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TURÁN PROBLEMS FOR PERIODIC POSITIVE DEFINITE FUNCTIONS

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Dedicated
to Professor F. Schipp on his 70th birthday and
to Professor P. Simon on his 60th birthday

Abstract. There are given a hole solution of integral Turán problem and a partial solution of pointwise Turán problem for periodic positive definite functions.

1. Introduction

Continuous and positive definite functions appear naturally in function theory, approximation theory, probability theory, discrete geometry, analytic number theory, time series analysis, optics, crystallography, signal processing. Optimization problems in these fields translate into extremal problems for such functions. We discuss some extremal problems for continuous positive definite functions on $\mathbb{R}$ and $\mathbb{T}$, known as integral and pointwise Turán problems. Turán problems admit equivalent reformulation as extremal problems for entire functions.

A lot of investigations were devoted to multidimensional Turán problems on $\mathbb{R}^n$ and on common locally compact abelian groups. Let us note the works of C.L. Siegel [1], D.V. Gorbachev [2]. V.V. Arestov and E.E. Berdysheva [3,4], M.N. Kolountzakis and Sz.Gy. Révész [5,6,7], W. Ehm, T. Gneiting and D. Richards [8], Sz.Gy. Révész [9]. We are not going to touch these researches.

The part of the paper, devoted to the solution of the integral Turán problem on $\mathbb{T}$, is written by V.I. Ivanov. The other part of the paper, devoted to the pointwise Turán problem, is written by A.V. Ivanov.

2. Positive definite functions

Let G be locally compact abelian group, $P(G)$ the class of continuous integrable positive definite functions, $P_{\mathbb{R}}(G)$ its subset of real functions. Function f is positive definite if for every collection $\{x_i\}_{i=1}^m \subset G$, $\{\alpha_i\}_{i=1}^m \subset \mathbb{C}$

$$\sum_{i,j=1}^m f(x_i - x_j)\alpha_i\overline{\alpha}_j \geq 0.$$

From Bochner-Weil results [10] follow that

$$P(G) = \left\{ f \in C(G) \bigcap L(G) : \widehat{f} \geq 0 \text{ on } \widehat{G} \right\},$$

$P_{\mathbb{R}}(G)$ is its subset of even functions. Here $\widehat{f}$, defined on dual group $\widehat{G}$, is Fourier transform of f.

Let $\mathbb{T}$ be the one dimensional torus $[0,1)$, $\mathbb{R}$ the set of real numbers, $\mathbb{Z}$ the set of integer numbers.

We have

$$P(\mathbb{R}) = \left\{ f(x) \in C(\mathbb{R}) \bigcap L(\mathbb{R}) : \widehat{f} \geq 0 \text{ on } \mathbb{R} \right\},$$

$$P_{\mathbb{R}}(\mathbb{R}) = \left\{ f \in P(\mathbb{R}) : f - \text{ even} \right\},$$

$$P(\mathbb{T}) = \left\{ f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}_k \exp(2\pi i k x) : \widehat{f}_k \geq 0,\ \sum_{k\in\mathbb{Z}} \widehat{f}_k < \infty \right\},$$

$$P_{\mathbb{R}}(\mathbb{T}) = \left\{ f(x) = \widehat{f}_0 + 2\sum_{k=1}^{\infty} \widehat{f}_k \cos(2\pi k x) : \widehat{f}_k \geq 0,\ \widehat{f}_0 + 2\sum_{k=1}^{\infty} \widehat{f}_k < \infty \right\}.$$

Let us give some examples of positive definite functions

$$e^{-\pi x^2} = \int_{\mathbb{R}} e^{-\pi y^2} \cos(2\pi x y)dy \in P_{\mathbb{R}}(\mathbb{R}),$$

(1) $$f^*(x) = (1-|x|)_+ = \int\limits_{\mathbb{R}} \left(\frac{\sin(\pi y)}{\pi y}\right)^2 \cos(2\pi xy)dy \in P_{\mathbb{R}}(\mathbb{R})$$

$$((x)_+ = \max(x,0)),$$

$$0 < h \le 1/2, \quad f_h(x) = (1-|x/h|)_+ \quad (|x| \le 1/2), \quad f_h(x+1) = f_h(x),$$

(2) $$f_h(x) = h + 2h\sum_{k=1}^{\infty}\left(\frac{\sin(\pi kh)}{\pi kh}\right)^2 \cos(2\pi kx) \in P_{\mathbb{R}}(\mathbb{T}).$$

3. The classes of even functions

Let $f : \mathbb{R} \to \mathbb{R}$ be even function, $\widehat{f}(x) = \int\limits_{\mathbb{R}} f(y)\cos(2\pi yx)\,dy$ its Fourier transform, $f(x) = \int\limits_{\mathbb{R}} \widehat{f}(y)\cos(2\pi yx)\,dy$ its inverse Fourier transform.

For $0 < h < 1/2$ let us define some classes of even continuous positive definite functions:

$$K_{\mathbb{R}}(h) = \left\{f \in C(\mathbb{R})\bigcap L(\mathbb{R}) : f(0) = 1,\ \widehat{f} \ge 0 \ \text{ on } \ \mathbb{R},\ \text{supp}\, f \subset [-h,h]\right\},$$

$$K_{\mathbb{T}}(h) = \left\{f \in C(\mathbb{T}) : f(0) = 1,\ \widehat{f}_k \ge 0 \ \text{ on } \ \mathbb{Z}_+,\ \text{supp}\, f \subset [-h,h]\right\}.$$

We need as well to define some classes of even entire functions of exponential type.

Let $E_{\mathbb{R}}(h)$ ($E_{\mathbb{Z}}(h)$) be the class of even entire functions $F(z)$ of exponential type $2\pi h$ which satisfy the conditions $F(x) \ge 0$ on $\mathbb{R}$ (on $\mathbb{Z}$), $\widehat{F}(0) = \int\limits_{\mathbb{R}} F(x)dx = F(0) + 2\sum\limits_{\nu=1}^{\infty} F(\nu) = 1$.

It is evident that $E_{\mathbb{R}}(h) \subset L(\mathbb{R})$. According to Plancherel-Polya theorem [11] $E_{\mathbb{Z}}(h) \subset L(\mathbb{R})$, too.

Between classes $K_{\mathbb{R}}(h)$ and $E_{\mathbb{R}}(h)$, $K_{\mathbb{T}}(h)$ and $E_{\mathbb{Z}}(h)$ it is possible to establish bijections. According to Paley-Wiener theorem [12] if $f(x) \in K_{\mathbb{R}}(h)$, then $F(z) = \widehat{f}(z) \in E_{\mathbb{R}}(h)$ and $F(0) = \widehat{f}(0) = \int\limits_{\mathbb{R}} f(y)dy$. Conversely, if $F(z) \in E_{\mathbb{R}}(h)$, then $f(x) = \widehat{F}(x) \in K_{\mathbb{R}}(h)$ and $f(0) = \widehat{F}(0)$.

Similarly if

$$f(x)=\widehat{f}_0+2\sum_{\nu=1}^{\infty}\widehat{f}_\nu\cos(2\pi\nu x)\in K_{\mathbb{T}}(h),$$

then

$$F(z)=\int_{-h}^{h}f(x)\cos(2\pi zx)dx\in E_{\mathbb{Z}}(h)$$

and $F(\nu)=\widehat{f}_\nu$. If $F(z)\in E_{\mathbb{Z}}(h)$, then according to Paley-Wiener theorem and Poisson summation formula [12]

$$f(x)=\sum_{k\in\mathbb{Z}}\widehat{F}(x+k)=F(0)+2\sum_{\nu=1}^{\infty}F(\nu)\cos(2\pi\nu x)\in K_{\mathbb{T}}(h).$$

4. Extremal problems

The integral Turán problem. Let us define the integral Turán problems for $\mathbb{R}$ and $\mathbb{T}$. To calculate the following values:

$$A_{\mathbb{R}}(h)=\sup\left\{\int_{-h}^{h}f(x)dx : f\in K_{\mathbb{R}}(h)\right\}, \tag{3}$$

$$A_{\mathbb{T}}(h)=\sup\left\{\int_{-h}^{h}f(x)dx : f\in K_{\mathbb{T}}(h)\right\}. \tag{4}$$

Taking account the connections between classes $K_{\mathbb{R}}(h)$ and $E_{\mathbb{R}}(h)$, $K_{\mathbb{T}}(h)$ and $E_{\mathbb{Z}}(h)$, we get

$$A_{\mathbb{R}}(h)=\sup\left\{F(0) : F\in E_{\mathbb{R}}(h)\right\}, \tag{5}$$

$$A_{\mathbb{T}}(h)=\sup\left\{F(0) : F\in E_{\mathbb{Z}}(h)\right\}. \tag{6}$$

The pointwise Turán problem. Let us define the pointwise Turán problems for $\mathbb{R}$ and $\mathbb{T}$. For $0 < x < h$ to calculate the following values

$$A_{\mathbb{R}}(x,h) = \sup\left\{f(x) : f \in K_{\mathbb{R}}(h)\right\}, \tag{7}$$

$$A_{\mathbb{T}}(x,h) = \sup\left\{f(x) : f \in K_{\mathbb{T}}(h)\right\}. \tag{8}$$

As in the case of integral Turán problems we have

$$A_{\mathbb{R}}(x,h) = \sup\left\{\int\limits_{\mathbb{R}} F(y)\cos(2\pi xy)dy : F \in E_{\mathbb{R}}(h)\right\}, \tag{9}$$

$$A_{\mathbb{T}}(x,h) = \sup\left\{\int\limits_{\mathbb{R}} F(y)\cos(2\pi xy)dy : F \in E_{\mathbb{Z}}(h)\right\}. \tag{10}$$

5. The case of line

The problems (3), (7) were solved by Boas and Kac in 1945 in [13]. The solutions were based on the representation for positive definite function $f(x)$ from $K_{\mathbb{R}}(h)$ in the form of

$$f(x) = \int\limits_{\mathbb{R}} u(x+y)\overline{u}(y)dy, \tag{11}$$

where $u(x) = 0,\ |x| \geq h/2,\ u \in L_2(\mathbb{R}),\ f(0) = \|u\|_2^2 = 1,\ \widehat{f}(0) = \left|\int\limits_{-h/2}^{h/2} udx\right|^2$. The function $u(x)$ is called a Boas-Kac convolution root of $f(x)$. With (11), by the Cauchy-Bunyakovsky inequality, we get the estimation

$$\left|\int\limits_{-h}^{h} f(x)dx\right| = \left|\int\limits_{-h/2}^{h/2} udx\right|^2 \leq h\int\limits_{-h/2}^{h/2} |u|^2dx = hf(0) = h.$$

It is attained at function $f^*(x/h)$ defined in (1). Boas-Kac convolution root for the function $f^*(x/h)$ is $u_h^*(x) = (1/\sqrt{h})\chi_{[-h/2,h/2]}(x)$, where $\chi_{[-h/2,h/2]}$ is the characteristic function of the interval $[-h/2, h/2]$. The corresponding entire function is $F_h^*(z) = h\left(\frac{\sin(\pi hz)}{\pi hz}\right)^2$. Thus

$$A_{\mathbb{R}}(h) = h.$$

They also proved that

$$A_{\mathbb{R}}(x,h) = \cos\left(\frac{\pi}{]\frac{h}{x}[+1}\right),$$

where $]x[$ is a least integer not lower than x.

6. The integral Turán problem in the case of torus

The problem (4) was set up by P. Turán in 1970 in a private conversation with S.B. Stechkin. It has application in the analytic number theory. The periodic positive definite function from $K_{\mathbb{T}}(h)$ does not admit the representation (11) that is why the problem (4) is much more complicated.

The function (2) $f_h(x) \in K_{\mathbb{T}}(h)$, so $A_{\mathbb{T}}(h) \geq h$. S.B. Stechkin [14] found that $A_{\mathbb{T}}(1/q) = 1/q$, $q \in \mathbb{N}$ and $A_{\mathbb{T}}(h) = h + O(h^2)$ $(h \to 0)$. A.Yu. Popov proved that $A_{\mathbb{T}}(h) > h$, $h \neq 1/q$. Earlier this inequality was proved by G. Halász. D.V. Gorbachev [2] made it more precise proving that $A_{\mathbb{T}}(h) = h + O(h^3)$, $h \to 0$. It was a hypothesis of A.Yu. Popov. D.V. Gorbachev and A.S. Manoshina [15] showed the problem (4) for rational h can be reduced to a discrete variant of a well known Fejér problem about the greatest value at zero of nonnegative trigonometric polynomial with fixed average value.

The first discrete Fejér problem. For $p, q \in \mathbb{N}$, $p \leq q/2$, $(p,q) = 1$ calculate the value

$$\lambda(p,q) = \sup t_{p-1}(0), \tag{12}$$

if

$$t_{p-1}(x) = 1 + 2\sum_{k=1}^{p-1} \widehat{t}_k \cos\left(\frac{2\pi kx}{q}\right) \geq 0, \quad x \in \mathbb{Z}_q = \{0, 1, \ldots, q-1\}.$$

Here $(p, q) = 1$ means, that p and q are relatively prime.

The problem (12) is the discrete variant of well known classic Fejér problem. Calculate the value

$$\Lambda(p) = \sup t_{p-1}(0),$$

if

$$t_{p-1}(x) = 1 + \sum_{k=1}^{p-1} \widehat{t}_k \cos(2\pi kx) \geq 0, \quad x \in \mathbb{T}.$$

This problem was set and solved by L. Fejér [16].

In [15] the equality

$$A_{\mathbb{T}}\left(\frac{p}{q}\right) = \frac{\lambda(p,q)}{q} \tag{13}$$

is proved and the values $\lambda(p, q)$ are calculated for small p.

In 2004 V.I. Ivanov and Yu.D. Rudomazina [17, 18, 19] solved the discrete analog of Fejér problem and thus the solution of integral Turán problem for rational h was achieved. In 2006 V.I. Ivanov [20] managed to solve the integral Turán problem for irrational h.

In what follows are outlined the main points of the solution of problems (4), (12).

7. Special partition of a set of natural numbers

Let us denote for the real number x: $[x]$ is its integral part, $\{x\}$ its fractional part, $\langle x\rangle$ the distance to its nearest integer. We have

$$x = [x] + \{x\}, \qquad \langle x\rangle = \min_{\nu\in\mathbb{Z}} |x-\nu| = \min\{\{x\}, 1-\{x\}\},$$

$$[x] \in \mathbb{Z}, \qquad \{x\} \in [0,1), \qquad \langle x\rangle \in [0, 1/2].$$

For $h \in (0, 1/2)$ let us define the partition of a set of natural numbers $\mathbb{N}$ by

$$S_0(h) = \{\nu \in \mathbb{N} : \langle \nu h\rangle = 0\}, \quad S_1(h) = \{\nu \in \mathbb{N} : \langle \nu h\rangle \in (0, h)\},$$

$$S_2\left(h\right)=\left\{\nu\in\mathbb{N}:\langle\nu h\rangle\geq h\right\}.$$

These sets do not intersect and $\mathbb{N}=S_0\left(h\right)\cup S_1\left(h\right)\cup S_2\left(h\right)$.

Note that for $h=p/q$ (irreducible fraction)

$$S_0\left(\frac{p}{q}\right)=\{\nu q:\nu\in\mathbb{N}\},\quad S_1\left(\frac{p}{q}\right)=\left\{\left[\frac{q\nu}{p}\right],\left[\frac{q\nu}{p}\right]+1:\nu\in\mathbb{N},\ \nu\neq ps\right\}.$$

For irrational h

$$S_0\left(h\right)=\emptyset,\qquad S_1\left(h\right)=\left\{\left[\frac{\nu}{h}\right],\quad\left[\frac{\nu}{h}\right]+1:\nu\in\mathbb{N}\right\}.$$

8. The case of rational h

The solutions of (4) for rational $h=p/q$ and of (12) are based on constructing of special trigonometric polynomial.

Lemma 1. *If $p,q\in\mathbb{N}$, $p\leq q/2$, $(p,q)=1$, then there is an even trigonometric polynomial*

$$f_{p,q}\left(x\right)=\widehat{f}_0^{p,q}+2\sum_{k=1}^{p-1}\widehat{f}_k^{p,q}\cos\left(\frac{2\pi kx}{q}\right),$$

satisfying the conditions

1) $\widehat{f}_k^{p,q}>0\quad(k=0,1,\ldots,p-1)$,

2) $f_{p,q}\left(\nu\right)=1\quad\left(\nu\in\{0\}\cup S_0\left(p/q\right)\right),\quad f_{p,q}\left(\nu\right)=0\quad\left(\nu\in S_1\left(p/q\right)\right)$, $0<$ $<f_{p,q}\left(\nu\right)<1\quad\left(\nu\in S_2\left(p/q\right)\right)$.

The polynomial $f_{p,q}$ has zeros on interval $1\leq x\leq q/2$ in the points of the set

$$S_{p,q}=\left\{\left[\frac{qi}{p}\right]:i=1,\ldots,\left[\frac{p}{2}\right]\right\}\bigcup\left\{\left[\frac{qi}{p}\right]+1:i=1,\ldots,\left[\frac{p-1}{2}\right]\right\}.$$

The set $S_{p,q}/q$ approximate the zeros i/p of the well known Fejér polynomial

$$F_{p-1}(x) = 1 + 2\sum_{k=1}^{p-1}\left(1-\frac{k}{p}\right)\cos(2\pi kx) = \frac{1}{p}\left(\frac{\sin(\pi px)}{\sin(\pi x)}\right)^2$$

on the subgroup of the torus $\{i/q\}$. The set $S_{p,q}$ has the arithmetic structure

$$S_{p,q} = \{q\langle \bar{r}k/q\rangle : k = 1,\ldots,p-1\},$$

where $\bar{r}p = 1$ or $\bar{r}(q-p) = 1$ in $\mathbb{Z}_q$.

Lemma 2. *If $p, q \in \mathbb{N}$, $p \le q/2$, $(p,q) = 1$, then for every even polynomial*

$$f(x) = \sum_{k=0}^{p-1} \widehat{f}_k \cos\left(\frac{2\pi kx}{q}\right)$$

the quadrature formula

$$\widehat{f}_0 = \widehat{f}_0^{p,q} f(0) + 2\sum_{k=1}^{p-1}\widehat{f}_k^{p,q} f(\bar{r}k)$$

is fulfilled.

From Lemma 2 we have the upper estimation for (12)

$$\lambda(p,q) \le \frac{1}{\widehat{f}_0^{p,q}}.$$

This estimation is achieved for the polynomial

$$F_{p,q}(x) = \frac{f_{p,q}(x)}{\widehat{f}_0^{p,q}}.$$

The polynomial in (12) $F_{p,q}$ is called the discrete extremal Fejér polynomial.

According to (13) the following theorem is proved.

Theorem 1. *For every $p, q \in \mathbb{N}$, $p \le q/2$, $(p,q) = 1$*

$$\lambda(p,q) = F_{p,q}(0) = \frac{1}{\widehat{f}_0^{p,q}},$$

$$A_{\mathbb{T}}\left(\frac{p}{q}\right)=\frac{\lambda(p,q)}{q}=\frac{F_{p,q}(0)}{q}=\frac{1}{q\widehat{f}_0^{p,q}}.$$

Let us put the solution of problem (4) that does not use (13).

Lemma 3. *If $p,q\in\mathbb{N}$, $p\le q/2$, $(p,q)=1$, then there is an even trigonometric polynomial*

$$g_{p,q}(x)=\widehat{g}_0^{p,q}+2\sum_{k=p}^{[q/2]}\widehat{g}_k^{p,q}\cos\left(\frac{2\pi kx}{q}\right),$$

for which the following conditions are satisfied

1) $\widehat{g}_0^{p,q}=\frac{1}{q\widehat{f}_0^{p,q}}$, $\quad\widehat{g}_k^{p,q}\ge 0\quad(k=p,...,[q/2])$,

2) $g_{p,q}(\nu)=1\ \ (\nu\in\{0\}\cup S_0(p/q))$, $g_{p,q}(\nu)=0\ \ (\nu\in S_2(p/q))$, $0<$ $<g_{p,q}(\nu)<1\ \ (\nu\in S_1(p/q))$.

The coefficients of polynomial $g_{p,q}(x)$ are calculated by the following formula

$$\widehat{g}_k^{p,q}=\frac{f_{p,q}(\bar{r}k)}{q\widehat{f}_0^{p,q}}.$$

Lemma 3 allows to write down quadrature formulas that will give upper bound for the function (4). Lemma 1 allows to construct extremal functions.

Lemma 4. *For every function*

$$f(x)=\widehat{f}_0+2\sum_{k=1}^{\infty}\widehat{f}_k\cos(2\pi kx),\qquad\sum_{k=1}^{\infty}\left|\widehat{f}_k\right|<\infty$$

the quadrature formula

$$\frac{1}{q\widehat{f}_0^{p,q}}f(0)+2\sum_{k=p}^{[q/2]}\widehat{g}_k^{p,q}f\left(\frac{k}{q}\right)=\widehat{f}_0+2\sum_{k\in S_0(p/q)}\widehat{f}_k+2\sum_{k\in S_1(p/q)}\widehat{f}_k g_{p,q}(k)$$

is fulfilled.

From Lemma 4 we have the following upper estimation for (4)

$$A_{\mathbb{T}}\left(\frac{p}{q}\right)\le\frac{1}{q\widehat{f}_0^{p,q}}.\tag{14}$$

Their accuracy is checked at function

$$\varphi_{p,q}(x) = \frac{1}{q\widehat{f}_0^{p,q}} \left\{ 1 + 2 \sum_{k\in S_2(p/q)} \left(\frac{\sin\left(\frac{\pi k}{q}\right)}{\frac{\pi k}{q}} \right)^2 f_{p,q}(k) \cos(2\pi kx) \right\},$$

for which the following conditions are satisfied

$$\varphi_{p,q}(x) \in K_{\mathbb{T}}\left(\frac{p}{q}\right), \quad \varphi_{p,q}(x) > 0 \text{ and decrease on } [0, p/q).$$

This function was suggested in [15]. Let us note that this function is piecewise linear.

Lemma 5. *For every even entire function $F(z) \in L(\mathbb{R})$ of exponential type the following quadrature formula is fulfilled*

$$\frac{1}{q\widehat{f}_0^{p,q}} \widehat{F}(0) + 2 \sum_{k=p}^{[q/2]} \widehat{g}_k^{p,q} \widehat{F}\left(\frac{k}{q}\right) =$$

$$= F(0) + 2 \sum_{k\in S_0(p/q)} F(k) + 2 \sum_{k\in S_1(p/q)} F(k)\, g_{p,q}(k).$$

The upper bound (14) for (4) arises from Lemma 5 and (6). Its accuracy is checked at function

$$G_{p,q}(z) = \frac{1}{q\widehat{f}_0^{p,q}} \left(\frac{\sin\left(\frac{\pi z}{q}\right)}{\frac{\pi z}{q}} \right)^2 f_{p,q}(z) =$$

$$= \frac{1}{q\widehat{f}_0^{p,q}} \prod_{k\in S_0(p/q)} \left(1 - \left(\frac{z}{k}\right)^2\right)^2 \prod_{k\in S_1(p/q)} \left(1 - \left(\frac{z}{k}\right)^2\right) \in E_{\mathbb{Z}}(p/q).$$

9. The case of irrational h

For calculating values $A_{\mathbb{T}}(h)$, with irrational h it is sufficient to prove their continuity at h.

Theorem 2. *At the interval* $(0,1/2]$ *the function* $A_{\mathbb{T}}(h)$ *is continuous and increasing.*

The proof of continuity $A_{\mathbb{T}}(h)$ is based on the approach suggested by V.S. Balagansky [21] while proving the continuity of an exact constant in the Jackson inequality in the space $L_2(\mathbb{T})$ as function of argument at the module of continuity.

The continuity of function $A_{\mathbb{T}}(h)$ and equality (6) allows to obtain values for irrational h by using their values for rational h and with the help of the passage to the limit. Thus we get the following theorem.

Theorem 3. *If* $h \in (0,1/2)$ *is irrational, then*

$$A_{\mathbb{T}}(h) = \left(1 + 2\sum_{\nu\in S_2(h)}\prod_{k\in S_1(h)}\left(1-\left(\frac{\nu}{k}\right)^2\right)\right)^{-1}.$$

The extremal function $G_h(z)$ *in* (6) *has zeros at set* $S_1(h)$ *and the extremal function in the integral Turán problem* (4) $\varphi_h(x)$ *is positive and does not decrease at the interval* $[0,h)$.

The extremal functions in Theorem 3 are of the form

$$G_h(z) = A_{\mathbb{T}}(h)\prod_{k\in S_1(h)}\left(1-\left(\frac{z}{k}\right)^2\right),$$

$$\varphi_h(x) = G_h(0) + 2\sum_{k=1}^{\infty} G_h(k)\cos(2\pi kx). \tag{15}$$

We get for rational $h = p/q \quad G_h(x) = G_{p,q}(x), \quad \varphi_h(x) = \varphi_{p,q}(x)$.

Lemma 6. *For every even function*

$$f(x) = \widehat{f}_0 + 2\sum_{k=1}^{\infty}\widehat{f}_k\cos(2\pi kx), \qquad \sum_{k=1}^{\infty}\left|\widehat{f}_k\right| < \infty$$

the quadrature formula

$$G_h(0)f(0) + 2\sum_{k\in S_2(h)} G_h(k)f(\langle kh\rangle) = \widehat{f}_0 + 2\sum_{k\in S_0(h)}\widehat{f}_k + 2\sum_{k\in S_1(h)}\widehat{f}_k\varphi_h(\langle kh\rangle)$$

is fulfilled.

Lemma 7. *For every even entire function $F \in L(\mathbb{R})$ of exponential type the quadrature formula*

$$G_h(0)\,\widehat{F}(0) + 2\sum_{k\in S_2(h)} G_h(k)\,\widehat{F}(\langle kh\rangle) =$$

$$= F(0) + 2\sum_{k\in S_0(h)} F(k) + 2\sum_{k\in S_1(h)} F(k)\,\varphi_h(\langle kh\rangle)$$

is fulfilled.

10. The characterization of extremal functions in (4) and (6)

It is sufficient to characterize the extremal functions in problem (6). Extremal functions in Turán problem (4) will be obtained by means of the correspondence between problems (4) and (6) mentioned above. For rational $h = p/q \in (0, 1/2)$ the solution of (6) is not unique. All extremal functions have the form of

$$G(z,p,q) = A_{\mathbb{T}}\left(\frac{p}{q}\right) f_{p,q}(z) \prod_{k=1}^{\infty}\left(1-\left(\frac{z}{qk}\right)^2\right)\left(1-\left(\frac{z}{z_k}\right)^2\right),$$

where $z_k \in [qk-1, qk+1]$.

For irrational h the extremal function $G_h(z)$ is the only possible solution in (15).

The extremal function can be written in the form

$$G(z,h) = A_{\mathbb{T}}(h) \prod_{k=1}^{\infty}\left(1-\left(\frac{z}{[k/h]}\right)^2\right)\left(1-\left(\frac{z}{[k/h]+1}\right)^2\right)$$

for every $h \in (0, 1/2)$.

11. The pointwise Turán problem for the case of torus

Let us consider the pointwise Turán problem (8). This problem is investigated in the paper of V.V. Arestov, E.E. Berdysheva and H. Berens [22]. They solved this problem in the case $h = 1/2$, $0 < x < 1/2$

$$A_{\mathbb{T}}\left(x, \frac{1}{2}\right) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x = \frac{r}{2q},\ r \text{ is even,} \\ \frac{1}{2}\left(1 + \cos\frac{\pi}{q}\right) & \text{if } x = \frac{r}{2q},\ r \text{ is odd.} \end{cases}$$

The paper [7] is devoted to the problem (8), too.

Like as in the case of integral Turán problem there is a connection between problem (8) for rational $x = \nu/q$, $h = p/q$ and some discrete Fejér problem.

The second discrete Fejér problem. For $\nu, p, q \in \mathbb{N}$, $p \le q/2$, $\nu \le p-1$ calculate the value

$$\lambda(\nu, p, q) = \sup \widehat{t}_\nu, \tag{16}$$

if

$$t_{p-1}(x) = 1 + 2\sum_{k=1}^{p-1} \widehat{t}_k \cos\left(\frac{2\pi k x}{q}\right) \ge 0, \quad x \in \mathbb{Z}_q.$$

In the continuous case, when

$$t_{p-1}(x) = 1 + 2\sum_{k=1}^{p-1} \widehat{t}_k \cos(2\pi k x) \ge 0, \quad x \in \mathbb{T},$$

this problem was solved by L. Fejér [16] ($\nu = 1$), G. Szegö [23], E. Egerváry and O. Szasz [24] ($\nu > 1$).

Theorem 4. *If $\nu, p, q \in \mathbb{N}$, $p \le q/2$, $\nu \le p-1$, $(\nu, p, q) = 1$, then*

$$A_{\mathbb{T}}\left(\frac{\nu}{q}, \frac{p}{q}\right) = \lambda(\nu, p, q).$$

If the polynomial $t^*_{p-1}(x)$ is extremal for the problem (16), *then the function*

$$\psi_{p,q}(x) = \frac{1}{q}\left\{1 + 2\sum_{k=1}^{\infty}\left(\frac{\sin\left(\frac{\pi k}{q}\right)}{\frac{\pi k}{q}}\right)^2 t^*_{p-1}(k)\cos(2\pi k x)\right\}$$

is extremal for the problem (8).

Lemma 8. *The following equality*

$$\lambda(\nu d, p, qd) = \lambda\left(\nu, \left[\frac{p-1}{d}\right] + 1, q\right)$$

is true.

We can assume, that $(\nu, q) = 1$. The next modification of Theorem 4 follows from monotonicity of function $A_{\mathbb{T}}(x, h)$ as function of h and the Lemma 8.

Theorem 5. *If $\nu, p, q \in \mathbb{N}$, $p \le q/2$, $\nu \le p-1$, $(\nu, q) = 1$, then*

$$A_{\mathbb{T}}\left(\frac{\nu}{q}, h\right) = \lambda(\nu, p, q), \quad h \in \left(\frac{p-1}{q}, \frac{p}{q}\right].$$

We have managed to solve the problem (16) only for the highest coefficient $\nu = p-1$. In this case the extremal polynomial in the continuous problem is $1 + c((p-1)x)$. But in this elementary case the discrete approximation of continuous problem is rather complicated. The following theorem is true.

Theorem 6. *If $(p-1, q) = 1$, q is odd, then*

$$\lambda(p-1, p, q) = \frac{1}{2\cos(\pi/q)} \cdot \frac{F_{p-1,q}(0)}{F_{p-1,q}(1)}.$$

If $(p-1, q) = 1$, q is even, then

$$\lambda(p-1, p, q) = \frac{F_{p-1,q}(0)}{2F_{p-1,q}(1)}.$$

Here $F_{p-1,q}(x)$ is the extremal polynomial of order $p-2$ in the first discrete Fejér problem.

Upper estimation for the value (16) is done with the help of quadrature formula in Theorem 7.

Theorem 7. *If $(p-1, q) = 1$, q is odd, then for every even polynomial*

$$f(x) = \sum_{k=0}^{p-1} \widehat{f}_k \cos\left(\frac{2\pi k x}{q}\right)$$

the quadrature formula

$$\frac{1}{2\cos(\pi/q)} \cdot \frac{F_{p-1,q}(0)}{F_{p-1,q}(1)}\widehat{f}_0 - \widehat{f}_{p-1} = \sum_{k=0}^{p-2} A_k f(\bar{r}(2k+1))$$

is true, where

$$2\bar{r}(p-1) = 1 \ \ in \ \ \mathbb{Z}_q, \quad A_k > 0 \quad (k = 0, 1, \ldots, p-2).$$

If $(p-1, q) = 1$*,* q *is even, then*

$$\frac{F_{p-1,q}(0)}{2F_{p-1,q}(1)}\widehat{f}_0 - \widehat{f}_{p-1} = \sum_{k=q/2-p+2}^{q/2} B_k f(\bar{r}k),$$

where

$$\bar{r}(p-1) = 1 \ \ in \ \ \mathbb{Z}_q, \quad B_k > 0 \quad (k = q/2-p+2, \ldots, q/2).$$

Zeros of extremal polynomials in the problem (16) with $\nu = p-1$ coincide with nodes of quadrature formulas in Theorem 7.

If $(p-1, q) = 1$, then Theorems 5, 6 allow to calculate $A_{\mathbb{T}}\left(\frac{p-1}{q}, h\right)$ for $h \in \left(\frac{p-1}{q}, \frac{p}{q}\right]$.

12. Conclusion

It will be interesting to investigate Fejér and Turán problems for finite abelian groups

$$G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_n}$$

and for compact abelian zero-dimensional groups

$$G = \prod_{k=1}^{\infty} \mathbb{Z}_{m_k}.$$

There are some results in [19] and [25].

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