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THE θ–SUMMATION ON LOCAL FIELDS

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Dedicated to
Prof. Ferenc Schipp on the occasion of his 70th birthday
and to
Prof. Péter Simon on the occasion of his 60th birthday

Abstract. We will introduce the definition of the θ-summation on local fields. It will be given a convolution form of the partial sums of the θ-summation. It will be shown examples for the θ-summation in the dyadic and in the arithmetic case.

1. Preliminaries

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}_+$ represent the set of nonnegative real numbers and denote $(\mathbb{B}, \overset{\circ}{+}, \circ)$ the dyadic (logical) field, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ the arithmetic field and $(\mathbb{R}, +, \cdot)$ the usual real field.

Recall that $\mathbb{B}$ is the set of binary infinite sequence $\underline{x} = (x_n, n \in \mathbb{Z})$, $x_n \in \in \{0, 1\}$ satisfying $\lim\limits_{n \to -\infty} x_n = 0$, $\mathbb{B}^* := \mathbb{B} \setminus \{\underline{\Theta}\}$ ($\underline{\Theta} = (\Theta_n = 0, n \in \mathbb{Z})$).

The *norm* on $\mathbb{B}$ is defined by

$$\|\underline{x}\| := 2^{-\pi(\underline{x})} \quad (\underline{x} \in \mathbb{B}), \tag{1}$$

where $\pi(\underline{\Theta}) := +\infty$ and for $\underline{x} \in \mathbb{B}^*$, $\pi(\underline{x}) := n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$.

The dyadic sum and dyadic product of $\underline{x}, \underline{t} \in \mathbb{B}$ is defined as

(2)
$$(\underline{x} \overset{\circ}{+} \underline{t})_n := x_n + t_n \pmod 2, \quad (\underline{x} \circ \underline{t})_n = \sum_{j=-\infty}^{\infty} x_j t_{n-j} \pmod 2 \quad (n \in \mathbb{Z}).$$

The arithmetic sum $\underline{x} \overset{\bullet}{+} \underline{t}$ of $\underline{x}, \underline{t} \in \mathbb{B}$ is defined as

(3)
$$(\underline{x} \overset{\bullet}{+} \underline{t})_n := r_n \qquad (n \in \mathbb{Z}),$$

where the bits $q_n, r_n \in \{0, 1\}$ $(n \in \mathbb{Z})$ are defined recursively using the additive digits of $\underline{x}, \underline{t}$ as follows

$$q_n = r_n = 0 \quad \text{for} \quad n < m := \min\{\pi(\underline{x}), \pi(\underline{t})\},$$

and

$$x_n + t_n + q_{n-1} = 2q_n + r_n \quad \text{for} \quad n \geq m.$$

The arithmetic product $\underline{x} \bullet \underline{t}$ of the elements $\underline{x}, \underline{t} \in \mathbb{B}^*$ is defined as

(4)
$$(\underline{x} \bullet \underline{t})_n := r_n \qquad (n \in \mathbb{Z}),$$

where the bits $q_n, r_n \in \{0, 1\}$ $(n \in \mathbb{Z})$ are defined recursively using the additive digits of $\underline{x}, \underline{t}$ as follows

$$q_n = r_n = 0 \quad \text{for} \quad n < m := \min\{\pi(\underline{x}), \pi(\underline{t})\},$$

and

$$\sum_{j=-\infty}^{\infty} x_j t_{n-j} + q_{n-1} = 2q_n + r_n \quad \text{for} \quad n \geq m.$$

Also define $\underline{x} \bullet \underline{\Theta} = \underline{\Theta}$ for all $\underline{x} \in \mathbb{B}$.

The fundamental sequence in $\mathbb{B}$ is the sequence $\underline{e}^n := (\delta_{nj}, j \in \mathbb{Z})$, defined for each $n \in \mathbb{Z}$, where δ_{nj} represents the Kronecker delta.

The zero element of $\mathbb{B}$ is the sequence $\underline{\Theta}$ in both cases. The dyadic additive inverse of $\underline{x} \in \mathbb{B}$ is itself, the arithmetic additive inverse of $\underline{x} \in \mathbb{B}$ is its reflection, $\underline{x}^-$ ($\underline{x}_j^- = 0$ for $j \leq \pi(x)$ and $\underline{x}_j^- = 1 - \underline{x}_j$ for $j > \pi(x)$). The identity of the dyadic and arithmetic multiplication is $\underline{e} := \underline{e}^0$ (see Schipp-Wade [1]). The multiplicative inverse of $\underline{x} \in \mathbb{B}^* := \mathbb{B}\backslash\{\underline{\Theta}\}$ will be denoted by x° and $x^\bullet$, respectively.

The norm (1) is non-Archimedean and multiplicative in both cases, i.e.

$$\begin{aligned} &\|\underline{x} \overset{\circ}{+} \underline{t}\| \leq \max\{\|\underline{x}\|, \|\underline{t}\|\}, \quad \|\underline{x} \circ \underline{t}\| = \|\underline{x}\| \cdot \|\underline{t}\| \quad (\underline{x}, \underline{t} \in \mathbb{B}), \\ &\|\underline{x} \overset{\bullet}{+} \underline{t}\| \leq \max\{\|\underline{x}\|, \|\underline{t}\|\}, \quad \|\underline{x} \bullet \underline{t}\| = \|\underline{x}\| \cdot \|\underline{t}\| \quad (\underline{x}, \underline{t} \in \mathbb{B}). \end{aligned}$$

The functions $\rho^\circ(\underline{x}, \underline{t}) := \|\underline{x} \overset{\circ}{+} \underline{t}\|$ $(\underline{x}, \underline{t} \in \mathbb{B})$, $\rho^\bullet(\underline{x}, \underline{t}) := \|\underline{x} \overset{\bullet}{+} \underline{t}^-\|$ $(\underline{x}, \underline{t} \in \mathbb{B})$ are metrics on $\mathbb{B}$. The dyadic addition and dyadic multiplication are continuous with respect to the metric ρ°, arithmetic addition and arithmetic multiplication are continuous with respect to the metric $\rho^\bullet$.

Thus $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ are locally compact topological fields (see Schipp-Wade [1]).

The interval with rank $n \in \mathbb{Z}$, and center $\underline{x} \in \mathbb{B}$ is defined as

$$I_n(\underline{x}) := \{\underline{t} \in \mathbb{B} : \ x_j = t_j \ (j < n)\}.$$

It is known (see Schipp-Wade [1]) that the image of every interval under an addition and under a multiplication is again an interval in both cases. In fact, if $J = I_n(\underline{b})$ $(n \in \mathbb{Z},\ \underline{a}, \underline{b} \in \mathbb{B})$, then

$$\begin{aligned} &\underline{a} \overset{\circ}{+} J := \{\underline{a} \overset{\circ}{+} \underline{x} : \underline{x} \in J\} = I_n(\underline{a} \overset{\circ}{+} \underline{b}), \\ &\underline{a} \circ J := \{\underline{a} \circ \underline{x} : \underline{x} \in J\} = I_{n+\pi(\underline{a})}(\underline{a} \circ \underline{b}), \\ &\underline{a} \overset{\bullet}{+} J := \{\underline{a} \overset{\bullet}{+} \underline{x} : \underline{x} \in J\} = I_n(\underline{a} \overset{\bullet}{+} \underline{b}), \\ &\underline{a} \bullet J := \{\underline{a} \bullet \underline{x} : \underline{x} \in J\} = I_{n+\pi(\underline{a})}(\underline{a} \bullet \underline{b}). \end{aligned} \tag{5}$$

Thus the measure μ defined on the set of intervals by

$$\mu(I_n(\underline{x})) := 2^{-n} \quad (n \in \mathbb{Z}, \underline{x} \in \mathbb{B})$$

is translation invariant and the extension of μ to the Borel-sets of $\mathbb{B}$ coincides with the Haar-measure of $(\mathbb{B}, \overset{\circ}{+})$ and $(\mathbb{B}, \overset{\bullet}{+})$ normalized by $\mu(I_0(\Theta)) = 1$.

The set $\mathbb{B}$ can be identified with the set of nonnegative real numbers $\mathbb{R}_+$ and $\mathbb{B}^*$ with $(0, \infty)$. This identification gives a useful interpretation of the concepts defined above. Define the map $\alpha : \underline{x} \mapsto x$ $(\underline{x} \in \mathbb{B}, x \in \mathbb{R}_+)$ by

$$\alpha(\underline{x}) := x := \sum_{n=-\infty}^{\infty} x_n 2^{-n-1}.$$

From this point of view, we call x_n $(n \in \mathbb{Z})$ the binary coefficients of x. The map α takes $I_n(\underline{x})$ to a dyadic interval of the form $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$ $(n \in \mathbb{Z})$, where $k = \sum_{j=-\infty}^{n-1} x_j 2^{-j-1}$. The map α is 1-1 except for a denumerable set of $\mathbb{B}$. All definitions above (dyadic addition, multiplication, etc.) correspond to a similar one defined on $\mathbb{R}_+$. That is why we can and will work on $\mathbb{R}_+$ except for $\mathbb{B}$ in the following.

In the sequel w_x $(x \in \mathbb{R}_+)$ stands for the generalized Walsh function defined on $\mathbb{R}_+$. We recall (see Schipp-Wade-Simon [2], Chapter 9 and [1], Chapter 3) that the generalized Walsh function w_x is defined by

$$(6) \qquad w_x(t) := (-1)^{(x \circ t)_{-1}}, \quad (x \circ t)_{-1} = \sum_{n=-\infty}^{\infty} x_n t_{-1-n} \pmod 2,$$

where $x_n, t_n \in \{0,1\}$ are the binary coefficients in the dyadic representation. If $x = n \in \mathbb{N} := \{0,1,2,\cdots\}$ is a natural number, then w_n coincides with the 1-periodic extension of the corresponding Walsh-Paley function. The functions w_y $(y \in \mathbb{R}_+)$ behave like characters with respect to dyadic addition, i.e. if $x, t \in \mathbb{R}_+$ and $x \overset{\circ}{+} t$ is dyadic irrational then

$$(7) \qquad w_y(x \overset{\circ}{+} t) = w_y(x) w_y(t).$$

The function

$$(8) \qquad w(t) := e^{2\pi\imath \frac{t_{-1}}{2}} = (-1)^{t_{-1}} \quad (t \in \mathbb{R}_+)$$

is called the basic character of the additive group of dyadic field. The Walsh-Paley functions can be expressed by the basic character in the form

$$(9) \qquad w_x(t) := w(x \circ t) \quad (x, t \in \mathbb{R}_+)$$

and consequently

$$w_y(x) = w_x(y), \quad w_{t\circ x}(y) = w_x(t \circ y), \quad w_x(y) = w_{[x]}(y) w_{[y]}(x) \quad (x, y, t \in \mathbb{R}_+),$$

where $[x]$ is the integer part of $x \in \mathbb{R}_+$.

In the sequel v_x $(x \in \mathbb{R}_+)$ defined on $\mathbb{R}_+$ stands for the characters of the additive group of arithmetic field. We recall (see Schipp-Wade [1], Chapter 2) that the basic character in this case is

$$(10) \qquad v(t) := e^{2\pi\imath \left(\frac{t_{-1}}{2} + \frac{t_{-2}}{2^2} + \cdots\right)} \quad (t \in \mathbb{R}_+)$$

and v_x $(x \in \mathbb{R}_+)$ can be expressed in the form

$$v_x(t) := v(x \bullet t) = e^{2\pi\imath\left(\frac{(x\bullet t)_{-1}}{2}+\frac{(x\bullet t)_{-2}}{2^2}+\cdots\right)} \quad (t \in \mathbb{R}_+), \tag{11}$$

where $(x \bullet t)_k$ $(k \in \mathbb{Z})$ are the binary coefficients of $x \bullet t$ in the dyadic representation. If $x = n \in \mathbb{N} := \{0, 1, 2, \cdots\}$ is a natural number, then v_n coincides with the 1-periodic extension of the characters of the subgroup $([0, 1), \overset{\bullet}{+})$. The functions v_x has also the character property, if $x, t \in \mathbb{R}_+$ and $x \overset{\bullet}{+} t$ is dyadic irrational, then

$$v_y(x \overset{\bullet}{+} t) = v_y(x)v_y(t). \tag{12}$$

Moreover

$$v_y(x) = v_x(y), \quad v_{t\bullet x}(y) = v_x(t \bullet y), \quad v_x(y) = v_{[x]}(y)v_{[y]}(x') \quad (x, y, t \in \mathbb{R}_+),$$

where $[x]$ is the integer part of $x \in \mathbb{R}_+$, $x' = x - [x]$.

In the sequel ϵ_x $(x \in \mathbb{R})$ defined on $\mathbb{R}$ stands for the characters of the real field $(\mathbb{R}, +)$. We recall (see Schipp-Wade [1], Chapter 3) that in this case the basic character is

$$\epsilon(t) := e^{2\pi\imath t} \quad (t \in \mathbb{R}) \tag{13}$$

and ϵ_x $(x \in \mathbb{R})$ is defined by

$$\epsilon_x(t) := \epsilon(x \cdot t) = e^{2\pi\imath xt} \quad (t \in \mathbb{R}). \tag{14}$$

If $x = n \in \mathbb{Z} := \{0, \pm1, \pm2, \cdots\}$ is an integer, then ϵ_n coincides with the 1-periodic extension of the characters of the subgroup $([0, 1), +\text{mod } 1)$. The functions ϵ_x have also the character property, if $x, t \in \mathbb{R}$, then

$$\epsilon_y(x + t) = \epsilon_y(x)\epsilon_y(t). \tag{15}$$

Let $L^p = L^p(\mathbb{R}_+)$ $(1 \le p \le \infty)$ denote the usual Banach spaces. We have introduced three fields (the real field, the logical field and the arithmetic field) and thus have three distinct Fourier transforms. Let $(\mathbb{F}, +, \cdot)$ represent one of the fields. Let $\mathbb{F}^\sharp$ denote $\mathbb{Z}$, when $\mathbb{F} = \mathbb{R}$, $\mathbb{N}$, when $\mathbb{F} = \mathbb{R}$.

Given $f \in L^1 := L^1(\mathbb{F})$ the Fourier transform of f is the function on $\mathbb{F}$ defined by

$$(\mathcal{F}f)(x) := \int_{\mathbb{F}} f(t)\overline{u}(x \cdot t)dt \quad (x \in \mathbb{F}), \tag{16}$$

where u is the basic character of $(\mathbb{F}, +)$, and $\overline{u}$ is the complex conjugation of u. Specially the trigonometric Fourier transform is denoted by

$$\hat{f}(x) := \int_{-\infty}^{\infty} f(t)\overline{\epsilon}(x \cdot t)dt \quad (x \in \mathbb{R}), \tag{17}$$

the Walsh-Fourier transform is denoted by

$$f^{\circ}(x) := \int_{0}^{\infty} f(t)w(x \circ t)dt \quad (x \in \mathbb{R}_{+}), \tag{18}$$

and the Fourier transform in the arithmetic case is denoted by

$$f^{\bullet}(x) := \int_{0}^{\infty} f(t)\overline{v}(x \bullet t)dt \quad (x \in \mathbb{R}_{+}). \tag{19}$$

It is known that, if $f, \mathcal{F}f \in L^1$, then the inversion formula is true. Namely,

$$f(x) = \int_{\mathbb{F}} (\mathcal{F}f)(t)u(x \cdot t)dt \quad \text{for } a.a. \ \ x \in \mathbb{F} \tag{20}$$

(see Schipp-Wade-Simon [2]).

The **dilation operator** generated by $b \in \mathbb{F}^*$ is defined by

$$(\delta_b f)(x) := f(x \cdot b) \quad (x \in \mathbb{F}),$$

and the **translation operator** generated by $h \in \mathbb{F}$ is defined by

$$(\tau_h f)(x) := f(x + h) \quad (x \in \mathbb{F}).$$

It is known (see Schipp-Wade [1]) that

$$(\mathcal{F} \circ \delta_b)(f) = \frac{1}{\|b\|} (\delta_{b^{-1}} \circ \mathcal{F})(f) \quad (f \in L^1(\mathbb{F}), \ b \in \mathbb{F}^*). \tag{21}$$

2. The θ-summation

We begin with the definition of the *θ-summation.* We will do it with a function $\theta : [0, \infty) \to \mathbb{R}$. We consider only functions θ with the property:

$$\sum_{k=0}^{\infty} |\theta(k \circ 2^{-n})| = \sum_{k=0}^{\infty} |\theta(k \cdot 2^{-n})| < \infty \qquad (n \in \mathbb{N}). \tag{22}$$

Definition 1. *Let $z = (z_n, n \in \mathbb{N})$ be a bounded real sequence, and θ a function with the property (22). The series*

$$\sum_{k=0}^{\infty} z_k \tag{23}$$

*is called θ-**summable**, if the sequence*

$$t_n := \sum_{k=0}^{\infty} \theta\left(k \cdot 2^{-n}\right) z_k \quad (n \in \mathbb{N}) \tag{24}$$

*is convergent. The limit of the sequence $(t_n, n \in \mathbb{N})$ is called the θ-**sum** of the series (23).*

We examine, when will this summation be permanent. Assume that the series (23) is convergent, and use the following notations:

$$s_0 := 0, \quad s_k := \sum_{j=0}^{k-1} z_j \quad (k \in \mathbb{N}^*). \tag{25}$$

After applying the Abel-rearrangment for (24), we get

$$t_n = \sum_{k=1}^{\infty} \left(\theta\left((k-1) \cdot 2^{-n}\right) - \theta\left(k \cdot 2^{-n}\right)\right) s_k \quad (n \in \mathbb{N}). \tag{26}$$

Applying the theorem of Toeplitz (see Zygmund [6]) we get, that the summation (24) is permanent if and only if for the double series

$$\alpha_{nk} := \theta\left((k-1) \cdot 2^{-n}\right) - \theta\left(k \cdot 2^{-n}\right) \quad (k \in \mathbb{N}^*, n \in \mathbb{N}), \tag{27}$$

the conditions

$$(28)\quad \sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}|\alpha_{nk}|<\infty,\qquad \lim_{n\to\infty}\sum_{k=1}^{\infty}\alpha_{nk}=1,\qquad \lim_{n\to\infty}\alpha_{nk}=0\qquad (k\in\mathbb{N}^*),$$

hold. Since

$$(29)\qquad \sum_{k=1}^{\infty}\alpha_{nk}=\theta(0),$$

it follows that if for the function θ condition (22) hold and

$$(30)\qquad i)\ \ \theta\in BV(\mathbb{R}_+),\qquad ii)\ \ \theta(0)=1,\qquad iii)\ \ \lim_{t\to 0+}\theta(t)=\theta(0)$$

are satisfied then the summation, which is generated by θ is permanent. Here BV denotes the set of functions with bounded variation on $\mathbb{R}^+$.

Assume now that the double sequence $\theta(k\cdot 2^{-n})$ has the following property:

$$(31)\qquad \theta\left(2^k\cdot 2^{-n}\right)=\theta\left((2^k+\ell)\cdot 2^{-n}\right)\qquad (0\le \ell<2^k,\ \ k,n\in\mathbb{N}).$$

In such a case using again the Abel arrangement, we get

$$(32)\quad t_n=\theta(0)s_1+\sum_{k=0}^{\infty}\theta(2^k\cdot 2^{-n})\sum_{\ell=2^k}^{2^{k+1}-1}z_\ell=\theta(0)s_1+\sum_{k=0}^{\infty}\theta(2^k\cdot 2^{-n})(s_{2^{k+1}}-s_{2^k})$$

for all $n\in\mathbb{N}$, and so

$$(33)\qquad t_n=\theta(0)s_1+\sum_{k=1}^{\infty}\left(\theta(2^{k-1}\cdot 2^{-n})-\theta(2^k\cdot 2^{-n})\right)s_{2^k}-\theta(2^{-n})s_1.$$

It follows from the Toeplitz-theorem that, if the function θ fulfils the conditions (30) and (27), then

$$(34)\qquad \lim_{n\to\infty}s_{2^n}=s\quad\Rightarrow\quad \lim_{n\to\infty}t_n=s\quad (s\in\mathbb{R}).$$

It is easy to see that, if $\theta\,(x)=\theta\,(\|x\|)$ $(x\in\mathbb{R}^+)$, then θ has the property (31).

3. θ-summation of Fourier-series

Let $f : [0,1) \to \mathbb{R}$ be an integrable function, and θ a function with the property (22). We will denote the Walsh-Fourier coefficients of f with $c_k^w(f)$ ($k \in \mathbb{N}$), the Fourier-coefficients of f with respect to the system $(v_k, k \in \mathbb{N})$ with $c_k^v(f)$, and the trigonometric Fourier coefficients of f with $c_k^\epsilon(f)$ ($k \in \mathbb{Z}$).

Let $(u_k, k \in \mathbb{F}^\sharp)$ represent one of the three systems and denote

$$c_k(f) = \int\limits_0^1 f(t)\overline{u}_k(t)dt \quad (k \in \mathbb{F}^\sharp)$$

the Fourier coefficient of f with respect to this system. The linear combinations of the functions u_k ($k \in \mathbb{F}^\sharp$) are called the trigonometric polynomial of $\mathbb{F}$.

For $1 \le p \le \infty$ denote X_p the closure in L^p-norm of the trigonometric polynomial of $\mathbb{F}$. It is known that $X_p = L^p[0,1)$ if $p < \infty$. In the case $\mathbb{F} = \mathbb{B}$ the space X_∞ is the set of W-continuous functions and if $\mathbb{F} = \mathbb{R}$ then X_∞ is the collection of one periodic continuous functions. The norm of X_p will be denoted by $\|\cdot\|_p$.

The Fourier series of f is called **θ-summable**, if the sequence

$$\sigma_n^\theta(f) := \sum_{k\in\mathbb{F}^\sharp} c_k(f)\theta\left(k\cdot 2^{-n}\right)u_k \quad (n \in \mathbb{F}^\sharp) \tag{35}$$

converges. For the trigonometric system see [3], [4].

We will show that σ_n^θ can be expressed as a convolution operator:

Theorem 1. *Assume that the function θ fulfils the conditions (30) and (22). If $\theta, \hat{\theta} \in L^1_{\mathbb{F}}$ and $f \in X_p$ $(1 \le p \le \infty)$, then*

$$\left(\sigma_n^\theta f\right)(x) = 2^n \int\limits_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t)dt \qquad (x \in [0,1)), \tag{36}$$

and for all $f \in X_p$

$$\|\sigma_n^\theta f - f\|_p \to 0 \qquad (n \to \infty). \tag{37}$$

Proof. We will use the following notation:

$$\left(V_n^\theta f\right)(x) = 2^n \int\limits_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t)dt \qquad (x \in \mathbb{F}).$$

The operators V_n^θ, $\sigma_n^\theta : X_p \to X_p$ are bounded. Indeed, since $|c_n(f)| \le \|f\|_p$ for the operator σ_n^θ we get

$$\|\sigma_n^\theta f\|_p \le \sum_{k=0}^{\infty} |c_k(f)| \left|\theta\left(k \cdot 2^{-n}\right)\right| \le \|f\|_p \sum_{k=0}^{\infty} \left|\theta\left(k \cdot 2^{-n}\right)\right|.$$

Applying the generalized Minkowsky-inequality for V_n^θ by (22) we have

$$\begin{aligned}
\|V_n^\theta f\|_p &\le \left(\int_0^1 \left|2^n \int_{\mathbb{F}} f(x+t)(\mathcal{F}\theta)(2^n t)dt\right|^p dx\right)^{1/p} \le \\
&\le \int_{\mathbb{F}} \left(\int_0^1 |2^n f(x+t)(\mathcal{F}\theta)(2^n t)|^p \, dx\right)^{1/p} dt = \\
&= \|f\|_p \int_{\mathbb{F}} |2^n \mathcal{F}\theta(2^n t)| \, dt = \|f\|_p \int_{\mathbb{F}} |(\mathcal{F}\theta)(t)| \, dt = \|f\|_p \cdot \|\mathcal{F}\theta\|_{L^1_{\mathbb{F}}}.
\end{aligned}$$

Since for all fixed $n \in \mathbb{F}^\sharp$ the operators V_n^θ, $\sigma_n^\theta : X_p \to X_p$ are bounded, and X_p is the closure of the trigonometric polynomials it is enough to prove our statements for the system $(u_k, k \in \mathbb{F}^\sharp)$. Let $f = u_k$ $(k \in \mathbb{F}^\sharp)$. By the definition of σ_n^θ (35) and V_n^θ we get that

$$\begin{aligned}
&\left(\sigma_n^\theta f\right)(x) = \theta\left(k \cdot 2^{-n}\right) u_k(x), \\
&\quad V_n^\theta(x) = 2^n \int_{\mathbb{F}} u_k(x+t)(\mathcal{F}\theta)(2^n t)dt = 2^n \int_{\mathbb{F}} u_k(x)u_k(t)(\mathcal{F}\theta)(2^n t)dt = \\
&\qquad = 2^n u_k(x) \int_{\mathbb{F}} u_k(t)(\mathcal{F}\theta)(2^n t)dt = u_k(x) \int_{\mathbb{F}} u_k(2^{-n} t)(\mathcal{F}\theta)(t)dt = \\
&\qquad = u_k(x) \int_{\mathbb{F}} u_{k \cdot 2^{-n}}(t)(\mathcal{F}\theta)(t)dt = \theta(k \cdot 2^{-n})u_k(x).
\end{aligned}$$

We used the character property of the functions $(u_k, k \in \mathbb{F}^\sharp)$ and the inversion formulae (20). From this we can easy estimate

(38)
$$\|\sigma_n^\theta u_k - u_k\|_p \le \|\theta(k \cdot 2^{-n})u_k - u_k\|_p \le |\theta(k \cdot 2^{-n}) - 1| \cdot \|u_k\|_p = |\theta(k \cdot 2^{-n}) - 1|.$$

Because of (30), $|\theta(k \cdot 2^{-n}) - 1| \to 0$ $(n \to \infty)$ and so $\|\sigma_n^\theta u_k - u_k\|_X \to 0$ $(n \to \infty)$. Using the Banach-Steinhaus theorem we get the statements of the theorem.

3.1. Estimations for the L^1-norm of the Walsh-Fourier transform

First we investigate functions with compact support. It is known that if $\operatorname{supp}\theta \subset [0,1]$, then

$$\|\theta^\circ\|_1 = \sum_{k=0}^{\infty} |c_k^w(\theta)| \quad \left(c_k^w(\theta) = \int_0^1 \theta(t) w_k(t)\,dt, \quad k \in \mathbb{N} \right).$$

We will use the estimation of the functions

$$J_n^{(1)}(x) := \int_0^x w_k(t)\,dt, \quad J_n^{(2)}(x) := \int_0^x J_n^{(1)}(t)\,dt \quad (x \in [0,1], n \in \mathbb{N}).$$

It is known that
(39)
i) $J_{2^n+k}^{(1)} := w_k J_{2^n}^{(1)}, \quad J_{2^n+k}^{(1)}(1) := 0, \quad \|J_{2^n+k}^{(1)}\|_\infty \le 2^{-n-2} \quad (n \in \mathbb{N}),$
ii) $J_{2^n+2^m+k}^{(2)}(1) := 0, \quad \|J_{2^n+2^m+k}^{(2)}\|_\infty = 2^{-n-m-1} \quad (0 \le k < 2^m, 0 \le m < N)$

(see Schipp-Wade-Simon [2]).

Using these estimations, we can prove the following

Lemma 1. *Assume that the function $\theta \in C^1[0,1]$ is twice differentiable on the open interval $(0,1)$ and $\theta'' \in L^1[0,1)$. Then*

$$\|\theta^\circ\|_1 = \sum_{k=0}^{\infty} |c_k^w(\theta)| < N(\theta) := \|\theta\|_1 + \|\theta'\|_1 + \|\theta''\|_1 < \infty.$$

Proof. Let $\ell = 2^n + 2^m + k$ $(0 \le k < 2^m, 0 \le m < n)$. Then $J_\ell^{(1)}(0) = = J_m^{(1)}(1) = 0$, $J_\ell^{(2)}(0) = J_m^{(2)}(1) = 0$. Integrating by parts we get

$$\int_0^1 \theta(t) w_\ell(t)\,dt = -\int_0^1 \theta'(t) J_\ell^{(1)}(t)\,dt = -[\theta'(t) J_\ell^{(2)}(t)]_{t=0}^1 + \int_0^1 \theta''(t) J_\ell^{(2)}(t)\,dt,$$

and so

$$|c_\ell^w(\theta)| \le \|\theta''\|_1 \|J_\ell^{(2)}\|_\infty \le \|\theta''\|_1 2^{-n-m-1}.$$

If $\ell = 2^n$, then

$$\int_0^1 \theta(t) w_\ell(t)\, dt = -\int_0^1 \theta'(t) J_\ell^{(1)}(t)\, dt,$$

so

$$|c_\ell^w(\theta)| \le \|\theta'\|_1 \|J_\ell^{(1)}\|_\infty \le \|\theta'\|_1 2^{-n-2}.$$

From this follows that

$$\|\theta^\circ\|_1 \le |c_0^w(\theta)| + \sum_{n=0}^{\infty} |c_{2^n}^w(\theta)| + \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} |c_{2^n+2^m+k}^w(\theta)| = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

From this follows that
(40)

i) $|\Sigma_1| \le \|\theta\|_1$,

ii) $|\Sigma_2| \le \|\theta'\|_1/2$,

iii) $|\Sigma_3| \le \|\theta''\|_1 \sum_{n=0}^{\infty} 2^{-n-1} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} 2^{-m} = \|\theta''\|_1 \sum_{n=0}^{\infty} n 2^{-n-1} \le \|\theta''\|_1.$

In the general case we suppose that $\theta \in L^1(\mathbb{R}_+)$. Then for $x \in [k, k+1)$ $(k \in \mathbb{N})$

$$\theta^\circ(x) := \int_0^\infty \theta(t) w_x(t)\, dt = \int_0^\infty \theta(t) w_{[x]}(t) w_{[t]}(x)\, dt = \sum_{\ell=0}^{\infty} w_\ell(x) \int_\ell^{\ell+1} \theta(t) w_k(t)\, dt.$$

Introduce the functions

$$\theta_\ell(t) := \theta(t+\ell) \quad (\ell \in \mathbb{N},\ t \in [0,1)),$$

the function θ° can be written in the form

$$\theta^\circ(x) = \sum_{\ell=0}^{\infty} w_\ell(x) c_k^w(\theta_\ell) \quad (x \in [k, k+1),\ k \in \mathbb{N}).$$

From this follows that

$$\|\theta^\circ\|_1 \le \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} |c_k^w(\theta_\ell)| < \sum_{\ell=0}^{\infty} N(\theta_\ell).$$

We proved the following

Lemma 2. *Assume that* $\theta \in C^2(\mathbb{R}_+)$. *Then*

$$\|\theta^\circ\|_1 \leq \sum_{\ell=0}^{\infty} N(\theta_\ell).$$

3.2. The Fourier-coefficients with respect UDMD system

By a dyadic interval of rank n in $[0,1)$ we mean an interval of the form $[p/2^n, (p+1)/2^n)$, where $0 \leq p < 2^n$ and $n \in \mathbb{N}$. Given $a \in [0,1)$ and $n \in \mathbb{N}$, there is one and only one interval of rank n which contains a. We denote it with $I_n(a)$. We denote the σ-algebra generated by the intervals $I_n(a)$ $(a \in [0,1))$ by $\mathcal{A}_n$.

Let φ_n $(n \in \mathbb{N})$ $\mathcal{A}_n$-measurable functions with $|\varphi_n| = 1$. Then

$$\phi_n := r_n \varphi_n \quad (n \in \mathbb{N})$$

is an UDMD sequence, where r_n is the nth Rademacher function. We denote by $\Psi = (\psi_m, m \in \mathbb{N})$ the product system of $\Phi = (\phi_n, n \in \mathbb{N})$:

$$\psi_m := \prod_{n=0}^{\infty} \phi_n^{m_n} \quad \left(m = \sum_{n=0}^{\infty} m_n 2^n \in \mathbb{N},\ m_n \in \{0,1\}\right).$$

It is called the UDMD product system.

In this section we will examine the order of the Fourier coefficients

$$\hat{f}(m) := \langle f, \psi_m \rangle \quad (m \in \mathbb{N})$$

of the function $f \in L^1[0,1)$ with respect to the system Ψ.

Let

$$R_n(x) := \int_0^x r_n(t)dt \quad (x \in [0,1),\ n \in \mathbb{N}). \tag{41}$$

Obviously R_n is linear on dyadic intervals with rank $n+1$, and

$$R_n(k2^{-n}) = 0, \quad R_n((k+1/2) \cdot 2^{-n}) = 2^{-(n+1)} \quad (k, n \in \mathbb{N}). \tag{42}$$

The basis of the estimation is the following

Lemma 3. *Assume that the function $f \in L^1[0,1]$ is absolutely continuous, and let $\gamma_n := R_n\varphi_n$ $(n \in \mathbb{N})$. Then*

$$\hat{f}(m) = -\widehat{(f'\overline{\gamma}_n)}(m') \quad (m = 2^n + m', 0 \le m' < 2^n). \tag{43}$$

Proof. Let

$$\mathcal{J}_{m,0} := \psi_m, \quad \mathcal{J}_{m,\ell+1}(x) := \int_0^x \mathcal{J}_{m,\ell}(t)dt \quad (x \in [0,1], m, \ell \in \mathbb{N}).$$

Integrating by parts we get that

(44)
$$\hat{f}(m) := \int_0^1 f(t)\overline{\mathcal{J}'_{m,1}(t)}dt = \left[f\overline{\mathcal{J}_{m,1}}\right]_0^1 - \int_0^1 f'(t)\overline{\mathcal{J}_{m,1}(t)}dt = -\int_0^1 f'(t)\overline{\mathcal{J}_{m,1}(t)}dt,$$

because $\int_0^1 \psi_m(t)dt = 0$ for all $m \in \mathbb{N}, m > 0$. It follows from the definition of ψ_m that if $m = 2^n + m', 0 \le m' < 2^n$, $n \in \mathbb{N}$, then

$$\mathcal{J}_{m,1}(x) = \int_0^x \psi_m(t)dt = \int_0^x \phi_n(t)\psi_{m'}(t)dt = \int_0^x r_n(t)\varphi_n(t)\psi_{m'}(t)dt =$$

$$= \varphi_n(x)\psi_{m'}(x)\int_0^x r_n(t)dt = \varphi_n(x)\psi_{m'}(x)R_n(x) = \gamma_n(x)\psi_{m'}(x).$$

Substitute this in (44), we get our statement.

With the aid of Lemma 3 we get

Theorem 2. *Let f be absolutely continuous function, and $f' \in L^2([0,1))$. Then $\hat{f} \in \ell^1$.*

Proof. Using the Cauchy inequality, Lemma 3, the Parseval formula and (42) we get that

$$\sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)| \le 2^{n/2}\left(\sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)|^2\right)^{1/2} =$$

$$= 2^{n/2}\left(\sum_{m'=0}^{2^n-1} |\widehat{(f'\overline{\gamma}_n)}(m')|^2\right)^{1/2} \le 2^{n/2}||f'\overline{\gamma}_n||_2 \le 2^{-n/2-1}||f'||_2$$

for all $n \in \mathbb{N}$. Consequently

$$||\hat{f}||_{\ell^1} = \sum_{n=0}^{\infty} |\hat{f}(n)| =$$

$$= |\hat{f}(0)| + \sum_{n=0}^{\infty} \sum_{m=2^n}^{2^{n+1}-1} |\hat{f}(m)| \leq |\hat{f}(0)| + \sum_{n=0}^{\infty} 2^{-n/2-1} ||f'||_2 < \infty.$$

It is known that the systems $(w_n, n \in \mathbb{N})$ and $(v_n, n \in \mathbb{N})$ are UDMD product systems. So, Theorem 2 gives an other condition for the function θ in the dyadic case:

Corollary 1. *If the absolutely continuous function θ has compact support with* supp $\theta \subset [0,1]$, *and $\theta' \in L^2[0,1)$, then in the logical and in the arithmetic case $\hat{\theta} \in L^1[0,\infty)$.*

4. Examples

In this section we consider some known examples for the θ-summation. We will examine the conditions of Theorem 1, and in certain cases the conditions of Corollary 1. For the trigonometric case see [6].

1. The 2^nth partial sum. If we use the function

$$\theta(x) = \chi_{[0,1)}(x), \tag{45}$$

where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0,1)$, the θ-summation will yield the 2^n-th partial sum. The Walsh-Fourier transform of (45) is itself, and it is in $L^1[0,\infty)$, so the conditions of Theorem 1 are met.

θ is absolutely continuous with compact support and its derivative is in $L^2[0,1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0,1)$ and the conditions of Theorem 1 are fulfilled in the arithmetic and in the logical case, too.

2. The (C,1)-summation. If we use the function

$$\theta(x) = \begin{cases} 1-x & \text{if } x \in [0,1), \\ 0 & \text{if } x \in [1,\infty), \end{cases} \tag{46}$$

then the θ-summation results in the 2^nth subsequence of the (C,1)-sums. The function θ is of course in $L^1[0,\infty)$. It follows from Lemma 1 that θ° is in $L^1[0,\infty)$, so the conditions of Theorem 1 are satisfied.

On the other hand θ is absolutely continuous with compact support and its derivative is in $L^2[0,1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0,\infty)$ and the conditions of Theorem 1 are fulfilled in the arithmetic and in the logical case, too.

3. De La Vallée-Poussin summation. If we use the function

$$\theta(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ 2(1-x) & \text{if } x \in [1/2, 1), \\ 0 & \text{if } x \in [1, \infty), \end{cases} \tag{47}$$

then the θ-summation results in the 2^nth subsequence of the De La Vallée-Poussin-sums. The function θ is of course in $L^1[0,\infty)$. It follows from Lemma 1 that θ° is in $L^1[0,\infty)$, so the conditions of Theorem 1 are fulfilled.

Furthermore θ is absolutely continuous function with compact support and its derivative is in $L^2[0,1)$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0,\infty)$ and the conditions of Theorem 1 are satisfied in the arithmetic and in the logical case, too.

4. The Riesz-summation. If we use the function

$$\theta(x) = \begin{cases} (1-x^\gamma)^\alpha & \text{if } x \in [0, 1), \\ 0 & \text{if } x \in [1, \infty), \end{cases} \tag{48}$$

where $1 \le \alpha < \infty$ and $0 < \gamma < \infty$, then the θ-summation results in the 2^nth subsequence of the Riesz-sums. The function θ is of course in $L^1[0,\infty)$. Since $\alpha \cdot \gamma \ge 1$, so $\theta'' \in L^1[0,1)$ and so Lemma 1 holds.

Furthermore θ is absolutely continuous with compact support and its derivative is in $L^2[0,1)$ if $\gamma > 1/2$, so by Corollary 1, $\theta^\circ, \theta^\bullet \in L^1[0,\infty)$ and the conditions of Theorem 1 are met in the arithmetic and in the logical case, too, if $\gamma > 1/2$.

5. Weierstrass summation. If we use the function

$$\theta(x) = e^{-x^\gamma} \quad (0 \le x < \infty, \gamma > 0),$$

then the θ-summation results in the 2^nth subsequence of the Weierstrass-sums. The function θ is of course in $L^1[0,\infty)$. In this case we get by Lemma 2 and Lemma 1 that

$$\|\theta^\circ\|_{L^1} \le \sum_{\ell=0}^{\infty} N(\ell) \le \sum_{\ell=0}^{\infty} \left(K \cdot e^{-\ell^\gamma} + e^{-\ell^\gamma} + \frac{\gamma \ell^{\gamma-1}}{e^{\ell^\gamma}} \right) < \infty,$$

where K is a constant, which depends only on γ.

6. Abel summation. If we use the function

$$\theta(x) = e^{-\kappa x} \quad (0 \le x < \infty,\ \kappa > 0)$$

then the limit of σ_n^θ is equal to the Weierstrass-sum. The function θ is of course in $L^1[0,\infty)$. In this case we get by Lemma 2 and Lemma 1 that

$$\|\theta^\circ\|_{L^1} \le \sum_{\ell=0}^{\infty} N(\ell) \le \sum_{\ell=0}^{\infty} \left(\frac{1}{\kappa} + 1 + \kappa\right) e^{-\ell\cdot\kappa} < \infty.$$

7. Norm-depending θ functions. Assume that $\theta(x) = \theta(\|x\|)$ $(x \in \in \mathbb{R}^+)$. Then its L^1-norm is

$$\|\theta\|_1 = \int_0^\infty |\theta(t)|\, dt = \sum_{k=-\infty}^{\infty} \left|\theta(2^k)\right| \cdot 2^k. \tag{49}$$

Its Walsh-Fourier transform is

$$\begin{aligned}
\theta^0(x) &= \int_0^\infty \theta(t) w_x(t) dt = \int_0^1 \theta(t) w_x(t) dt + \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \theta(t) w_x(t) dt = \\
&= \int_0^1 \theta(t) w_x(t) dt + \sum_{k=0}^{\infty} \theta(2^k) \int_{2^k}^{2^{k+1}} w_x(t) dt = \\
&= \int_0^1 \theta(t) w_{[x]}(t) w_{[t]}(x) dt + \sum_{k=0}^{\infty} \theta(2^k) \sum_{\ell=0}^{2^k-1} \int_{2^k+\ell}^{2^k+\ell+1} w_{[x]}(t) w_{[t]}(x) dt = \\
&= \int_0^1 \theta(t) w_{[x]}(t) dt + \sum_{k=0}^{\infty} \theta(2^k) \sum_{\ell=0}^{2^k-1} w_{2^k+\ell}(x) \int_{2^k+\ell}^{2^k+\ell+1} w_{[x]}(t) dt
\end{aligned} \tag{50}$$

for all $x \in \mathbb{R}^+$. The integral of the Walsh-functions are zero over every interval with length 1 except the w_0, so

$$\begin{aligned}
\theta^\circ(x) &= \begin{cases} \theta^0([x]) & \text{if} \quad x > 1, \\ \theta^0([x]) + \sum\limits_{k=0}^{\infty} \theta(2^k) \sum\limits_{\ell=0}^{2^k-1} w_{2^k+\ell}(x) & \text{if} \quad x \in [0,1), \end{cases} \\
&= \begin{cases} \theta^\circ([x]) & \text{if} \quad x > 1, \\ \theta^\circ(0) + \sum\limits_{k=0}^{\infty} \theta(2^k) r_k(x) D_{2^k}(x) & \text{if} \quad x \in [0,1). \end{cases}
\end{aligned} \tag{51}$$

From this expression we can estimate the L^1-norm of θ°:

$$\|\theta^\circ\|_1 = \int\limits_0^\infty |\theta^\circ(t)|\,dt =$$
$$= \int\limits_0^1 \left|\theta^\circ(0) + \sum_{k=0}^\infty \theta(2^k) r_k(x) D_{2^k}(x)\right| dx + \int\limits_1^\infty |\theta^\circ([x])|\,dx \le$$
$$\le |\theta^\circ(0)| + \sum_{k=0}^\infty 2^k |\theta(2^k)| + \sum_{k=1}^\infty |\theta^\circ(k)| = \sum_{k=0}^\infty 2^k |\theta(2^k)| + \sum_{k=0}^\infty |\theta^\circ(k)| \le$$
$$\le \|\theta\|_1 + \sum_{k=0}^\infty |\theta^\circ(k)|.$$

From this follows that, if θ is integrable ($\theta \in L^1$), and its Walsh-Fourier series is absolutely convergent, then it Walsh-Fourier transform is in L^1, too.

If $\theta\,(x) = \theta\,(\|x\|)$ $(x \in \mathbb{R}^+)$, then θ has the form

$$\theta\,(x) = \sum_{n=0}^\infty \alpha_n \frac{2 \cdot D_{2^n}(x) - D_{2^{n+1}}(x)}{2^{n+1}} \qquad (x \in (0,1)), \tag{52}$$

on $(0,1)$, where the $(\alpha_n, n \in \mathbb{N})$ is a real sequence. If $\lim\limits_{x\to 0+} \theta(x) = \theta(0) = 1$, then the sequence $(\alpha_n, n \in \mathbb{N})$ is convergent, and its limit is 1 ($\lim\limits_{n\to\infty} \alpha_n = 1$). Let us count the Fourier-coefficients of θ.

$$\theta(x) = \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(2 \sum_{k=0}^{2^n-1} w_k(x) - \sum_{k=0}^{2^{n+1}-1} w_k(x) \right) =$$
$$= \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(\sum_{k=0}^{2^n-1} w_k(x) - \sum_{k=2^n}^{2^{n+1}-1} w_k(x) \right) = \tag{53}$$
$$= \sum_{n=0}^\infty \frac{\alpha_n}{2^{n+1}} \left(\sum_{k=0}^{2^{n+1}-1} w_k(x) \widetilde{\operatorname{sgn}}(2^n - k) \right),$$

where

$$\widetilde{\operatorname{sgn}}(t) = \begin{cases} 1, & \text{if} \quad t > 0, \\ -1, & \text{if} \quad t \le 0. \end{cases}$$

If we replace the order of the summation, we get for all $x \in [0,1)$ that

(54)

$$\theta(x) = \sum_{k=0}^{\infty} w_k(x) \sum_{n\geq \log_2(k+1)-1,\, n\in\mathbb{N}} \frac{\alpha_n}{2^{n+1}} \cdot \widetilde{\operatorname{sgn}}(2^n - k) =$$

$$= w_0(x)\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{n+1}} +$$

$$+ \sum_{j=0}^{\infty}\sum_{\ell=0}^{2^j-1} w_{2^j+\ell}(x) \sum_{n\geq \log_2(2^j+\ell+1)-1,\, n\in\mathbb{N}} \frac{\alpha_n}{2^{n+1}} \cdot \widetilde{\operatorname{sgn}}(2^n - 2^j - \ell) =$$

$$= w_0(x)\sum_{n=0}^{\infty} \frac{\alpha_n}{2^{n+1}} + \sum_{j=0}^{\infty}\sum_{\ell=0}^{2^j-1} w_{2^j+\ell}(x) \left(\sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}}\right).$$

If $\alpha_n \to 1$ $(n\to\infty)$, then $\gamma_n := 1-\alpha_n \to 0$ $(n\to\infty)$, and so

$$2^j \sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}} = 2^j \sum_{n=j+1}^{\infty} \frac{\gamma_n}{2^{n+1}} - \frac{\gamma_j}{2^{j+1}}. \tag{55}$$

The condition of the absolute convergence is

(56)

$$\sum_{k=0}^{\infty} |\theta^\circ(k)| = \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty}\sum_{\ell=0}^{2^j-1} \left|\sum_{n=j+1}^{\infty} \frac{\alpha_n}{2^{n+1}} - \frac{\alpha_j}{2^{j+1}}\right| =$$

$$= \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty} 2^j \left|\sum_{n=j+1}^{\infty} \frac{\gamma_n}{2^{n+1}} - \frac{\gamma_j}{2^{j+1}}\right| \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{j=0}^{\infty} 2^j \sum_{n=j}^{\infty} \frac{|\gamma_n|}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{|\gamma_n|}{2^{n+1}} \sum_{j=0}^{n} 2^j \leq$$

$$\leq \sum_{n=0}^{\infty} \frac{|\alpha_n|}{2^{n+1}} + \sum_{n=0}^{\infty} |\gamma_n| < \infty.$$

That means, if $\sum\limits_{n=0}^{\infty} |\gamma_n| < \infty$, then $\theta^\circ \in L^1[0,+\infty)$.

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