role for Walsh functions. The reasons for this are three fold. 1) Each $I_n(k)$ is compact in $\mathbf{G}$; 2) for each n, $\mathbf{G}$ is the disjoint union of $I_n(k)$ for $k = 0, 1, \dots, 2^n - 1$; and 3) the Walsh functions are constant on the $I_n(k)$'s with values ± 1. Specifically, if $n, m \in \mathbf{N}$ satisfy $2^n \le m < 2^{n+1}$, then w_m is constant on $I_{n+1}(k)$ for all k, and changes signs exactly once on each $I_n(k)$. In particular, a convergent Walsh series usually does so because of cancellation within dyadic intervals.

The dyadic partial sums of the Walsh-Dirichlet kernel satisfy

$$D_{2^n}(x) = \begin{cases} 2^n & x \in I_n(0), \\ 0 & \text{otherwise.} \end{cases}$$

Since $(Sf)_2^n = D_{2^n} * f$, we have by (1) that if $f \in L^p(\mathbf{G})$, for some $1 \le p \le \infty$, and if $\epsilon_k = 1$ for all $k \in \mathbf{N}$, then the dyadic partial sums of $Sf := \sum_{k=0}^{\infty} \epsilon_k \widehat{f}(k) w_k$ satisfy

$$\sup_{n \in \mathbf{N}} \left\| \sum_{k=0}^{2^n - 1} \epsilon_k \widehat{f}(k) w_k \right\|_p \le \|f\|_p, \tag{3}$$

and it is well-known that if $f \in L^p(\mathbf{G})$ for some $p > 1$, then the full partial sums of Sf satisfy

$$\lim_{n\to\infty} \sum_{k=0}^{n} \epsilon_k \widehat{f}(k) w_k = h(x) \quad \text{almost everywhere on } \mathbf{G}, \tag{4}$$

where $h = f$ (see [3], p. 142).

Watari [4] examined random Walsh-Fourier, i.e., $\sum\limits_{k=0}^{\infty} \epsilon_k \widehat{f}(k) w_k$, where $\epsilon = \pm 1$. He showed that if $f \in L^2(\mathbf{G})$ and if

$$\sum_{k=0}^{\infty} |\widehat{f}(k)|^2 \log^{1+\epsilon} k < \infty \tag{5}$$

for some $\varepsilon > 0$, then there is a continuous function h such that $\sum\limits_{k=0}^{\infty} \epsilon_k \widehat{f}(k) w_k = h(x)$ uniformly on $\mathbf{G}$ for almost all sign changes $\epsilon_k = \pm 1$. In particular, (3) and (4) hold not only when all the ϵ_k's are 1 but also can hold when some of the ϵ_k's are -1.

What happens to filtered Walsh-Fourier series? That is, what happens if some of the ϵ_k's are zero? In view of the cancellation that takes place within dyadic intervals, it seems unlikely that either (3) or (4) will still hold in this case. Nevertheless, we will show that they both do hold in a wide variety of cases. Here are two sample results (see (6) and (7), and Theorems 1 and 4 below):

1) For all $f \in L^p(\mathbf{G})$ for $p > 1$, a condition much more general than (5), if $\epsilon_k = 1$ for all even k and $\epsilon_k = 0$ for all odd k, or vice versa, then (3) and (4) hold.
2) We will also obtain the same results when $\epsilon_k = 0$ on increasingly larger intervals, e.g., $\epsilon_k = 1$ for $k = 0,\ 1,\ 2^{10},\ 2^{10}+1,\ 2^{10^2},\ 2^{10^2}+1,\ 2^{10^2}+2^{10},\ 2^{10^2}+2^{10}+1,\ 2^{10^3},\dots$, and $\epsilon_k = 0$ otherwise.

2. Filtered Walsh-Fourier series

For each strictly increasing sequence of nonnegative integers $\mathbf{m} = (m_1, m_2, \dots)$, define the *modified Walsh-Dirichlet kernel* by

$$W^{(\mathbf{m})} := \sum_{k=0}^{\infty} \epsilon_k^{(\mathbf{m})} w_k,$$

where

$$\epsilon_k^{(\mathbf{m})} := \begin{cases} 1 & \text{there exist } j \in \mathbf{N} \text{ and } \eta_\ell\text{'s in } \{0,1\} \text{ such that } k = \sum_{\ell=1}^{j} \eta_\ell 2^{m_\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $m_j = j$, then

$$W^{(\mathbf{m})} = \sum_{k=0}^{\infty} w_{2k}, \tag{6}$$

and if

$$m_j = \begin{cases} 0 & j = 1, \\ 10^{j-1} & \text{for } j \geq 2, \end{cases}$$

then

(7)

$$W^{(\mathbf{m})} = w_0 + w_1 + w_{2^{10}} + w_{2^{10}+1} + w_{2^{10^2}} + \sum_{j \in \{1, 2^{10}, 2^{10}+1\}} w_{2^{10^2}+j} + w_{2^{10^3}} + \ldots.$$

Given a Walsh series $S = \sum_{k=0}^{\infty} a_k w_k$, define the **m**-*filtering* of S by

$$\widetilde{S} := \widetilde{S}^{(\mathbf{m})} := \sum_{k=0}^{\infty} \epsilon_k^{(\mathbf{m})} a_k w_k.$$

If $f \in L^1(\mathbf{G})$,then it is easy to check that the *filtered Walsh-Fourier series* $\widetilde{Sf}$ satisfies

$$(\widetilde{Sf})_n(x) = (f * W_n^{(\mathbf{m})})(x) := \int_{\mathbf{G}} f(x \dot{+} t) \sum_{k=0}^{n-1} \epsilon_k^{(\mathbf{m})} w_k(t) dt. \tag{8}$$

Our first result shows that a filtered Walsh-Fourier series always satisfies (3).

Theorem 1. *Suppose that $1 \leq p \leq \infty$ and that $f \in L^p(\mathbf{G})$. If $\mathbf{m}$ is a strictly increasing sequence of nonnegative integers, then the $\mathbf{m}$-filtering of the Walsh-Fourier series of f satisfies*

$$\sup_{n \in \mathbf{N}} \|(\widetilde{Sf})_{2^n}\|_p \leq \|f\|_p.$$

Proof. Define a sequence $q_j \in \{0,1\}$ by

$$q_j := \begin{cases} 1 & \text{if } j = m_k \text{ for some } k, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$F_n := \prod_{k=0}^{n-1} (1 + r_k)^{q_k}, \qquad n \in \mathbf{N}. \tag{9}$$

If we expand this product, using the definition of dyadic addition and the w_k's, we see that

$$F_n = (1 + r_{m_0})(1 + r_{m_1}) \cdots (1 + r_{m_{n-1}}) = \sum_{k=0}^{2^{m_n}-1} \epsilon_k^{(\mathbf{m})} w_k = W_{2^{m_n}}^{(\mathbf{m})}.$$

In particular, (2) implies that $W_{2^{m_n}}^{(\mathbf{m})} = F_n \geq 0$ everywhere on $\mathbf{G}$. Since F_n always includes the constant function $w_0 \equiv 1$, we also have, by orthogonality, that $\|W_{2^{m_n}}\|_1 = \sum\limits_{k=0}^{2^{m_n}-1} \epsilon_k^{(\mathbf{m})} \int\limits_{\mathbf{G}} w_k(x)dx = 1$ for all $n \in \mathbf{N}$. Because of the gaps, however, it is also clear that if $m_n < N \leq m_{n+1}$, then $\widetilde{S}_{2^N} = \widetilde{S}_{2^{m_n}}$. We conclude by (8) and (1) that

$$\sup_{N \in \mathbf{N}} \|(\widetilde{Sf})_{2^N}\|_p = \sup_{n \in \mathbf{N}} \|(\widetilde{Sf})_{2^{m_n}}\|_p = \sup_{n \in \mathbf{N}} \|f * W_{2^{m_n}}^{(\mathbf{m})}\|_p \leq$$

$$\leq \sup_{n \in \mathbf{N}} \|f\|_p \|W_{2^{m_n}}^{(\mathbf{m})}\|_1 = \|f\|_p.$$

Thus any $\mathbf{m}$-filtering of a Walsh-Fourier series satisfies (3).

To investigate (4), recall that the *Walsh-Fourier-Stieltjes* coefficients of a finite Borel measure μ on $\mathbf{G}$ are defined by $\widehat{\mu}(k) := \int\limits_{\mathbf{G}} w_k d\mu$ and the *Walsh-Fourier-Stieltjes series* of μ is defined by

$$(S\mu)(x) = \sum_{k=0}^{\infty} \widehat{\mu}(k) w_k(x).$$

The usual relationship between Walsh-Fourier-Stieltjes series and Dirichlet kernels holds. Indeed, by interchanging the order of summation, it is easy to verify that

$$(S\mu)_n(x) = \int_{\mathbf{G}} D_n(x \dot{+} t)\,d\mu(t), \qquad n \in \mathbf{N}. \tag{10}$$

The next result, which is in some sense dual to Theorem 1, shows that the dyadic partial sums of Walsh series behave like Cesàro summability (compare with Fine [2]).

Theorem 2. *Let $S = \sum_{k=0}^{\infty} a_k w_k$ be a Walsh series. Then S is the Walsh-Fourier-Stieltjes series of a finite Borel measure μ on $\mathbf{G}$ whose total variation satisfies $\|\mu\| \le M$ if and only if there exists a strictly increasing sequence of nonnegative integers $\{m_n\}$ such that*

$$\sup_{n \in \mathbf{N}} \|S_{2^{m_n}}\|_1 \le M. \tag{11}$$

Proof. Let $n \in \mathbf{N}$ and suppose that $S = S\mu$. By (10), S satisfies

$$\|S_{2^n}\|_1 = \|W_{2^n} * \mu\|_1 \le \|W_{2^n}\|_1 \|\mu\| \le 1 \cdot M = M.$$

Conversely, suppose that S is a Walsh series which satisfies (11) for some integers $0 \le m_1 < m_2 < \cdots$. Then the operator

$$T_n(f) := \int_{\mathbf{G}} S_{2^{m_n}}(x) f(x)\,dx$$

is a bounded, linear functional on $\mathcal{C}(\mathbf{G})$, with $|T_n(f)| \le \|f\|_\infty \|S_{2^{m_n}}\|_1 \le$ $\le M\|f\|_\infty$ for $n = 1, 2, \ldots$. It follows from the Banach-Alaoglu Theorem that there is a subsequence n_j such that T_{n_j} converges pointwise on $\mathcal{C}(\mathbf{G})$ to a bounded linear operator T on $\mathcal{C}(\mathbf{G})$ whose operator norm satisfies $\|T\| \le M$. Hence, by the Riesz Representation Theorem, there is a finite Borel measure μ on $\mathbf{G}$ which satisfies $\|\mu\| \le M$ such that

$$Tf = \int_{\mathbf{G}} f\,d\mu$$

for all $f \in \mathcal{C}(\mathbf{G})$. Since each Walsh function is continuous on the group, it follows from the definition of T and orthogonality that for any $k \in \mathbf{N}$,

$$\widehat{\mu}(k) := \int_{\mathbf{G}} w_k d\mu = T(w_k) = \lim_{j\to\infty} \int_{\mathbf{G}} S_{2^{m_{n_j}}}(x) w_k(x)dx = a_k.$$

Thus $S = S\mu$ as promised.

We now show (see (13) below) that a filtered Walsh-Fourier series always satisfies an averaged version of (4).

Theorem 3. *Suppose that* $\mathbf{m}$ *is a strictly increasing sequence in* $\mathbf{N}$ *and that* $f \in L^1(\mathbf{G})$. *Then there is a finite Borel measure* $\widetilde{\mu}$ *on* $\mathbf{G}$ *such that the* $\mathbf{m}$*-filtering* $\widetilde{Sf}$ *satisfies*

$$\lim_{n\to\infty} \frac{(\widetilde{Sf})_n(x)}{n} = \widetilde{\mu}(\varphi^{-1}(\{x\})), \qquad x \in \mathbf{I}\backslash\mathbf{Q}. \tag{12}$$

Moreover, if F *is the distribution function of* $\widetilde{\mu}$, *i.e., if* $F(x) := \widetilde{\mu}(\varphi^{-1}([0,x)))$ *for* $x \in \mathbf{I}$, *then*

$$\lim_{n\to\infty} \int_0^x (\widetilde{Sf})_n(t)dt = F(x) \tag{13}$$

for all $x \in \mathbf{Q}$ *and for all* x *which are points of continuity of* F.

Proof. By Theorems 1 and 2, the $\mathbf{m}$-filtering of the Walsh-Fourier series Sf is the Walsh-Fourier-Stieltjes series of some finite, Borel measure $\widetilde{\mu}$ on $\mathbf{G}$. Hence, (12) and (13) follow immediately from Theorems 3 and 7 in Fine [2].

3. Pointwise convergence of filtered Walsh-Fourier series.

If $\widetilde{Sf}$ is itself a Walsh-Fourier series of some $g \in L^1(\mathbf{G})$, then we will say that the $\mathbf{m}$-*conjugate function* of f exists, and denote it by $\widetilde{f} := g$.

The following result shows that if $1 < p < \infty$, then the $\mathbf{m}$-conjugate map, $f \mapsto \widetilde{f}$ takes $f \in L^p(\mathbf{G})$ into $L^p(\mathbf{G})$. It also proves that the $\mathbf{m}$-filtering of a Walsh-Fourier series of an $f \in L^p(\mathbf{G})$ always satisfies (3) and (4).

Theorem 4. *Suppose that* $\mathbf{m}$ *is a strictly increasing sequence in* $\mathbf{N}$ *and that* $f \in L^p(\mathbf{G})$ *for some* $p > 1$. *Then* $\widetilde{f}$ *exists and the* $\mathbf{m}$*-filtering* $\widetilde{Sf}$ *converges* *to $\widetilde{f}$ almost everywhere on* $\mathbf{G}$. *Moreover, if* $1 < p < \infty$, *then the* $\mathbf{m}$-*conjugate function* $\widetilde{f}$ *belongs to* $L^p(\mathbf{G})$ *and there is a constant* C_p, *which depends only on* p, *such that*

$$\|\widetilde{f}\|_p \leq C_p \|f\|_p. \tag{14}$$

Proof. Suppose that $1 < p < \infty$. By the proof of Theorem 1,

$$\|\widetilde{Sf}_{2^n}\|_p \leq \|f\|_p$$

for all $n \in \mathbf{N}$, i.e. $\widetilde{Sf}$ satisfies (3). Repeat the proof of Theorem 2. The only essential change is that the operators T_j are uniformly bounded on $L^{p'}(\mathbf{G})$, where p' is the index conjugate to p, so by the Riesz Representation Theorem, that there is an $h \in L^p(\mathbf{G})$ such that

$$Tf = \int_{\mathbf{G}} f(x)h(x)dx$$

for all $f \in L^{p'}(\mathbf{G})$. As before, orthogonality implies that $\widehat{h}(k) = a_k$ for $k \in \mathbf{N}$, i.e., $\widetilde{Sf}$ is the Walsh-Fourier series of some $\widetilde{f} := h \in L^p(\mathbf{G})$. Hence, $\widetilde{f}$ exists, $(\widetilde{Sf})_n := (S\widetilde{f})_n \to \widetilde{f}$ almost everywhere on $\mathbf{G}$ and in L^p norm, and

$$\|(\widetilde{Sf})_n\|_p \leq C_p \|f\|_p \tag{15}$$

for all $n \in \mathbf{N}$. (See, e.g., [3], pp. 135 and 142.) Taking the limit of (15), as $n \to \infty$, verifies (14).

Finally, if $f \in L^\infty(\mathbf{G})$, then $f \in L^p(\mathbf{G})$ for all $1 \leq p \leq \infty$. Hence by what we just proved, $\widetilde{f}$ exists and belongs to $L^p(\mathbf{G})$ for all $1 \leq p < \infty$.

It follows that if $f \in L^p(\mathbf{G})$ for some $1 < p < \infty$, then

$$g := \sum_{k=0}^{\infty} \widehat{f}(2k) w_{2k} \quad \text{and} \quad h := \sum_{k=0}^{\infty} \widehat{f}(2k+1) w_{2k+1}$$

are both $L^p(\mathbf{G})$ functions and converge almost everywhere because g is a conjugate function and $h = f - g$.

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