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FOURIER–VILENKIN SERIES AND ANALOGS OF BESOV AND SOBOLEV CLASSES

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Dedicated to professor Ferenc Schipp on his 70th birthday and to professor Péter Simon on his 60th birthday

Abstract. In this work we prove several theorems connected with embeddings of $\mathbf{P}$-adic generalized Besov spaces and Sobolev spaces in each other. The sharpness of these results in a certain sense is shown. Trigonometrical analogs of two main results were previously proved by M.K. Potapov.

1. Introduction

Let $\mathbf{P} = \{p_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that $2 \le p_n \le N$, $m_0 = 1$ and $m_n = p_1 \dots p_n$ for $n \in \mathbf{N} = \{1, 2, \dots\}$. Every number $x \in [0, 1)$ can be represented as

$$(1) \qquad x = \sum_{n=1}^{\infty} x_n/m_n, \quad x_n \in \mathbb{Z}, \quad 0 \le x_n < p_n.$$

If $x = k/m_i$, $k, i \in \mathbb{N}$, then we take extension with finite number of nonzero x_n. Every $k \in \mathbf{Z}_+ = \{0, 1, \dots\}$ can be expressed uniquely in the form

$$(2) \qquad k = \sum_{i=1}^{\infty} k_i m_{i-1}, \quad k_i \in \mathbb{Z}, \quad 0 \le k_i < p_i.$$

For $x \in [0,1)$ and $k \in \mathbb{Z}_+$, let us define $\chi_k(x)$ by the formula

$$\chi_k(x) = \exp\left(2\pi i\left(\sum_{j=1}^{\infty} x_j k_j/p_j\right)\right).$$

It is well known that the Vilenkin system $\{\chi_k(x)\}_{k=0}^{\infty}$ is an orthonormal and complete system in $L[0,1)$ (see [5, §1.5]). In the case $p_n \equiv 2$ it coincides with the Walsh system. Let by definition for $f \in L[0,1)$

$$\hat{f}(n) = \int_0^1 f(t)\overline{\chi_n(t)}\,dt, \quad n \in \mathbb{Z}_+, \quad S_n(f)(x) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k(x), \quad n \in \mathbb{N},$$

$$\Delta_n(f)(x) = S_{m_n}(f)(x) - S_{m_{n-1}}(f)(x), \quad n \in \mathbb{N}, \quad \Delta_0(f)(x) = \hat{f}(0).$$

The sum $\sum\limits_{k=0}^{n-1} \chi_k(x) =: D_n(x)$ is called the n-th Dirichlet kernel. By the generalized Paley lemma $D_{m_n}(x) = m_n X_{[0,1/m_n)}$, where $n \in \mathbb{Z}_+$ and X_E is the indicator of the set E. From this identity we deduce that

$$S_{m_n}(f)(x) = m_n \int_{I_k^n} f(t)\,dt$$

$$\text{for} \quad x \in I_k^n = [k/m_n, (k+1)/m_n), \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, m_n - 1.$$

In addition, $|D_n(x)| \le C_1 \min(n, 1/x)$ for $x \in (0,1)$ (see [5, §1.5] or [1, Ch. 4, §3]). If $\|f\|_p = \left(\int\limits_0^1 |f(t)|^p\,dt\right)^{1/p}$ is the usual norm in $L^p[0,1)$, $1 \le p < \infty$, then we have for $n \in \mathbb{Z}_+$ and $1 < p < \infty$

$$\|D_n\|_p^p \le C_1 \left(\int_0^{1/n} n^p dt + \int_{1/n}^1 t^{-p} dt\right) \le C_2 n^{p-1}. \tag{3}$$

The maximal function $M(f)$ is defined for $f \in L^1[0,1)$ by $M(f)(x) = \sup\limits_{n \in \mathbb{Z}_+} |S_{m_n}(f)(x)|$. The **P**-adic Hardy space $H(\mathbf{P}, [0,1))$ consists of functions $f \in L^1[0,1)$ such that $\|f\|_H = \|M(f)\|_1 < \infty$. If $x, y \in [0,1)$ are represented in the form (1), then $x \oplus y = z = \sum\limits_{i=1}^{\infty} z_i/m_i$, where $z_i \in \mathbb{Z}$, $0 \le z_i < p_i$

and $z_i = x_i + y_i \pmod{p_i}$. The inverse operation $\ominus$ is defined similarly. Let us introduce a modulus of continuity in $L^p[0,1)$, $1 \le p < \infty$, by the formula $\omega^*(f,t)_p = \sup\{\|f(x \ominus h) - f(x)\|_p : 0 < h < t\}$, $t \in [0,1]$. In addition, we will denote $\omega^*(f, 1/m_n)_p$ by $\omega_n(f)_p$. If $\{\omega_n\}_{n=0}^{\infty}$ is decreasing to zero, then we define $H_p^\omega = \{f \in L^p[0,1) : \omega_n(f)_p \le C\omega_n, n \in \mathbb{Z}_+\}$. Let $\mathcal{P}_n = \{f \in L[0,1) : \hat{f}(k) = 0, k \ge n\}$, $E_n(f)_p = \inf\{\|f - t_n\|_p : t_n \in \mathcal{P}_n\}$ for $n \in \mathbb{N}$. Further, we will often use A.V. Efimov's inequality [5, §10.5]

(4)
$$E_{m_n}(f)_p \le \|f - S_{m_n}(f)\|_p \le \omega_n(f)_p \le 2E_{m_n}(f)_p, \quad 1 \le p < \infty, \quad n \in \mathbb{Z}_+.$$

In a similar way we define $\omega^*(f,t)_H$, $\omega_n(f)_H$, H_H^ω and $E_n(f)_H$, and have (see [18])

$$(4') \qquad E_{m_n}(f)_H \le \|f - S_{m_n}(f)\|_H \le \omega_n(f)_H \le 2E_{m_n}(f)_H, \quad n \in \mathbb{Z}_+.$$

Let $\alpha(t)$ be a measurable and positive function on $(0,1)$ such that $\alpha \in L[\delta,1)$ for all $0 < \delta < 1$. Then we can introduce two sequences $\{\beta(n)\}_{n=0}^{\infty}$, $\{\mu(n)\}_{n=1}^{\infty}$ by formulas $\beta(n) = \int\limits_{1/(n+1)}^{1} \alpha(t)dt$ for $n \in \mathbb{N}$, $\beta(0) = 1$, and $\mu(n) = \int\limits_{1/m_n}^{1/m_{n-1}} \alpha(t)dt$, $n \in \mathbb{N}$. If $f \in L^p[0,1)$, $1 \le p, \theta < \infty$ and the series $\sum\limits_{n=1}^{\infty} \beta^{1/\theta}(n)\hat{f}(n)\chi_n(x)$ is Fourier–Vilenkin series of a function $\varphi(f) = \varphi(\theta, f) \in L^p[0,1)$, then $f \in W(\theta,p,\alpha) = W(\theta,p,\alpha,\mathbf{P})$. Similarly, if $f \in H(\mathbf{P},[0,1))$ and the series $\sum\limits_{n=1}^{\infty} \beta^{1/\theta}(m_n - 1) \sum\limits_{k=m_n}^{m_{n+1}-1} \hat{f}(k)\chi_k(x)$ is the Fourier–Vilenkin series of a function $\psi(f) \in H(\mathbf{P},[0,1))$, then $f \in W(\theta,H,\alpha)$. By definition, for $p, \theta \in [1,\infty)$

$$B(\theta,p,\alpha) = \left\{ f \in L^p[0,1) : I_{\theta,p,\alpha} := \left(\int_0^1 \alpha(t)(\omega^*(f,t)_p)^\theta \, dt \right)^{1/\theta} < \infty \right\}.$$

The quantity $I_{\theta,H,\alpha}$ and the space $B(\theta,H,\alpha)$ are introduced in a similar way.

Further we assume that for $\alpha(t)$ the δ_2-condition

$$(5) \qquad \int_{\delta/2}^{\delta} \alpha(t)dt \le C \int_{\delta}^{2\delta} \alpha(t)dt \le C \int_{\delta}^{1} \alpha(t)dt, \quad \delta \in (0,1/2),\ C > 0,$$

is satisfied. If $p_n \le N \le 2^a$, $n \in \mathbb{N}$, then it is easy to see that the δ_2-condition (5) implies the inequality

$$\mu(n+1) \le \int\limits_{2^{-a}/m_n}^{1/m_n} \alpha(t)dt \le \sum_{i=1}^{a} C^i \int\limits_{1/m_n}^{2/m_n} \alpha(t)dt \le A(C)\mu(n). \tag{6}$$

Finally, from (6) one can deduce that for $m_k \le n < m_{k+1}$, $k \in \mathbb{Z}_+$,

$$\begin{aligned} \beta(n) < \beta(m_{k+1}) \le (A^k + \ldots + 1)\mu(1) \le C_1 A^k \le \\ \le C_1 2^{k\gamma} \le C_1 m_k^\gamma \le C_1 n^\gamma, \quad \gamma = \log_2 A. \end{aligned} \tag{7}$$

We will consider several classes of generalized monotone sequences. If $\lim\limits_{n\to\infty} a_n = 0$ and $a_n n^{-\tau}$ decreases for some $\tau \ge 0$ and for all $n \ge 1$, then $\{a_n\}_{n=0}^\infty$ is called quasi-monotone ($\{a_n\}_{n=0}^\infty \in A_\tau$). If $\lim\limits_{n\to\infty} a_n = 0$ and $a_n n^\tau$ increases for some $\tau > 0$ and for all $n \in \mathbb{Z}_+$, then $\{a_n\}_{n=0}^\infty \in A_{-\tau}$. The classes A_τ were introduced by O. Szász [17] and A.A. Konyushkov [8] in the case $\tau \ge 0$ and by G.K. Lebed' [9] in the case $\tau < 0$. If $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{k=n}^{\infty} |a_k - a_{k+1}| \le C a_n$ for all $n \in \mathbb{Z}_+$, then $\{a_n\}_{n=0}^\infty$ belongs to the class $RBVS$ introduced by L. Leindler [10]. It is easy to see that condition $\{a_n\}_{n=0}^\infty \in RBVS$ implies the inequality $a_n \le C a_m$ for all $m \le n$.

The trigonometric counterparts of $B(\theta, p, \alpha)$ and $W(\theta, p, \alpha)$ are generalizations of O.V. Besov and S.L. Sobolev classes of 2π-periodic functions. These classes were studied by M.K. Potapov [12], [13]. So, in [12] he investigated embeddings between generalized Besov and Sobolev classes while interrelations between generalized Besov classes may be found in [13]. In this paper we obtain sufficient conditions for embeddings of $B(\theta, p, \alpha)$ and $W(\theta, p, \alpha)$ in each other and show that these conditions are sharp in a certain sense. A criterion for functions with generalized monotone Fourier-Vilenkin coefficients to be in $B(\theta, p, t^{-r\theta-1})$ is also given. Note that δ_2-condition in the present paper replaces two conditions used by M.K. Potapov.

1. Auxiliary propositions

The first lemma has been proved by C. Watari [21] and generalizes the famous Paley theorem for the Walsh system.

Lemma 1. *1) Let* $f \in L^p[0,1)$, $1 < p < \infty$, $\hat{f}(0) = 0$ *and* $Q(f) = \left(\sum_{n=1}^{\infty} |\Delta_n(f)(x)|^2\right)^{1/2}$. *Then*

$$C_1\|Q(f)\|_p \leq \|f\|_p \leq C_2\|Q(f)\|_p.$$

2) If for $p \in (1,\infty)$ *and for the series* $\sum_{n=1}^{\infty} a_n\chi_n(x)$ *it is true that*

$$I_p = \left\| \left(\sum_{n=1}^{\infty} \left| \sum_{j=m_{n-1}}^{m_n - 1} a_j \chi_j(x) \right|^2 \right)^{1/2} \right\|_p < \infty,$$

then this series is the Fourier–Vilenkin series of a function $f \in L^p[0,1)$. *Moreover,* $\|f\|_p \leq C_3 I_p$.

Lemma $1'$ extends Lemma 1 to the **P**-adic Hardy space corresponding to the case $p = 1$. In the dyadic case Lemma $1'$ may be found in [16, p. 101, Corollary 4].

Lemma $1'$. *If* $f \in L^1[0,1)$, $\hat{f}(0) = 0$, *then*

$$C_1\|Q(f)\|_1 \leq \|f\|_H \leq C_2\|Q(f)\|_1.$$

The following Lemma is an analog of the Marcinkiewicz theorem on multiplicators.

Lemma 2 ([3]). *If* $\{\lambda_k\}_{k=0}^{\infty} \subset \mathbb{C}$ *and there exists* $M > 0$ *with the property*

$$1) \quad |\lambda_n| \leq M, \quad 2) \quad \sum_{k=m_n}^{m_{n+1}-1} |\lambda_k - \lambda_{k+1}| \leq M, \quad n \in \mathbb{Z}_+,$$

then for every function $f \in L^p[0,1)$, $1 < p < \infty$, *the series* $\sum_{k=0}^{\infty} \lambda_k \hat{f}(k)\chi_k(x)$ *is the Fourier-Vilenkin series of a function* $f_\lambda \in L^p[0,1)$. *Moreover,*

$$\|f_\lambda\|_p \leq C(p,N)\|f\|_p.$$

Corollary 1. *Set* $\lambda_k = (\beta(k)/\beta(m_{n-1}-1))^{1/\theta}$ *and* $\gamma_k = (\beta(m_{n-1}-1)/\beta(k))^{1/\theta}$ *for* $m_{n-1} \le k < m_n$, $n \in \mathbb{N}$ *with* λ_0, γ_0 *arbitrary. Then the sequences* $\{\lambda_k\}_{k=0}^\infty$ *and* $\{\gamma_k\}_{k=0}^\infty$ *satisfy the conditions of Lemma 2. In particular, functions* $\varphi(f)$ *and* $\psi(f)$ *belong to* $L^p[0,1)$, $1 < p < \infty$, *simultaneously.*

Proof. Since $\alpha(t) > 0$ and $\{\beta(k)\}_{k=1}^\infty$ increases, we see that $\{\lambda_k\}_{k=0}^\infty$ increases and $\{\gamma_k\}_{k=0}^\infty$ increases in every interval of the form $[m_{n-1}, m_n)$, $n \in \mathbb{N}$. The boundedness of $\{\lambda_k\}_{k=0}^\infty$ follows from the δ_2-condition, while the boundedness of $\{\gamma_k\}_{k=0}^\infty$ is evident. The boundedness and monotonicity imply the fulfilment of property 2) in Lemma 2. The Corollary is proved.

There are different forms of Minkowski inequality in the spaces L^p and l^p. The two following statements will be used later.

Lemma 3 ([14]). *Let* $1 \le p < \infty$, $a_{nk} \ge 0$, $n, k \in \mathbb{N}$. *Then the inequalities*

$$\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} a_{nk}\right)^p\right)^{1/p} \le \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} a_{nk}^p\right)^{1/p}, \tag{8}$$

$$\left(\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty} a_{nk}\right)^p\right)^{1/p} \le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{nk}^p\right)^{1/p}, \tag{9}$$

are valid.

Lemma 4 ([4]). *Let* $\mathbf{g} = \{g_k\}_{k=1}^\infty$, *where* $g_k \in L^p[0,1)$, $k \in \mathbb{N}$, *and*

$$\|\mathbf{g}\|_{L^p(l^q)} = \left\|\left(\sum_{k=1}^{\infty} |g_k|^q\right)^{1/q}\right\|_p, \quad \|\mathbf{g}\|_{l^q(L^p)} = \left(\sum_{k=1}^{\infty} \|g_k\|_p^q\right)^{1/q}.$$

Then the inequality $\|\mathbf{g}\|_{L^p(l^2)} \ge \|\mathbf{g}\|_{l^2(L^p)}$ *is valid for* $1 < p \le 2$. *If* $p \ge 2$, *then we have*

$$\|\mathbf{g}\|_{L^p(l^2)} \le \|\mathbf{g}\|_{l^2(L^p)}, \quad \|\Delta(f)\|_{L^p(l^p)} \le \|f\|_p, \quad \Delta(f) = \{\Delta_n(f)\}_{n=1}^\infty.$$

Remark 1. The last inequality of Lemma 4 is proved in [4] for the Walsh system with help of interpolation and its proof is translated to the case of an arbitrary system $\{\chi_n\}_{n=0}^\infty$ of bounded type.

Lemma 5. *Let* $\{\varphi_n\}_{n=0}^\infty$ *be a subsystem of* $\{\chi_k\}_{k=0}^\infty$ *such that* $\varphi_n = \chi_{k_n}$, $m_n \le k_n < m_{n+1}$ *and* $\sum\limits_{n=0}^{\infty} |a_n|^2 < \infty$. *Then the series* $\sum\limits_{n=0}^{\infty} a_n\varphi_n(x)$ *converges* *in every* $L^p[0,1)$, $1 \le p < \infty$, *to a function* f *and the following two double inequalities are valid:*

$$C_1 \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \le \|f\|_p \le C_2 \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}, \tag{10}$$

$$C_1 \left(\sum_{n=m_k}^{\infty} |a_n|^2 \right)^{1/2} \le \omega_k(f)_p \le 2C_2 \left(\sum_{n=m_k}^{\infty} |a_n|^2 \right)^{1/2}, \quad k \in \mathbb{N}. \tag{11}$$

Proof. The inequality (10) has been proved by N.Ya. Vilenkin [19]. According to (10) and (4) we have

$$\omega_k(f)_p \le 2\|f - S_{m_k}(f)\|_p \le 2C_2 \left(\sum_{n=m_k}^{\infty} |a_n|^2 \right)^{1/2}.$$

The left inequality in (11) is obtained in a similar way. The lemma is proved.

Lemma 6. *Let* $1 < p < \infty$, $f \in L^p[0,1)$ *and either* $\{\hat{f}(n)\}_{n=0}^{\infty} \in A_\tau$, $\tau \in \mathbb{R}$, *or* $\{\hat{f}(n)\}_{n=0}^{\infty} \in RBVS$. *Then*

$$C_1 \sum_{i=m_{n+1}}^{\infty} |\hat{f}(i)|^p i^{p-2} \le \omega_n^p(f)_p \le$$

$$\le C_2 \left(m_n^{p-1} |\hat{f}(m_n)|^p + \sum_{i=m_n}^{\infty} |\hat{f}(i)| i^{p-2} \right), \quad n \in \mathbb{N}, \tag{12}$$

(13)

$$C_3 \left(|\hat{f}(0)|^p + \sum_{i=1}^{\infty} |\hat{f}(i)|^p i^{p-2} \right) \le \|f\|_p^p \le C_4 \left(|\hat{f}(0)|^p + \sum_{i=1}^{\infty} |\hat{f}(i)|^p i^{p-2} \right).$$

Proof. The right inequality in (12) has been proved by N.Yu. Agafonova [2]. If $1 < p \le 2$, then the left inequality (12) follows from the famous Paley theorem (see [7, Theorem [6.3.2]]). If $p \ge 2$, then by Lemma 4 we have

$\|f-S_{m_n}(f)\|_p^p \geq \sum\limits_{k=n+1}^{\infty}\|\Delta_k\|_p^p$. From conditions $\{\hat{f}(n)\}_{n=0}^\infty \in A_\tau$, $\tau \geq 0$, or $\{\hat{f}(n)\}_{n=0}^\infty \in RBVS$ we deduce that ($k\in\mathbb{N}$)

$$\|\Delta_k(f)\|_p^p \geq \int\limits_0^{1/m_k} |\Delta_k(f)(x)|^p\,dx =$$

$$= \int\limits_0^{1/m_k}\left|\sum_{i=m_{k-1}}^{m_k-1}\hat{f}(i)\right|^p dx \geq C_5 m_k^{p-1}|\hat{f}(m_k)|^p.$$

Summing these inequalities over k from $n+1$ to ∞, we obtain

$$\|f-S_{m_n}(f)\|_p^p \geq C_5\sum_{k=n+1}^{\infty} m_k^{p-1}|\hat{f}(m_k)|^p \geq C_6\sum_{i=m_{n+1}}^{\infty}|\hat{f}(i)|^p i^{p-2}.$$

For $\{\hat{f}(n)\}_{n=0}^\infty \in A_\tau$, $\tau<0$, we have similarly $\|\Delta_k(f)\|_p^p \geq C_7 m_k^{p-1}|\hat{f}(m_{k-1})|^p$ and $\|f-S_{m_n}(f)\|_p^p \geq C_8\sum\limits_{i=m_n}^{\infty}|\hat{f}(i)|^p i^{p-2}$. Since $|\hat{f}(i)| \leq \|f\|_p$, $i\in\mathbb{Z}_+$, $p \in \in [1,\infty)$, the inequality (13) is obtained in a similar way. The lemma is proved.

Lemma 7. *Let $1\leq p,\ \theta<\infty$, $f\in L^p[0,1)$. Then for $n,q\in\mathbb{Z}_+$, $n<q$, the inequality*

$$\sum_{k=n+1}^{q}\mu(k)E_{m_k}^\theta(f)_p \leq \int\limits_{1/m_q}^{1/m_n}\alpha(t)(\omega^*(f,t)_p)^\theta\,dt \leq C_1\sum_{k=n}^{q-1}\mu(k)E_{m_k}^\theta(f)_p \tag{14}$$

holds. This statement is also valid for $E_n(f)_H$ and $\omega^*(f,t)_H$.

Proof. By (4) and by the fact that $\omega^*(f,t)_p$ increasing we obtain

$$\mu(k)E_{m_k}^\theta(f)_p \leq \int\limits_{1/m_k}^{1/m_{k-1}}\alpha(t)(\omega^*(f,1/m_k)_p)^\theta\,dt \leq \int\limits_{1/m_k}^{1/m_{k-1}}\alpha(t)(\omega^*(f,t)_p)^\theta\,dt.$$

Summing these inequalities over k from $n+1$ to q yields the left inequality from (14). Using (4) and (6), we have for all $k\geq 0$

$$\mu(k)E_{m_k}^\theta(f)_p \geq C_2\mu(k+1)(\omega^*(f,1/m_k)_p)^\theta \geq C_2\int\limits_{1/m_{k+1}}^{1/m_k}\alpha(t)(\omega^*(f,t)_p)^\theta dt.$$

Summing these inequalities over k from n to $q-1$ we establish the right inequality from (14). The lemma is proved.

Corollary 2. *If the conditions of Lemma 7 are valid, then*

$$\sum_{k=1}^{\infty}\mu(k)E^{\theta}_{m_k}(f)_p \le \int_0^1 \alpha(t)(\omega^*(f,t)_p)^{\theta}\,dt \le C_1\sum_{k=0}^{\infty}\mu(k)E^{\theta}_{m_k}(f)_p,$$

$$2^{-\theta}\sum_{k=1}^{\infty}\mu(k)\omega^{\theta}_k(f)_p \le \int_0^1 \alpha(t)(\omega^*(f,t)_p)^{\theta}\,dt \le C_1\sum_{k=0}^{\infty}\mu(k)\omega^{\theta}_k(f)_p.$$

Similar results are valid for $E_n(f)_H$ and $\omega^*(f,t)_H$.

Lemma 8. *1) Let $n\in\mathbb{N}$, $\tau>0$, $1<p<\infty$. Then*

$$\left\|\sum_{k=0}^{n-1}k^{\tau}a_k\chi_k(x)\right\|_p \le C(p)n^{\tau}\left\|\sum_{k=0}^{n-1}a_k\chi_k(x)\right\|_p,$$

$$\left\|\sum_{k=0}^{n-1}k^{\tau}a_k\chi_k(x)\right\|_H \le C(p)n^{\tau}\left\|\sum_{k=0}^{n-1}a_k\chi_k(x)\right\|_H.$$

2) Let $n\in\mathbb{N}$, $\tau>0$, $1<p<\infty$, $i\in[m_n,m_{n+1})$. Then

$$\left\|\sum_{k=m_n}^{i}k^{-\tau}\chi_k(x)\right\|_p \le C(p)m_n^{1-1/p-\tau}.$$

Proof. 1) Both inequalities may be proved by the method of [20]. In the case $1<p<\infty$ the proof is simpler. Set $t_n=\sum\limits_{k=0}^{n-1}a_k\chi_k$. By analog of M.Riesz theorem $\|S_n(f)\|_p\le C_1(p)\|f\|_p$ (see [16, §3.3, Corollary 6] in the dyadic case) and summation by parts we find that

$$\left\|\sum_{k=0}^{n-1}k^{\tau}a_k\chi_k\right\|_p \le \sum_{k=0}^{n-2}((k+1)^{\tau}-k^{\tau})\|S_{k+1}(t_n)\|_p+(n-1)^{\tau}\|S_n(t_n)\|_p \le$$
$$\le C_2(p)n^{\tau}\|t_n\|_p.$$

2) Using (3), we obtain $\|D_i - D_{m_n}\|_p \le C_3 m_n^{1-1/p}$ for $i \in [m_n, m_{n+1}]$ and $1 < p < \infty$. Summation by parts yields

$$\left\| \sum_{k=m_n}^{i} k^{-\tau}\chi_k \right\|_p \le \sum_{k=m_n}^{i-1} (k^{-\tau} - (k+1)^{-\tau})\|D_{k+1} - D_{m_n}\|_p +$$

$$+ i^{-\tau}\|D_{i+1} - D_{m_n}\|_p \le C_3 m_n^{1-1/p} m_n^{-\tau}.$$

The lemma is proved.

2. Embeddings between generalized Besov and Sobolev classes

Theorem 1.

1) Let $1 < p < \infty$, $\theta = \min(2, p)$, $f \in B(\theta, p, \alpha)$. Then $f \in W(\theta, p, \alpha)$ and

$$\|\varphi(f)\|_p \le C(I_{\theta,p,\alpha}(f) + E_1(f)_p) \le C(I_{\theta,p,\alpha}(f) + \|f\|_p).$$

2) If $f \in B(1, H, \alpha)$, then $f \in W(1, H, \alpha)$ and $\|\psi(f)\|_H \le C I_{1,H,\alpha}$.

Proof. 1) Remember that $\psi(f) = \sum_{n=1}^{\infty} \beta^{1/\theta}(m_n - 1)\Delta_{n+1}(f)(x)$. Set $\Delta_n(x) := \Delta_n(f)(x)$. Since $\theta = p$ for $1 < p \le 2$ and $\beta(m_n - 1) = \sum_{\nu=1}^{n} \mu(\nu)$, we obtain

$$S_1(x) = \left(\sum_{n=1}^{\infty} \beta^{2/p}(m_n - 1)|\Delta_{n+1}(x)|^2 \right)^{p/2} \le \sum_{\nu=1}^{\infty} \mu(\nu) \left(\sum_{n=\nu}^{\infty} |\Delta_{n+1}(x)|^2 \right)^{p/2}$$

according to (8). From Lemma 1 we deduce that

$$J_1 := \int_0^1 S_1(x)dx \le \sum_{\nu=1}^{\infty} \mu(\nu) \int_0^1 \left(\sum_{n=\nu+1}^{\infty} |\Delta_n(x)|^2 \right)^{p/2} dx \le \tag{15}$$

$$\le C_1 \sum_{\nu=1}^{\infty} \mu(\nu)\|f - S_{m_\nu}(f)\|_p^p.$$

Using Corollary 2, (4) and Lemma 1, we find that $J_1 \le C_2 \int\limits_0^1 \alpha(t)(\omega^*(f,t)_p)^p\,dt$ and $\psi(f) \in L^p[0,1)$. If $2 \le p < \infty$, then $\theta = 2$ and

$$J_2 = \left\{ \int\limits_0^1 \left(\sum_{n=1}^{\infty} \beta(m_n - 1)|\Delta_{n+1}(x)|^2 \right)^{p/2} dx \right\}^{2/p} =$$

$$= \left\{ \int\limits_0^1 \left(\sum_{n=1}^{\infty} \sum_{\nu=1}^{n} \mu(\nu)|\Delta_{n+1}(x)|^2 \right)^{p/2} dx \right\}^{2/p} =$$

$$= \left\{ \int\limits_0^1 \left(\sum_{\nu=1}^{\infty} \mu(\nu) \sum_{n=\nu+1}^{\infty} |\Delta_n(x)|^2 \right)^{p/2} dx \right\}^{2/p}.$$

Applying the triangle inequality in $L_{p/2}[0,1)$, $p \ge 2$, Lemma 1 and Corollary 2, we obtain

$$J_2 \le \sum_{\nu=1}^{\infty} \mu(\nu) \left\| \sum_{n=\nu+1}^{\infty} |\Delta_n|^2 \right\|_{p/2} \le C_3 \sum_{\nu=1}^{\infty} \mu(\nu) E^2_{m_\nu}(f)_p \le$$

$$\le C_3 \int\limits_0^1 \alpha(t)(\omega^*(f,t)_p)^2\,dt.$$

Thus, the function $\psi(f)$ belongs to $L^p[0,1)$ and $\|\psi(f)\|_p \le C_4 I_{\theta,p,\alpha}$. By Corollary 1 and inequalities $|\hat{f}(k)| \le E_k(f)_p$, $1 \le k < m_1$, we conclude that $\varphi(f)$ belongs to $L^p[0,1)$ and $\|\varphi(f)\|_p \le C_5(I_{\theta,p,\alpha} + E_1(f)_p)$.

2) As in 1) (see (15)) we have, due to Lemma $1'$

$$J_1 := \int\limits_0^1 \left(\sum_{n=1}^{\infty} \beta^2(m_n - 1)|\Delta_{n+1}|^2 \right)^{1/2} dx \le$$

$$\le C_1 \sum_{\nu=1}^{\infty} \mu(\nu) \int\limits_0^1 \left(\sum_{n=\nu+1}^{\infty} |\Delta_n(x)|^2 \right)^{1/2} dx \le C_1 \sum_{\nu=1}^{\infty} \mu(\nu) \|f - S_{m_\nu}\|_H.$$

Using $(4')$, Lemma $1'$ and Corollary 2, we obtain that $\psi(f) \in H(\mathbf{P}, [0,1))$ and $\|\psi(f)\|_H \le C_6 I_{1,H,\alpha}$. The theorem is proved.

Theorem 2.
1) Let $1 < p < \infty$, $\theta = \max(2, p)$, $f \in W(\theta, p, \alpha)$. *Then* $f \in B(\theta, p, \alpha)$ *and*

$$\left(\int_0^1 \alpha(t)(\omega^*(f,t)_p)^\theta \, dt\right)^{1/\theta} \le C(\|\varphi(f)\|_p + \|f\|_p).$$

2) Let $f \in W(2, H, \alpha)$. *Then* $f \in B(2, H, \alpha)$ *and*

$$\left(\int_0^1 \alpha(t)(\omega^*(f,t)_H)^2 \, dt\right)^{1/2} \le C(\|\psi(f)\|_H + \|f\|_H).$$

Proof. 1) Set $J = \sum_{k=1}^{\infty} \mu(k) E^\theta_{m_k}(f)_p$. Using Lemma 1 and (4), we find that

$$J \le C_1(p) \sum_{k=1}^{\infty} \mu(k) \left(\int_0^1 \left(\sum_{n=k+1}^{\infty} |\Delta_n(x)|^2\right)^{p/2} dx\right)^{\theta/p}.$$

In the case $2 \le p < \infty$ ($\theta = p$) by (9) we have

$$J \le C_1 \sum_{k=1}^{\infty} \mu(k) \int_0^1 \left(\sum_{n=k+1}^{\infty} |\Delta_n(x)|^2\right)^{p/2} dx =$$

$$= C_1 \int_0^1 \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \mu^{2/p}(k) |\Delta_{n+1}(x)|^2\right)^{p/2} dx \le$$

$$\le C_1 \int_0^1 \left(\sum_{k=1}^{\infty} |\Delta_n(x)|^2 \left\{\sum_{k=1}^{n} \mu(k)\right\}^{2/p}\right)^{p/2} dx = \tag{16}$$

$$= C_1 \int_0^1 \left(\sum_{n=1}^{\infty} |\Delta_{n+1}(x)|^2 \beta^{2/p}(m_n - 1)\right)^{p/2} dx.$$

In the case $1 < p \le 2$ we use the converse of the triangle inequality

$$\|f + g\|_{p/2} \ge \|f\|_{p/2} + \|g\|_{p/2}, \quad 0 < p/2 \le 1, \quad f, g \ge 0$$

and change of the summation order:

$$J=\sum_{k=1}^{\infty}\mu(k)E^2_{m_k}(f)_p\le C_2\sum_{k=1}^{\infty}\mu(k)\left(\int_0^1\left(\sum_{n=k+1}^{\infty}|\Delta_n(x)|^2\right)^{p/2}dx\right)^{2/p}=$$

$$\begin{aligned}&=C_2\sum_{k=1}^{\infty}\left(\int_0^1\left(\sum_{n=k}^{\infty}\mu(k)|\Delta_{n+1}(x)|^2\right)^{p/2}dx\right)^{2/p}\le\\ &\le C_2\left(\int_0^1\left(\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\mu(k)|\Delta_n(x)|^2\right)^{p/2}dx\right)^{2/p}=\\ &=C_2\left(\int_0^1\left(\sum_{n=1}^{\infty}|\Delta_{n+1}(x)|^2\beta(m_n-1)\right)^{p/2}dx\right)^{2/p}.\end{aligned}\tag{17}$$

From (16), (17) and Lemma 1 it follows that $J\le C_3(p)\|\psi(f)\|_p^{\theta}$. By Lemma 2 and Corollary 1 we have $\|\psi(f)\|_p\le C_4(p)\|\varphi(f)\|_p$. Applying Corollary 2 and inequality $E_1(f)_p\le\|f\|_p$, we finish the proof of 1).

2) Using Lemma 1′ we obtain similarly to (17)

$$J=\sum_{k=1}^{\infty}\mu(k)E^2_{m_k}(f)_H\le C_2\sum_{k=1}^{\infty}\mu(k)\left(\int_0^1\left(\sum_{n=k+1}^{\infty}|\Delta_n(x)|^2\right)^{1/2}dx\right)^{2}=$$

$$=C_2\left(\int_0^1\left(\sum_{n=1}^{\infty}|\Delta_{n+1}(x)|^2\beta(m_n-1)\right)^{1/2}dx\right)^{2}\le C_5\|\psi(f)\|_H^2.$$

Applying Corollary 2, we finish the proof of 2). The theorem is proved.

Corollary 3. *For $f\in L^2[0,1)$ conditions $f\in B(2,2,\alpha)$ and $f\in W(2,2,\alpha)$, are equivalent.*

Some particular cases of our results are connected with the Butzer-Wagner-Onneweer **P**-adic derivative (see [16, Appendix 0.7]). Let $\gamma>0$, $r\in\mathbb{Z}_+$, $T_r^{(\gamma)}(x)=\sum_{k=0}^{m_r-1}k^{\gamma}\chi_k(x)$, $f*g(x)=\int_0^1 f(x\ominus t)g(t)\,dt$ is the **P**-adic convolution of f and g. If for $f\in L^p[0,1)$, $1\le p<\infty$, there exists $g\in L^p[0,1)$ such that $\lim\limits_{r\to\infty}\|T_r^{(\gamma)}*f-g\|_p=0$, then function g is called the strong derivative of order γ in $L^p[0,1)$ for function f ($g=I^{(\gamma)}f$). It is easy to see that $(I^{(\gamma)}f)\hat{}(k)=k^\gamma\hat{f}(k)$ if $k\in\mathbb{Z}_+$. This definition comes from to He Zelin [20]. Since $\beta(n)=((n+1)^{pr}-1)/pr$ for $\alpha(t)=t^{-pr-1}$, $r>0$, $p\geq 1$, and $\lambda_n=(n^{pr}/((n+1)^{pr}-1))^{1/p}$ is increasing, it follows by Lemma 2 that in this case the condition $\varphi(p,f)\in L^p[0,1)$, $1<p<\infty$, is equivalent to the existence of $\eta(f)\in L^p[0,1)$ with Fourier series $\sum\limits_{n=1}^{\infty}n^r\hat{f}(n)\chi_n(x)$, that is to the existence of $I^{(r)}f\in L^p[0,1)$. Hence, the conditions $f\in W(p,p,t^{-pr-1})$ and $I^{(r)}f\in L^p[0,1)$ are also equivalent.

Corollary 4. *Let $1<p\leq 2$, $r>0$ and $f\in L^p[0,1)$ be such that $\int\limits_0^1(\omega^*(f,t)_p)^pt^{-pr-1}dt<\infty$. Then $I^{(r)}f$ exists and*

$$\|I^{(r)}f\|_p\leq C(p)\left(\left(\int\limits_0^1(\omega^*(f,t)_p)^pt^{-rp-1}dt\right)^{1/p}+\|f\|_p\right).$$

Corollary 5. *Let $p\geq 2$, $r>0$ and suppose that for $f\in L^p[0,1)$ there exists $I^{(r)}f\in L^p[0,1)$. Then $f\in B(p,p,t^{-pr-1})$ and*

$$\left(\int\limits_0^1(\omega^*(f,t)_p)^pt^{-rp-1}dt\right)^{1/p}\leq C(p)(\|I^{(r)}f\|_p+\|f\|_p).$$

Remark 2. Using Corollary 2, we can replace $\int\limits_0^1(\omega^*(f,t)_p)^pt^{-rp-1}dt$ by $\sum\limits_{k=0}^{\infty}m_k^{rp}E_{m_k}^p(f)_p$ in Corollaries 3 and 4.

3. The sharpness of the embedding conditions

Theorem 3. *1) Let $p\in(1,\infty)$, $\alpha(t)$ and $\omega_n\downarrow 0$ satisfy the condition*

$\sum\limits_{n=1}^{\infty} \mu(n)\omega_n^\theta < \infty$ *for* $\theta = \min(p,2)$. *Then there exists* $h \in H_p^\omega$ *such that*

$$\|\varphi(h)\|_p \geq C \left(\sum_{n=1}^{\infty} \mu(n)\omega_n^\theta \right)^{1/\theta}. \tag{18}$$

2) If $\alpha(t)$ *and* $\omega_n \downarrow 0$ *satisfy the condition* $\sum\limits_{n=1}^{\infty} \mu(n)\omega_n < \infty$, *then there exists* $h \in H_H^\omega$ *such that*

$$\|\psi(h)\|_p \geq C \sum_{n=1}^{\infty} \mu(n)\omega_n. \tag{18$'$}$$

Proof. 1) In the case $1 < p \leq 2$ $(\theta = p)$ we consider the function

$$h(x) = \sum_{k=1}^{\infty} (\omega_k^p - \omega_{k+1}^p)^{1/p} m_k^{1/p/1} (D_{m_{k+1}}(x) - D_{m_k}(x)).$$

(see [1, Chapter 4, §9]). According to (4), Lemma 1, Lemma 4, (3) and the Jensen inequality we obtain

$$\omega_n(h)_p \leq 2\|h - S_{m_n}(h)\|_p \leq C_1 \left\| \left(\sum_{k=n+1}^{\infty} |\Delta_k(h)|^2 \right)^{1/2} \right\|_p \leq$$

$$\leq C_1 \left(\sum_{k=n+1}^{\infty} \|\Delta_k(h)\|_p^2 \right)^{1/2} \leq C_2 \left(\sum_{k=n}^{\infty} (\omega_k^p - \omega_{k+1}^p)^{2/p} \right)^{1/2} \leq \tag{19}$$

$$\leq C_2 \left(\left(\sum_{k=n}^{\infty} (\omega_k^p - \omega_{k+1}^p) \right)^{2/p} \right)^{1/2} = C_2\omega_n, \quad n \in \mathbb{N}.$$

By (19) we get $h \in H_p^\omega$. If

$$\psi(h) = \sum_{k=1}^{\infty} (\omega_k^p - \omega_{k+1}^p)^{1/p} m_k^{1/p-1} \beta^{1/p}(m_k - 1)(D_{m_{k+1}}(x) - D_{m_k}(x)),$$

then according to Corollary 1 $\|\psi(h)\|_p \le C_3\|\varphi(h)\|_p$. By Paley theorem (see [7, Theorem [6.3.2]])

$$\|\psi(h)\|_p \ge C_4 \left(\sum_{k=1}^{\infty}(\omega_k^p - \omega_{k+1}^p) m_k^{p-1}\beta(m_k-1)m_k^{1-p}\right)^{1/p} =$$

(20) $$= C_4 \left(\sum_{k=2}^{\infty}\omega_k^p(\beta(m_k-1)-\beta(m_{k-1}-1)) + \omega_1^p\beta(m_1-1)\right)^{1/p} =$$

$$= C_4 \left(\sum_{k=1}^{\infty}\omega_k^p\mu(k)\right)^{1/p}.$$

From (20) it follows (18) in the case $1 < p \le 2$. If $p \ge 2$, then $\theta = 2$ and $h(x) := \sum\limits_{k=1}^{\infty}(\omega_k^2 - \omega_{k+1}^2)^{1/2}\chi_{m_k-1}(x)$. By Lemma 5 we have $h \in H_p^\omega$ for all $p \ge 1$. Applying (20) for $p = 2$ and Lemma 5, we obtain

$$\|\varphi(h)\|_p =\ge C_5\left(\sum_{k=1}^{\infty}(\omega_k^2-\omega_{k+1}^2)\beta(m_k-1)\right)^{1/2} \ge C_6\left(\sum_{k=1}^{\infty}\omega_k^2\mu(k)\right)^{1/2}.$$

2) Let us consider the function $h(x) = \sum\limits_{k=1}^{\infty}(\omega_k-\omega_{k+1})(D_{m_{k+1}}(x)-D_{m_k}(x))$. Using Lemma $1'$ similarly to (19) we find that $h \in H_H^\omega$. Instead of the Paley theorem we apply the analog of the Hardy inequality $\sum\limits_{n=1}^{\infty}|\hat{f}(n)/n \le C_7\|f\|_H$ (see [16, p. 109] in the dyadic case). As in (20) we obtain $\|\psi(h)\|_H \ge C_8\sum\limits_{k=1}^{\infty}\omega_k\mu(k)$. The theorem is proved.

Theorem 4. *1) If one of the following conditions*

(i) $p \ge 2$, $h(t) \in W(p,p,\alpha)$, $\{\hat{h}(n)\}_{n=0}^{\infty} \in A_\tau$, $\tau \in \mathbb{R}$, *or* $\{\hat{h}(n)\}_{n=0}^{\infty} \in \in RBVS$;

(ii) $1 < p < 2$, $h \in W(2,p,\alpha)$, *and* $\hat{h}(n) = 0$ *for all* $n \ne m_k - 1$, $k \in \mathbb{N}$ *holds, then for* $\gamma = \max(p,2)$ *the inequality*

$$\|\varphi(h)\|_p^\gamma \le C\left(\int_0^1 \alpha(t)(\omega^*(h,t)_p)^\gamma dt + \|h\|_p^\gamma\right)$$

is valid.

2) If $h \in W(2, H, \alpha)$ and $\hat{h}(n) = 0$ for all $n \neq m_k - 1$, $k \in \mathbb{N}$, then

$$\|\psi(h)\|_H^2 \leq C \left(\int_0^1 \alpha(t)(\omega^*(h,t)_p)^2 dt + \|h\|_p^\gamma \right).$$

Proof. 1) Let $p \geq 2$ and $h \in W(p, p, \alpha)$. By Paley theorem ([7, Theorem [6.3.2]]) and summation by parts we conclude that

$$\|\psi(h)\|_p^p \leq C_1 \sum_{n=1}^{\infty} \beta(m_n - 1) \sum_{k=m_n}^{m_{n+1}-1} |\hat{h}(k)|^p k^{p-2} =$$
$$= C_1 \left(\sum_{n=2}^{\infty} (\beta(m_n - 1) - \beta(m_{n-1} - 1)) \sum_{k=m_n}^{\infty} |\hat{h}(k)|^p k^{p-2} \right) +$$
$$+ C_1 \beta(m_1 - 1) \sum_{k=m_1}^{\infty} |\hat{h}(k)|^p k^{p-2}.$$

Using generalized monotonicity of $\{\hat{h}(n)\}_{n=0}^{\infty}$, Lemma 6, (6) and Corollary 2, we obtain

(21)
$$\|\psi(h)\|_p^p \leq C_2 \sum_{k=1}^{\infty} \mu(n) \omega_{n-1}^p(h) \leq C_3 \left(\mu(1)\|h\|_p + \sum_{n=1}^{\infty} \mu(n) \omega_n^p(h)_p \right) \leq$$
$$\leq C_4 \left(\|h\|_p + \int_0^1 \alpha(t)(\omega^*(h,t)_p)^p \, dt \right).$$

Since $|\hat{h}(k)| \leq \|h\|_p$ for all $k \in \mathbb{Z}_+$, $p \in [1, \infty)$, the inequality

(21′)
$$\|\varphi(h)\|_p \leq C_5 \left(\|h\|_p + \int_0^1 \alpha(t)(\omega^*(h,t)_p)^p \, dt \right)$$

is also valid due to Lemma 2 and Corollary 1. If $1 < p < 2$, then by Lemma 5

(22)
$$\|\varphi(h)\|_p \leq C_6 \left(\sum_{n=1}^{\infty} \beta(m_n - 1) |\hat{h}(m_n - 1)|^2 \right)^{1/2} =$$
$$= C_6 \sum_{n=2}^{\infty} (\beta(m_n - 1) - \beta(m_{n-1} - 1)) \sum_{k=n}^{\infty} |\hat{h}(m_k - 1)|^2 +$$
$$+ C_6 \beta(m_1) \sum_{k=1}^{\infty} |\hat{h}(m_k - 1)|^2)^{1/2}) \leq C_7 \left(\sum_{n=1}^{\infty} \mu(n) \omega_n^2(h)_p \right)^{1/2}.$$

Using (6) and Corollary 2, we finish the proof of 1).

2) Since $\|g\|_1 \le \|g\|_H \le C_7\|f\|_p$ for all $p>1$, from Lemma 5 we obtain $\omega_{n-1}^2(f)_H \ge \omega_{n-1}^2(f)_1 \ge \sum\limits_{k=n}^{\infty} |\hat{h}(m_k-1)|^2$ and $\|\psi(h)\|_H^2 \le C_8 \sum\limits_{n=1}^{\infty} \mu(n)\omega_n^2(h)_H$ (see (22)). Using (6) and Corollary 2 we prove 2). The theorem is proved.

Theorems 3 and 4 show that Theorems 1 and 2 are sharp in a certain sense.

The last theorem gives a criterion of $f \in B_{p,\theta}^r := B(p,\theta,t^{-r\theta-1})$ for functions f with generalized monotone Fourier–Vilenkin coefficients. One can find trigonometric analogs of the Theorem 5 in [15] for decreasing Fourier coefficients and in [11] for cosine and sine coefficients from the class $RBVS$.

Theorem 5. *Let* $1<p<\infty$, $\theta \ge 1$, $r>0$ *and* $f \in L^p[0,1)$ *be such that either* $\{\hat{f}(k)\}_{k=0}^{\infty} \in A_\tau$, $\tau \in \mathbb{R}$, *or* $\{\hat{f}(k)\}_{k=0}^{\infty} \in RBVS$. *Then* $f \in B_{p,\theta}^r$ *if and only if*

$$J := \sum_{n=1}^{\infty} |\hat{f}(n)|^\theta n^{r\theta+\theta-\theta/p-1} < \infty.$$

Proof. According to Corollary 2 we can consider $\sum\limits_{n=1}^{\infty} m_n^{r\theta}\omega_n^\theta(f)_p$ instead of $\int\limits_0^1 t^{-r\theta-1}\omega^\theta(f,t)_p dt$. By Lemma 6

$$\sum_{n=1}^{\infty} m_n^{r\theta}\omega_n^\theta(f)_p \le C_1\left(\sum_{n=1}^{\infty} m_n^{r\theta+\theta(1-1/p)}|\hat{f}(m_n)|^\theta + \right.$$

$$\left. + \sum_{n=1}^{\infty} m_n^{r\theta}\left(\sum_{i=m_n}^{\infty} |\hat{f}(i)|^p i^{p-2}\right)^{\theta/p}\right) = C_1(I_1+I_2).$$

If either $\{\hat{f}(k)\}_{k=0}^{\infty} \in A_\tau$, $\tau \ge 0$, or $\{\hat{f}(k)\}_{k=0}^{\infty} \in RBVS$, then $\hat{f}(m_{n+1}) \le$ $\le C_2\hat{f}(k)$, $m_n \le k < m_{n+1}$, and we obtain that the convergence of I_1 is equivalent to convergence of the series $\sum\limits_{n=1}^{\infty} |\hat{f}(n)|^\theta n^{r\theta+\theta-\theta/p-1}$. If $\{\hat{f}(k)\}_{k=0}^{\infty} \in A_\tau$, $\tau<0$, then $\hat{f}(m_n) \le C_3\hat{f}(k)$, $m_n \le k < m_{n+1}$, and we obtain the same conclusion. To estimate I_2 we must consider two cases. In the first case $\theta/p \le 1$ we use Jensen inequality and change the order of summation:

$$\sum_{n=1}^{\infty} m_n^{r\theta}\left(\sum_{k=n}^{\infty} |\hat{f}(m_k)|^p m_k^{p-1}\right)^{\theta/p} \le \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} m_n^{r\theta} m_k^{\theta(1-1/p)}|\hat{f}(m_k)|^\theta \le$$

$$\leq \sum_{k=1}^{\infty} m_k^{r\theta+\theta(1-1/p)} |\hat{f}(m_k)|^{\theta}.$$

Similarly to the case of I_1, convergence of the last series is equivalent to inequality $J < \infty$. In the second case $\theta/p > 1$ the inequality $I_2 < \infty$ is equivalent to

$$I_3 = \sum_{n=1}^{\infty} n^{r\theta-1} \left(\sum_{k=n}^{\infty} |\hat{f}(i)|^p i^{p-2} \right)^{\theta/p} < \infty.$$

According to Hardy–Littlewood inequality [6, Theorem 346]

$$I_3 \leq C_4 \sum_{n=1}^{\infty} (|\hat{f}(n)|^p n^{p-2} n)^{\theta/p} = C_4 \sum_{n=1}^{\infty} |\hat{f}(n)|^{\theta} n^{r\theta+\theta(1-/p)-1} = C_4 J.$$

Thus, the condition $f \in B_{p,\theta}^r$ follows from the finiteness of J in all cases.

Conversely, if $f \in B_{p,\theta}^r$, then the series $\sum_{n=1}^{\infty} m_n^{r\theta} \omega_n^{\theta}(f)_p$ converges. By Lemma 6 and by the conditions on $\hat{f}(i)$ we have

$$\sum_{n=1}^{\infty} m_n^{r\theta} \omega_n^{\theta}(f)_p \geq C_5 \sum_{n=2}^{\infty} m_n^{r\theta} \left(\sum_{i=m_n}^{\infty} |\hat{f}(i)|^p i^{p-2} \right)^{\theta/p} \geq$$

$$\geq C_6 \sum_{n=2}^{\infty} m_n^{r\theta} \left(\sum_{k=n+1}^{\infty} |\hat{f}(m_k)|^p m_k^{p-1} \right)^{\theta/p}. \tag{23}$$

In the case $\theta/p \geq 1$ we obtain by Jensen inequality

$$\sum_{n=1}^{\infty} m_n^{r\theta} \omega_n^{\theta}(f)_p \geq C_7 \sum_{n=2}^{\infty} \sum_{k=n+1}^{\infty} |\hat{f}(m_k)|^{\theta} m_k^{\theta(1-1/p)} m_n^{r\theta} =$$

$$= C_7 \sum_{k=3}^{\infty} \sum_{n=2}^{k-1} |\hat{f}(m_k)|^{\theta} m_k^{\theta(1-1/p)} m_n^{r\theta} \geq C_8 \sum_{k=3}^{\infty} |\hat{f}(m_k)|^{\theta} m_k^{\theta(1-1/p)+r\theta},$$

whence the finiteness of J easily follows. In the case $\theta/p < 1$ we use Theorem 346 from [6] as follows

$$\sum_{n=m_3}^{\infty} n^{r\theta-1} (\hat{f}(n) n^{p-1})^{\theta/p} \leq C_9 \sum_{n=m_3}^{\infty} n^{r\theta-1} \left(\sum_{k=n}^{\infty} |\hat{f}(i)|^p i^{p-2} \right)^{\theta/p} \leq$$

$$\leq C_{10} \sum_{n=2}^{\infty} m_n^{r\theta} \left(\sum_{i=m_n}^{\infty} |\hat{f}(i)|^p i^{p-2} \right)^{\theta/p}.$$

The last inequality and (21) imply $j < \infty$ in the case $\theta/p < 1$. The theorem is proved.

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