THE WALSH TRANSFORM BELONGING TO THE SPACE $L^p(1 < p < 2)$

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Dedicated to Prof. Ferenc Schipp on his 70th birthday and to Prof. Péter Simon on his 60th birthday

Abstract. The sufficient conditions on the functions under which their Walsh transforms belong to the space $L^p(R_+)$, $1 < p < 2$, are given.

1. Introduction

The properties of classical Fourier transform are presented, e.g., in the book of E. Titchmarsh [1]. The Walsh transform $F(f) \equiv \widetilde{f}$ was introduced by N.G. Fine [2] in 1950. This transform has the properties analogous to the ones of classical Fourier transform. For example, the inversion formula $F(F(f)) = f$ and Plansherel equality $\|\widetilde{f}\|_2 = \|f\|_2$ for functions $f \in L^2(R_+)$ are valid, where $R_+ = [0, +\infty)$ and $\|f\|_p$ denotes the usual $L^p(R_+)$-norm of the function f. If $f \in L^p(R_+)$, $1 \le p \le 2$, then $\widetilde{f} \in L^q(R_+)$ and $\|\widetilde{f}\|_q \le \|f\|_p$, where $1/p+1/q = 1$ (see [3], Ch. 9).

The Walsh transform was generalized by N.Ya. Vilenkin [4]. He introduced the notation of multiplicative Fourier transform for functions defined on locally compact Abelian group. M.S. Bespalov (see [5], [6] and [7], Ch. 6) proved some

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Mathematics Subject Classification: 42C10 results for multiplicative Fourier transform on R_+ similar to the ones of classical Fourier transform.

In this article we give the sufficient conditions on the functions under which their Walsh transforms belong to the space $L^p(R_+),\ 1 < p < 2$.

2. Definitions and auxiliary results

For a number $x \in R_+ \equiv [0, +\infty)$ and a natural n we set

$$x_n \equiv [2^n x] \pmod 2, \quad x_{-n} \equiv \left[2^{1-n} x\right] \pmod 2, \tag{2.1}$$

where $[a]$ denotes the integer part of the number a, and the numbers x_n and x_{-n} are equal to 0 or 1.

Let us note that x_{-n} (or x_n) is equal to n-th dyadic digit of integer part (or fractional part) of the number $x \in R_+$ and the dyadic-rational numbers $x \in R_+$ have finite expansions.

Since $x_{-n} = 0$ for $n \geq n(x)$ for $(x, y) \in R_+ \times R_+$ the series

$$t(x, y) = \sum_{n=1}^{\infty} (x_n y_{-n} + x_{-n} y_n)$$

is finite and its sum is a nonnegative integer.

The Walsh kernel $\psi(x, y)$ is defined by the equality

$$\psi(x, y) = (-1)^{t(x,y)}, \quad (x, y) \in R_+ \times R_+. \tag{2.2}$$

The Walsh transform $F[f] \equiv \widetilde{f}$ of a function $f \in L(R_+)$ is defined by the equality

$$\widetilde{f}(x) = \int\limits_{R_+} \psi(x, y) f(y) dy. \tag{2.3}$$

If $f \in L^p(R_+),\ 1 < p \leq 2$, we define $\widetilde{f}$ as the limit of the function sequence

$$\int\limits_0^{2^n} f(y) \psi(x, y) dy, \quad n \in Z_+$$

by the norm of the space $L^q(R_+),\ 1/p + 1/q = 1$.

For the Walsh transform the following theorems are valid.

Theorem A. *If* $f \in L^p(R_+)$, $1 \le p \le 2$, *then* $\widetilde{f} \in L^q(R_+),\ 1/p+1/q = 1$, *and* $\|\widetilde{f}\|_q \le \|f\|_p$.

The proof of this theorem see e.g. in ([3], p. 35).

Theorem B. *If* $\int\limits_{R_+} |f(x)|^q x^{q-2} dx < \infty,\ 2 < q < \infty$, *then* $\widetilde{f} \in L^q(R_+)$.

This theorem is a special case of the similar result for multiplicative Fourier transform (see [5] or [6]).

Let us introduce the operation $\oplus$ of dyadic addition on R_+ as follows

$$x \oplus y = z \quad \text{for } (x, y) \in R_+ \times R_+,$$

where the number z has dyadic digits

$$z_n \equiv x_n + y_n \pmod 2,\ n \in Z \setminus \{0\},$$

and $x_n,\ y_n$ are calculated by the rule (2.1).

Let us note that

$$z = \sum_{n=1}^{+\infty} \frac{z_n}{2^n} + \sum_{n=1}^{+\infty} 2^{n-1} z_{-n}$$

and the case $z_n = 1$ for $n \ge n(z)$ is not excluded.

For the Walsh kernel (2.2) the equality

$$\psi(x \oplus y, t) = \psi(x, t)\psi(y, t) \tag{2.4}$$

holds if $t, x, y \in R_+$ and $x \oplus y$ is dyadic irrational.

Lemma 2.1. *For* $n \in Z,\ \alpha > 0$ *and* $x > 0$ *the function*

$$W_n^{\{\alpha\}}(x) \equiv \lim_{m \to +\infty} \int\limits_{2^{-n}}^{2^m} \psi(x, y) y^{-\alpha} dy$$

is defined and the limit exists also in the space $L(R_+)$, *hence* $W_n^{\{\alpha\}} \in L(R_+)$.

For $\alpha = 1$ this statement is known (see. [3], p. 434), and for $\alpha > 0$ it was proved in our article [9].

Let us define the dyadic convolution $f * g$ of the functions $f, g \in L(R_+)$ as follows

$$(f * g)(x) = \int_{R_+} f(y)g(x \oplus y)dy, \quad x \in R_+.$$

It is known that $f * g \in L(R_+)$, $\widetilde{(f * g)} = \widetilde{f}(x)\widetilde{g}(x)$ for all $x \in R_+$ The dyadic convolution exists also in the case $f \in L^p(R_+)$ $1 \le p \le \infty$, $g \in L(R_+)$, moreover $f * g \in L^p(R_+)$ and $\widetilde{(f * g)} = \widetilde{f}(x)\widetilde{g}(x)$ almost everywhere on R_+ for $1 \le p \le 2$. This can be proved by using the equality (2.4) (see e.g. [10], Ch. 1, Theorem 2.6).

Definition 2.1. *Let* $\alpha > 0$, $f, g \in L^p(R_+)$, $1 \le p \le \infty$, *and*

$$\lim_{n \to +\infty} \|f * W_n^{\{\alpha\}} - g\|_{L^p(R_+)} = 0.$$

Then the function $g \equiv I_\alpha^p(f)$ *will be called the strong dyadic integral (SDI) of order* α *of the function* f *in the space* $L^p(R_+)$.

For $\alpha = 1$, $p = 1$, this definition was introduced by H.J. Wagner [11].

Lemma 2.2. *Let* $f, g \in L^p(R_+)$, $1 \le p \le 2$, *and let the function* g *be SDI of order* $\alpha > 0$ *of the function* f *in the space* $L^p(R_+)$. *Then* $\widetilde{g}(x) = \widetilde{f}(x)x^{-\alpha}$ *almost everywhere on* R_+.

Proof. It follows from the Lemma 2.1 that $\widetilde{W_n^{\{\alpha\}}}(x) = x^{-\alpha}X_{[2^{-n},+\infty)}(x)$ for $x \in R_+$. Therefore

$$(2.5) \qquad \widetilde{(f * W_n^{\{\alpha\}})}(x) = \widetilde{f}(x)x^{-\alpha}X_{[2^{-n},+\infty)}(x)$$

almost everywhere in R_+. From Theorem **A** we have

$$\|\widetilde{(f * W_n^{\{\alpha\}})} - \widetilde{g}\|_q \le \|f * W_n^{\{\alpha\}} - g\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By the conjecture of the lemma the right-hand side of this inequality tends to 0 as $n \to +\infty$. Therefore the left-hand side also tends to 0. Hence, there exists the subsequence $n_k \to +\infty$ such that $\lim\limits_{k \to +\infty} \widetilde{(f * W_n^{\{\alpha\}})}(x) = \widetilde{g}(x)$ almost everywhere on R_+. Now the assertion of the lemma follows from the equality (2.5).

Let us note that in the case $p = 1$ the assertion of the Lemma 2.2 can be made more precise. In this case $\widetilde{g}(x) = \widetilde{f}(x)x^{-\alpha}$ for $x > 0$, moreover $\widetilde{g}(0) = 0$. It was proved by H.J. Wagner [11] in the case $p = 1$, $\alpha = 1$ (see also [3], p. 435), and by the author [9] in the case $p = 1$, $\alpha > 0$.

Theorem C. *If a function f belongs to the space $L^p(R_+)$, $1 < p \le 2$, then the inequality*

$$\int\limits_{R_+} x^{p-2} \left|\widetilde{f}(x)\right|^p dx \le B_p \|f\|_p^p \tag{2.6}$$

is valid, where the constant $B_p > 0$ does not depend on the function f.

This theorem is a special case of the theorem, that was formulated by M.S. Bespalov (see [5], Theorem 4 or [6], Theorem 3) for multiplicative Fourier transforms. But he did not publish the proof of his theorem. He restricted himself by pointing out that the proof is similar to that for classical Fourier transform. For completeness we prove Theorem **C**, because we use it below for the proof of the Theorem 3.2. Our proof differs from that mentioned in [5] and [6].

For the proof of Theorem **C** we need some lemmas. For the Walsh kernel (2.2) we will use the notation $\psi(x,y) \equiv \psi_y(x)$.

Lemma 2.3. *The system*

$$\{\, 2^{-\frac{n}{2}} \psi_{m2^{-n}}(x) X_{[0,2^n)}(x) \,\}_{m=0}^{\infty} \equiv \{a_m(x)\}_{m=0}^{\infty} \tag{2.7}$$

is orthonormal on $D_n = [0, 2^n)$ for any fixed $n \in Z_+$.

This result is well known (see e.g. [8], Ch. 1, Proposition 5.1).

Lemma 2.4. *The equalities $\widetilde{a}_{m,n}(x) = 2^{\frac{n}{2}} X_{[m2^{-n},(m+1)2^{-n})}(x)$ are valid, where $m \in Z_+$, $n \in Z$, $x \in R_+$.*

This result is known also (see e.g. [8], Ch. 2, Lemma 1.1).

Let us denote by $\hat{f}_n(k)$, $n \in Z_+$, the Fourier coefficients of the function

$$f_n(x) = f(x), \quad x \in D_n, \tag{2.8}$$

with respect to orthonormal system (2.7), i.e. we set

$$\hat{f}_n(k) = 2^{-\frac{n}{2}} \int\limits_{D_n} f_n(t) \psi_{k2^{-n}}(t) dt, \quad k \in Z_+. \tag{2.9}$$

Lemma 2.5. *For any function $f \in L^p(D_n)$, $1 \le p < \infty$, $n \in Z_+$, the subsequence $\{S_{2^m}(f)\}_{m=0}^{\infty}$ of its Fourier sums with respect to the system (2.7)* *converges to the function f in the space $L^p(D_n)$ and almost everywhere on D_n. Moreover, the inequality*

$$\|S_{2^{n+m}}(f) - f\|_p \leq 2^{1/p}\omega_p(f, 2^{-m}) \tag{2.10}$$

holds, where

$$\omega_p(f,\delta) = \sup_{0\leq u\leq\delta} \int\limits_0^{2^n-u} |f(x+u) - f(x)|dx, \quad \delta \in D_n$$

is integral modulus of continuity of the function f in the space $L^p(D_n)$.

Proof. For the case $p = 1$ this statement was proved in [8], (Ch. 2, Lemma 5.3). For the case $1 \leq p < \infty$ we will prove it by similar way. In [8] (p. 64) the following equality

$$S_{2^{n+m}}(f,x) = 2^m \int\limits_{I_j^m} f(t)dt \tag{2.11}$$

has been proved, where I_j^m is dyadic interval of range m, containing the point $x \in D_n$. It follows from (2.11) that the subsequence $\{S_{2^m}(f)\}_{m=0}^{\infty}$ converges to f almost everywhere on D_n. Let us prove the inequality (2.10).

For $i = 0, 1, \ldots, 2^n - 1$ and $I_j^m = \left[i + \frac{j}{2^m}, i + \frac{j+1}{2^m}\right)$ it follows from the equality (2.11)

$$\int\limits_i^{i+1} |S_{2^{n+m}}(x,f) - f(x)|^p\, dx = \sum_{j=0}^{2^m-1} \int\limits_{I_j^m} |S_{2^{n+m}}(x,f) - f(x)|^p\, dx = \tag{2.12}$$

$$= \sum_{j=0}^{2^m-1} \int\limits_{I_j^m} \frac{1}{|I_j^m|} \left| \int\limits_{I_j^m} (f(t) - f(x))dt \right|^p dx \leq$$

$$\leq \sum_{j=0}^{2^m-1} \int\limits_{I_j^m} \frac{1}{|I_j^m|} \int\limits_{I_j^m} |(f(t) - f(x))|^p\, dtdx.$$

Using the identity of P.L. Ulyanov (see [7], p. 223])

$$\int\limits_a^b \int\limits_a^b |f(x) - f(y)|^p dxdy = 2 \int\limits_0^{b-a} \left\{ \int\limits_a^{b-t} |f(y+t) - f(y)|^p dy \right\} dt$$

in the case $[a, b) = I_j^m$, we have from (2.12) the inequality

$$\int\limits_i^{i+1} |S_{2^{n+m}}(x, f) - f(x)|^p \, dx \leq$$

$$\leq \sum_{j=0}^{2^m-1} 2^{m+1} \int\limits_0^{2^{-m}} \left\{ \int\limits_{i+\frac{j}{2^m}}^{i+\frac{j+1}{2^m}-u} |f(y+u) - f(y)|^p dy \right\} du =$$

$$= 2^{m+1} \int\limits_0^{2^{-m}} \left\{ \int\limits_i^{i+1-u} |f(y+u) - f(y)|^p dy \right\} du.$$

By summing these inequalities over $i = 0, 1, \ldots, 2^n - 1$, we obtain the inequality

$$\|S_{2^{n+m}}(f) - f\|_p^p \leq 2^{m+1} \int\limits_0^{2^{-m}} \left\{ \int\limits_0^{2^n-u} |f(y+u) - f(y)|^p dy \right\} du.$$

From here it follows easily

$$\|S_{2^{n+m}}(f) - f\|_p \leq 2^{1/p} \omega_p \left(f, 2^{-m}\right).$$

Since $f \in L^p(D_n)$, we conclude from the last inequality that the sequence $S_{2^m}(f)$ converges to the function f in the space $L^p(D_n)$.

The proof of the last Lemma 2.6 is based on the following theorem of Paley (see e.g. [13], p. 182).

Theorem D. *Let $\{\varphi_k(x)\}_{k=0}^\infty$ be orthonormal uniformly bounded system on the interval $[a, b]$ satisfying the inequality $|\varphi_k(x)| \leq M,\ x \in [a, b],\ k \in Z_+$. Then there exists a constant $0 < A_p < \infty$ such that the Fourier coefficients $\left\{\hat{f}(k)\right\}_{k=0}^\infty$ of any function $f \in L^p[a, b],\ 1 < p \leq 2$, with respect to this system satisfy the inequality*

$$\left|\hat{f}(0)\right|^p + \sum_{k=1}^\infty k^{p-2} \left|\hat{f}(k)\right|^p \leq A_p M^{2-p} \|f\|_{L^p[a,b]}^p,$$

where the constant A_p depends only on p.

Lemma 2.6. *If $f \in L^p(D_n)$, $1 < p \le 2$, $n \in Z_+$, then for the Fourier coefficients (2.9) of the function (2.8) the inequality*

$$\left|\hat{f}_n(0)\right|^p + \sum_{k=1}^{\infty} \frac{\left|\hat{f}_n(k)\right|^p}{k^{2-p}} \le A_p \cdot 2^{\left(\frac{p}{2}-1\right)n} \|f\|^p_{L^p(D_n)} \tag{2.13}$$

is valid.

Proof. The inequality (2.13) follows from Theorem D since the functions of the orthonormal system (2.7) are bounded by the constant $2^{-n/2}$ on the interval $D_n = [0, 2^n)$.

Proof of Theorem C. Let us set

$$f_n(x) = f(x) X_{D_n}(x), \quad x \in R_+, \tag{2.14}$$

where $X_E(x)$ is the characteristic function of the set $E \subset R_+$. The symbol f_n below will denote both the function (2.14) and its restriction (2.8) on the interval D_n. The meaning of the notation will be clear from the context.

By the Lemma 2.5 the equality

$$f_n(x) = \hat{f}_n(0) 2^{-n/2} X_{D_n}(x) + \sum_{i=0}^{\infty} \left\{ \sum_{k=2^i}^{2^{i+1}-1} \hat{f}_n(k) 2^{-\frac{n}{2}} \psi_{k2^{-n}}(x) \right\} X_{D_n}(x) \tag{2.15}$$

is valid, where the series on the right-hand side converges both in the space $L^p(R_+)$, $1 < p \le 2$, and almost everywhere on R_+. By applying the Walsh transform to both sides of the equality (2.15) and using Lemma 2.4, we obtain

$$\tilde{f}_n(x) = 2^{\frac{n}{2}} \hat{f}_n(0) X_{[0,2^{-n})}(x) + \sum_{i=0}^{+\infty} \left\{ 2^{\frac{n}{2}} \sum_{k=2^i}^{2^{i+1}-1} \hat{f}_n(k) X_{[k2^{-n},(k+1)2^{-n})}(x) \right\}, \tag{2.16}$$

where the series in the right-hand side of (2.16) converges to $\tilde{f}_n(x)$ everywhere on R_+. Since the intervals $I_k^n = [k2^{-n}, (k+1)2^{-n})$, $k \in Z_+$, are mutually disjoint, the brackets in the right-hand side of the equality (2.16) can be omitted. After that we obtain the equality

$$\tilde{f}_n(x) = 2^{\frac{n}{2}} \hat{f}_n(0) X_{[0,2^{-n})}(x) + 2^{\frac{n}{2}} \sum_{k=1}^{\infty} \hat{f}_n(k) X_{[k2^{-n},(k+1)2^{-n})}(x),$$

where the series in the right-hand side converges to $\tilde{f}_n(x)$ everywhere on R_+. From this equality we obtain easily

$$x^{p-2}\left|\tilde{f}_n(x)\right|^p = x^{p-2}\left|\hat{f}_n(0)\right|^p 2^{\frac{n}{2}p} X_{[0,2^{-n})}(x) +$$

$$+2^{\frac{n}{2}p}\sum_{k=1}^{\infty}\left|\hat{f}_n(k)\right|^p x^{p-2} X_{[k2^{-n},(k+1)2^{-n})}(x). \tag{2.17}$$

If we integrate the equality (2.17) over D_n, we have

$$\int\limits_{D_n}\frac{\left|\tilde{f}_n(x)\right|^p}{x^{2-p}}dx = 2^{n(1-p/2)}\left\{\frac{\left|\hat{f}_n(0)\right|^p}{p-1} + \sum_{k=1}^{2^n-1}\frac{\left|\hat{f}_n(k)\right|^p}{p-1}\left[(k+1)^{p-1} - k^{p-1}\right]\right\}.$$

Since $(k+1)^{p-1} - k^{p-1} \sim (p-1)k^{p-2}$ as $k \to +\infty$, we obtain from the previous equality

$$\int\limits_{D_n}\frac{\left|\tilde{f}_n(x)\right|^p}{x^{2-p}}dx \le C_p 2^{n(1-p/2)}\left\{\left|\hat{f}_n(0)\right|^p + \sum_{k=1}^{2^n-1}\frac{\left|\hat{f}_n(k)\right|^p}{k^{2-p}}\right\}. \tag{2.18}$$

By integrating the equality (2.17) over the set $[2^n, +\infty)$ we have

$$\int\limits_{2^n}^{+\infty} x^{p-2}\left|\tilde{f}_n(x)\right|^p dx = 2^{\frac{n}{2}p}\sum_{k=2^n}^{\infty}\left|\hat{f}_n(k)\right|^p \int\limits_{k2^{-n}}^{(k+1)2^{-n}} x^{p-2}dx =$$

$$= 2^{n(1-p/2)}\sum_{k=2^n}^{\infty}\left|\hat{f}_n(k)\right|^p \frac{\left[(k+1)^{p-1} - k^{p-1}\right]}{p-1} \le$$

$$\le C_p 2^{n(1-p/2)}\sum_{k=2^n}^{\infty}\left|\hat{f}_n(k)\right|^p k^{p-2}. \tag{2.19}$$

By summing the inequalities (2.18) and (2.19) we obtain

$$\int\limits_{R_+}\frac{\left|\tilde{f}_n(x)\right|^p}{x^{2-p}}dx \le C_p 2^{n(1-p/2)}\left\{\left|\hat{f}_n(0)\right|^p + \sum_{k=1}^{\infty}\frac{\left|\hat{f}_n(k)\right|^p}{k^{2-p}}\right\}. \tag{2.20}$$

From (2.20) by the Lemma 2.6 we have the inequality

$$(2.21) \qquad \int\limits_{R_+} \frac{\left|\tilde{f}_n(x)\right|^p}{x^{2-p}} dx \le A_p C_p \|f\|^p_{L^p(D_n)} \le A_p C_p \|f\|_{L^p(R_+)}.$$

Since $f \in L^p(R_+),\ 1 < p \le 2$, so $\lim\limits_{n\to\infty} \|\tilde{f}_n - \tilde{f}\|_q = 0$, where $1/p + 1/q = $ $= 1$. Hence there exists a sequence of natural numbers $n_k \to +\infty$, such that $\tilde{f}_{n_k}(x) \to \tilde{f}(x)$ as $k \to +\infty$ for almost all $x \in R_+$. If we substitute the index n by n_k in the left-hand side of the inequality (2.21) and apply the theorem of Fatou (see [14], p. 133), we obtain the inequality (2.6) with the constant $B_p = A_p C_p$.

For the proof of the Theorem 3.1 below we will use the following

Theorem E. *If $f \in L^p(R_+),\ 1 < p \le \infty$ and $F(x) = x^{-1} \int\limits_0^x f(t)dt$, then $F \in L^p(R_+)$.*

This theorem was proved by G.H. Hardy (see [12], Theorem 327).

3. Proof of main results

In Theorems 3.1 and 3.2 below the Walsh transform is defined by the equality (2.3), where the integral is understood as improper with singular point $+\infty$.

Theorem 3.1.* *Let the function $f(x)$ be defined and nonincreasing on R_+ and $f(x) \to 0$ as $x \to +\infty$. Then the improper integral in right-hand side of (2.3) converges at each point $x > 0$, and if $f(x)x^{1-2/p} \in L^p(R_+)$, where $1 < p < 2$, then $\tilde{f}(x) \in L^p(R_+)$.*

Proof. For the Walsh kernel (2.2) at each point (x, y), where $x > 0$ and $y \in R_+$ the inequality

$$(3.1) \qquad \left|\int\limits_0^y \psi(x,t)dt\right| \le 3/x$$

* This theorem is a counterpart to a theorem of Hardy and Littlewood on classical Fourier transform (see, e.g. [1], Theorem 82).

holds (see [3], p. 428). Therefore if the function $f(x)$ is nonincreasing on R_+ and $f(x) \to 0$ as $x \to +\infty$, the improper integral

$$\int_{R_+} f(y)\psi(x,y)dy$$

with singular point $+\infty$ converges at each point $x > 0$. Hence the Walsh transform $\tilde{f}(x)$ is defined by the equality (2.3) at all points $x > 0$.

Let us assume that the function f satisfies the condition

$$f(x)x^{1-2/p} \in L^p(R_+), \quad 1 < p < 2. \tag{3.2}$$

Let us write $\tilde{f}(x)$ as the sum of two summands as follows

$$\tilde{f}(x) = \int_0^{1/x} f(y)\psi(x,y)dy + \int_{1/x}^{+\infty} f(y)\psi(x,y)dy \equiv F_1(x) + F_2(x),\ x > 0. \tag{3.3}$$

Now we will prove that $F_1, F_2 \in L^p(R_+)$. According to the mean value theorem of Bonne there exists a point $\xi > 1/x$ such that

$$F_2(x) = f(1/x) \int_{1/x}^{\xi} \psi(x,y)dy.$$

Hence it follows from (3.1)

$$|F_2(x)| \le 6f(1/x)/x, \quad x > 0.$$

Therefore we obtain from (3.2)

$$\int_{R_+} |F_2(x)|^p dx \le 6^p \int_{R_+} [f(1/x)/x]^p dx = 6^p \int_{R_+} [f(y)]^p y^{p-2} dy < \infty, \tag{3.4}$$

that is $F_2 \in L^p(R_+)$.

Let us prove now that $F_1 \in L^p(R_+)$. Since $\psi(x,y) = \pm 1$ (see(2.2)), then the equality

$$|F_1(x)| \le \int_0^{1/x} f(1/y)dy, \quad x > 0$$

is valid. Hence we have

$$(3.5) \qquad \int\limits_{R_+} |F_1(x)|^p dx \le \int\limits_{R_+} \left(\int\limits_0^{1/x} [f(y)]dy \right)^p dx = \int\limits_{R_+} \left(\frac{1}{t^{2/p}} \int\limits_0^t [f(y)]dy \right)^p dt.$$

We note that the condition (3.2) is equivalent to

$$(3.6) \qquad g(x) :\equiv f(x)x^{1-2/p} \in L^p(R_+).$$

Since $f(x) = g(x)x^{2/p-1}$,

$$y^{-2/p} \int\limits_0^y [f(t)]dt = y^{-2/p} \int\limits_0^y g(t)t^{\frac{2}{p}-1}dt \le y^{-1} \int\limits_0^y g(t)dt \equiv G(y).$$

As long as $1 < p < 2$, by the Theorem E of Hardy it follows from the condition (3.6) that $G(y) \in L^p(R_+)$. Then from (3.5) we have $F_1 \in L^p(R_+)$.

Thus it is proved that both summands in right-hand side of the equality (3.3) belong to the space $L^p(R_+)$. Hence the inclusion $\tilde{f} \in L^p(R_+)$ is proved.

The following theorem may be considered as an analog of a theorem of Titchmarsh relating to classical Fourier transform (see [1], Theorem 83).

Theorem 3.2. *Let the function $\varphi \in L^p(R_+)$, $1 < p < 2$, has the strong dyadic integral $I^p_{(2/p)-1}(\varphi) \equiv f$. Then the inclusion $\tilde{f} \in L^p(R_+)$ holds.*

Proof. According to the Lemma 2.2 the equality $\tilde{f}(x) = \tilde{\phi}(x)x^{1-2/p}$ is valid almost everywhere on R_+. If $\phi \in L^p(R_+)$, $1 < p < 2$, then by the Theorem C we have $\tilde{\varphi}(x)x^{1-2/p} \in L^p(R_+)$, that is $\tilde{f} \in L^p(R_+)$.

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