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THE DUAL SPACES OF CERTAIN HARDY SPACES ON $\mathbb{R}^+$ AND ON $\mathbb{N}$

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Dedicated to Professor Ferenc Schipp on his 70th and Professor Péter Simon on his 60th birthdays

Abstract. In connection with the problem of integrability of trigonometric series several sufficient conditions have been given. One of the most famous and efficient is the one due to Telyakovskiĭ [9]. In the paper [3] of the second author it was shown that the transform that corresponds to Telyakovskiĭ's condition generates an atomic Hardy type space $H_{\mathbb{N}}$ on $\mathbb{N}$. The continuous version $H_{\mathbb{R}^+}$ of this Hardy space is defined on the half line. In this paper we characterize the dual spaces of $H_{\mathbb{R}^+}$ and $H_{\mathbb{N}}$.

1. Introduction

Let $\mathbf{a} = (a_k)$ be a null sequence of real numbers and let us take the corresponding cosine series $\sum\limits_{k=0}^{\infty} a_k \cos kx$. Then the following estimate holds

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concerning the integrability of the cosine series

$$\int_0^\pi \Big|\sum_{k=0}^\infty a_k \cos kx\Big| dx \le C\Big(\sum_{k=0}^\infty |\Delta a_k| + \sum_{n=2}^\infty \Big|\sum_{k=1}^{[n/2]} \frac{\Delta a_{n-k} - \Delta a_{n+k}}{k}\Big|\Big), \tag{1}$$

where $\Delta a_k = a_{k-1} - a_k$ $(k \ge 1)$, and $\Delta a_0 = 0$. (Throughout the paper C will always denote an absolute positive constant not necessarily the same in different occurrences.) This integrability conditions for cosine series was proved by Telyakovskiĭ in [9]. Introducing the so called discrete Telyakovskiĭ transform

$$(\mathcal{T}_{\mathbb{N}}\mathbf{a})_n = \sum_{k=1}^{[n/2]} \frac{a_{n-k} - a_{n+k}}{k} \quad (n \ge 2)$$

with $(\mathcal{T}_{\mathbb{N}}\mathbf{a})_0 = (\mathcal{T}_{\mathbb{N}}\mathbf{a})_1 = 0$ we have that the right side of (1) is nothing but $\|\Delta\mathbf{a}\|_{\ell^1} + \|(\mathcal{T}_{\mathbb{N}}\Delta\mathbf{a})\|_{\ell^1}$.

Let us take the continuous version $\mathcal{T}_{\mathbb{R}^+}$ of $\mathcal{T}_{\mathbb{N}}$, which is called called Telyakovskiĭ transform. To this order let $f : \mathbb{R}^+ \mapsto \mathbb{R}$ be a locally integrable function. Then

$$\begin{aligned}\mathcal{T}_{\mathbb{R}^+} f(x) &= \int_0^{x/2} \frac{f(x-t) - f(x+t)}{t} dt = \\ &= \lim_{\delta\to 0^+} \int_\delta^{x/2} \frac{f(x-t) - f(x+t)}{t} dt,\end{aligned} \tag{2}$$

or equivalently

$$\mathcal{T}_{\mathbb{R}^+} f(x) = \int_{x/2}^{3x/2} \frac{f(t)}{x-t} dt = \lim_{\delta\to 0^+} \int_{\delta\le|x-t|\le x/2} \frac{f(t)}{x-t} dt.$$

Recall that the Hilbert transform $\mathcal{H}$ is defined as

$$\begin{aligned}\mathcal{H} f(x) &= \lim_{\delta\to 0+} \int_{\delta\le|x-t|} \frac{f(t)}{x-t} dt = \\ &= \lim_{\delta\to 0+} \int_\delta^\infty \frac{f(x-t) - f(x+t)}{t} dt \qquad (f \in L^1(\mathbb{R})).\end{aligned}$$

For technical reasons we omitted the usual $1/\pi$ factor in the definition of $\mathcal{H}$.

The classical real Hardy space on $\mathbb{R}$ generated by the Hilbert transform will be denoted by $H_{\mathbb{R}}$. It is the space of integrable functions for which also $\mathcal{H}f$ is integrable and the norm is defined as $\|f\|_{H_{\mathbb{R}}} = \|f\|_1 + \|\mathcal{H}f\|_1$. If the Telyakovskiĭ transform is taken instead of the Hilbert transform then another Hardy type space is obtained. It will be denoted by $H_{\mathbb{R}^+}$.

Liflyand in [5], and [6] recognized that $H_{\mathbb{R}^+}$ is isomorphic to the closed subspace of odd function in $H_{\mathbb{R}}$. In [3] we showed that $H_{\mathbb{R}^+}$ is an atomic sequence Hardy space, and identified its atoms. Namely, two types of $H_{\mathbb{R}^+}$-atoms will be distinguished. $\mathfrak{f}$ will be called an $H_{\mathbb{R}^+}$-atom of first type if $\mathfrak{f} = \delta^{-1}\chi_{[0,\delta]}$ with some $\delta > 0$. An $\mathfrak{f} \in L^\infty(\mathbb{R}^+)$ will be said to be an $H_{\mathbb{R}^+}$-atom of second type if there exists a finite interval $I \subset \mathbb{R}^+$ such that

(i) $\operatorname{supp}\mathfrak{f} \subset I$,
(ii) $\int\limits_I \mathfrak{f} = 0$,
(iii) $\|\mathfrak{f}\|_{L^\infty(\mathbb{R}^+)} \le |I|^{-1}$,

where $|I|$ stands for the length of I. The collection of $H_{\mathbb{R}^+}$-atoms will be denoted by $\mathcal{A}_{\mathbb{R}^+}$. Then (see [3]) $f \in H_{\mathbb{R}^+}$ if and only if f can be decomposed as $f = \sum\limits_{k=0}^{\infty} \alpha_k \mathfrak{f}_k$, where $\mathfrak{f}_k \in \mathcal{A}_{\mathbb{R}^+}$, and $\alpha_k \in \mathbb{R}$ $(k \in \mathbb{N})$ with $(\alpha_k) \in \ell^1$. (The convergence in the decomposition is a.e. and in $L^1(\mathbb{R}^+)$ norm.) Moreover

$$\|f\|_{H_{\mathbb{R}^+}} \approx \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all decompositions of f.

Comparing the atomic decomposition in $H_{\mathbb{R}}$ and $H_{\mathbb{R}^+}$ we find that the atoms in $H_{\mathbb{R}}$ are analogous to the second type atoms in $H_{\mathbb{R}^+}$ but there are no atoms corresponding to the first type atoms in $\mathcal{A}_{\mathbb{R}^+}$. Indeed, a function $\mathfrak{g} \in L^\infty(\mathbb{R})$ is called an $H_{\mathbb{R}}$ atom, in notation $\mathfrak{g} \in \mathcal{A}_{\mathbb{R}}$, if there exists a finite interval $I \subset \mathbb{R}$ such that

(i) $\operatorname{supp}\,\mathfrak{g} \subset I$,
(ii) $\int\limits_I \mathfrak{g} = 0$,
(iii) $\|\mathfrak{g}\|_{L^\infty(\mathbb{R})} \le |I|^{-1}$.

Let us now turn back to the original Telyakovskiĭ transform, and let us define the Hardy type sequence space $H_{\mathbb{N}}$ as the collection of sequences $\mathbf{a}$ for which $\mathcal{T}_{\mathbb{N}}a \in \ell^1$. The norm is defined by $\|\mathbf{a}\|_{H_{\mathbb{N}}} = \|\mathbf{a}\|_{\ell^1} + \|\mathcal{T}_{\mathbb{N}}\mathbf{a}\|_{\ell^1}$. Since $H_{\mathbb{N}}$ is defined by $\mathcal{T}_{\mathbb{N}}$, the discrete analogue of $\mathcal{T}_{\mathbb{R}^+}$ we may consider $H_{\mathbb{N}}$ as the discrete space that corresponds to $H_{\mathbb{R}^+}$. Indeed, it is the natural discretization of $H_{\mathbb{R}^+}$ from other aspects as well. Namely, let $\mathcal{P}\mathbf{a}$ denote the step function associated to the real sequence $\mathbf{a}$ by

$$(\mathcal{P}\mathbf{a})(x) = a_{[x]} \qquad (x \in \mathbb{R}^+),$$

where $[x]$ stands for the integer part of x. Then (see [3]) $\mathbf{a} \in H_{\mathbb{N}}$ if and only if $\mathcal{P}\mathbf{a} \in H_{\mathbb{R}^+}$, and $\|\mathbf{a}\|_{H_{\mathbb{N}}} \approx \|\mathcal{P}\mathbf{a}\|_{H^+_{\mathbb{R}}}$. On the other hand $H_{\mathbb{N}}$ is an atomic Hardy space (see [3]) where the atoms can be given by means of $\mathcal{P}$. By definition the real sequence $\mathfrak{a}$ be called an $\mathbb{N}$-atom if $\mathcal{P}\mathfrak{a}$ is an $\mathbb{R}^+$-atom. If the collection of $\mathbb{N}$-atoms is denoted by $\mathcal{A}_{\mathbb{N}}$ then $\mathfrak{a} \in \mathcal{A}_{\mathbb{N}}$ if and only if

$$\mathfrak{a}_j = \begin{cases} 1/n, & \text{if } j = 0, \dots, n-1; \\ \\ 0, & \text{if } j \geq n \end{cases}$$

with some $n \in \mathbb{N}$, or there exist $k, n \in \mathbb{N}$ such that

(i) $\mathfrak{a}_j = 0$ if $j < n$ or $j > n + k$,

(ii) $\sum\limits_{j=n}^{n+k} \mathfrak{a}_j = 0$,

(iii) $\max_{n \leq j \leq n+k} |\mathfrak{a}_j| \leq 1/(k+1)$,

where the atoms are defined as follows. As we showed in [3] a sequence $\mathbf{a}$ belongs to $H_{\mathbb{N}}$ if and only if it can be decomposed as $\mathbf{a} = \sum\limits_{k=0}^{\infty} \alpha_k \mathfrak{a}^{(k)}$, where $\mathfrak{a}^{(k)} \in \mathcal{A}_{\mathbb{N}}$, and $\alpha_k \in \mathbb{R}$ $(k \in \mathbb{N})$ with $(\alpha_k) \in \ell^1$. (The convergence in the decomposition is taken in ℓ^1 norm.) Moreover

$$\|\mathbf{a}\|_{H_{\mathbb{N}}} \approx \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all decompositions of $\mathbf{a}$.

2. Results

It was proved by Feffermann [2] that the dual space of $H_{\mathbb{R}}$ is $BMO_{\mathbb{R}}$. For the definition of $BMO_{\mathbb{R}}$ let f be a locally integrable function on $\mathbb{R}$. The $BMO_{\mathbb{R}}$ seminorm is defined by

$$\|f\|_{BMO_{\mathbb{R}}} = \sup_I \frac{1}{|I|} \int_I \Big| f - \frac{1}{|I|} \int_I f \Big|.$$

Here, I denotes any finite subinterval of $\mathbb{R}$. Since $\|f - g\|_{BMO_{\mathbb{R}}} = 0$ if and only if $f - g$ is constant we can introduce equivalence classes. Then $BMO_{\mathbb{R}}$ is the collection of those equivalence classes whose members have finite $BMO_{\mathbb{R}}$ seminorm. Moreover, the norm of an equivalence class is defined by the seminorm of its members. Similarly to the case of L^p spaces we will call the elements of $BMO_{\mathbb{R}}$ functions. Based on the relation between $H_{\mathbb{R}}$ and $H_{\mathbb{R}^+}$ we can identify the space dual to $H_{\mathbb{R}^+}$. Namely, let $BMO_{\mathbb{R}^+}$ denote the collection of locally integrable functions for which

$$\sup_{\delta>0} \frac{1}{\delta} \int_0^\delta \Big| f \Big| + \sup_I \frac{1}{|I|} \int_I \Big| f - \frac{1}{|I|} \int_I f \Big| < \infty, \tag{3}$$

where the supremum is taken over all finite intervals $I \subset \mathbb{R}^+$. Define the norm in $BMO_{\mathbb{R}^+}$ by the quantity on the left side of (3).

Theorem 1. *The space dual to $H_{\mathbb{R}^+}$ is $BMO_{\mathbb{R}^+}$.*

Remark 1. We note that, even though the definitions of $BMO_{\mathbb{R}}$ and $BMO_{\mathbb{R}^+}$ are similar, the latter one is significantly different from the first one. This is because of the the additional term

$$\sup_{\delta>0} \frac{1}{\delta} \int_0^\delta |f|,$$

which in fact can be relaxed to $\sup_{\delta>0} \frac{1}{\delta} \Big| \int_0^\delta f \Big|$. This follows from

$$\frac{1}{\delta} \int_0^\delta |f| \le \frac{1}{\delta} \int_0^\delta \Big| f - \frac{1}{\delta} \int_0^\delta f \Big| + \frac{1}{\delta} \Big| \int_0^\delta f \Big| \qquad (\delta > 0)\,.$$

Remark 2. It will turn out from the proof of *Theorem 1* that $BMO_{\mathbb{R}^+}$ is isomorphic to the subspace of odd functions, i.e. the equivalence classes that have odd members, of $BMO_{\mathbb{R}}$. Let this space be denoted by $BMO_{\mathbb{R}}^-$.

Let $BMO_{\mathbb{N}}$, the discrete version of $BMO_{\mathbb{R}^+}$, be the collection of sequences $\mathbf{a}$ for which

$$\sup_{\ell\in\mathbb{N}} \frac{1}{\ell}\sum_{j=0}^{\ell-1}|a_j| + \sup_{n,k\in\mathbb{N}} \frac{1}{k}\sum_{j=0}^{k-1}\Big|a_{n+j} - \frac{1}{k}\sum_{\ell=0}^{k-1}a_{n+\ell}\Big| < \infty, \tag{4}$$

and the norm of $\mathbf{a}$ is defined by the quantity on the left side of (4). The following theorem shows the connection between $BMO_{\mathbb{N}}$ and $BMO_{\mathbb{R}^+}$, and the duality relation that we expect.

Theorem 2. (i) The space dual to $H_{\mathbb{N}}$ is $BMO_{\mathbb{N}}$.

(ii) A sequence $\mathbf{a} \in BMO_{\mathbb{N}}$ if and only if $\mathcal{P}\mathbf{a} \in BMO_{\mathbb{R}^+}$, and $\|\mathbf{a}\|_{BMO_{\mathbb{N}}} \approx \approx \|\mathcal{P}\mathbf{a}\|_{BMO_{\mathbb{R}^+}}$.

For any locally integrable function f on $\mathbb{R}^+$ let $\mathcal{E}f$ be the step function defined as

$$\mathcal{E}f(x) = \int_{[x]}^{[x]+1} f \qquad (x \in \mathbb{R}^+).$$

Remark 3. Let $BMO_{\mathbb{R}^+}^{\bullet}$ denote the closed subspace of those elements in $BMO_{\mathbb{R}^+}$ that take on constant values on each interval $[n, n+1)$ $(n \in \mathbb{N})$, i.e.

$$BMO_{\mathbb{R}^+}^{\bullet} = \{f \in BMO_{\mathbb{R}^+} :\ \mathcal{E}f = f\}.$$

Then (ii) of *Theorem 2* means that $\mathbf{BMO}_{\mathbb{N}}$ and $BMO_{\mathbb{R}^+}^{\bullet}$ are isomorphic. This is another reason why $BMO_{\mathbb{N}}$ can be considered as the discretization of $BMO_{\mathbb{R}^+}$.

It was shown by Coifman and Weiss [1] that $H_{\mathbb{R}}$ is a dual space itself. Namely, let $VMO_{\mathbb{R}}$, a closed subspace of $BMO_{\mathbb{R}}$, be defined as the collection of functions f in $BMO_{\mathbb{R}}$ for which

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I \Big|f - \frac{1}{|I|}\int_I f\Big| = 0.$$

Then $H_{\mathbb{R}}$ is the dual of $VMO_{\mathbb{R}}$. We note that a similar result for dyadic Hardy and VMO spaces was proved by Schipp [7]. In view of the relation between the

Hardy spaces $H_{\mathbb{R}}$ and $H_{\mathbb{R}^+}$, and *Theorem 1* on the corresponding BMO spaces it is logical to define $VMO_{\mathbb{R}^+}$ as the set of functions in $BMO_{\mathbb{R}^+}$ for which

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_0^{\delta}|f| + \lim_{|I|\to 0}\frac{1}{|I|}\int_I\Big|f-\frac{1}{|I|}\int_I f\Big| = 0.$$

Following a similar logic we obtain that $BMO_{\mathbb{N}}$ is its own VMO type space. Then the classical duality result, the relationship between $H_{\mathbb{R}^+}$, $H_{\mathbb{N}}$, $BMO_{\mathbb{R}^+}$, $BMO_{\mathbb{N}}$ and their classical counterparts along with *Theorems 1* and *2* imply the following duality results.

Theorem 3. *(i) The dual of $VMO_{\mathbb{R}^+}$ is $H_{\mathbb{R}^+}$.*
(ii) The dual of $BMO_{\mathbb{N}}$ is $H_{\mathbb{N}}$.

3. Proofs

Proof of Theorem 1. Our result will follow from the duality relation between $H_{\mathbb{R}}$ and $BMO_{\mathbb{R}}$ which was proved by Fefferman [2]. This says that if $h \in BMO_{\mathbb{R}}$ then

$$L(g) = \int_{\mathbb{R}} gh \qquad (g \in H_{\mathbb{R}}) \tag{5}$$

defines a bounded linear functional on $H_{\mathbb{R}}$. Here the integral should be suitable defined for it does not converge for general $g \in H_{\mathbb{R}}$ and $h \in BMO_{\mathbb{R}}$. Therefore, initially (5) is defined on a dense linear subspace of $H_{\mathbb{R}}$. This can be for example the subspace of finite linear combinations of $\mathbb{R}$-atoms. Then (5) has a unique extension to $H_{\mathbb{R}}$. For details see for example [10] or [1]. Moreover, any bounded linear functional L on $H_{\mathbb{R}}$ is of this form, and $\|L\| \approx \|h\|_{BMO_{\mathbb{R}}}$.

Now we will show that $BMO_{\mathbb{R}^+}$ is isomorphic to the closed subspace $BMO_{\mathbb{R}}^-$ consisting of the odd functions of $BMO_{\mathbb{R}}$. More precisely, we will show that if f is a function defined on $\mathbb{R}^+$ and f_O is its odd extension onto $\mathbb{R}$ then $f \in BMO_{\mathbb{R}^+}$ if and only if $f_O \in BMO_{\mathbb{R}}$, and $\|f\|_{BMO_{\mathbb{R}^+}} \approx \|f_O\|_{BMO_{\mathbb{R}}}$. Indeed, let us take the interval $I = [a, b]$ $(a, b \in \mathbb{R})$ and consider

$$\frac{1}{|I|}\int_I\Big|f_O-\frac{1}{|I|}\int_I f_O\Big|.$$

If $a > 0$ or $b < 0$ then

$$\frac{1}{|I|}\int\limits_I \left|f_O - \frac{1}{|I|}\int\limits_I f_O\right| = \frac{1}{|I'|}\int\limits_{I'} \left|f - \frac{1}{|I'|}\int\limits_{I'} f\right|, \tag{6}$$

where $I' = [\min\{|a|,|b|\},\ \max\{|a|,|b|\}] \subset \mathbb{R}^+$.

If $a < 0 < b$ then

$$\frac{1}{|I|}\int\limits_I \left|f_O - \frac{1}{|I|}\int\limits_I f_O\right| \le 2\frac{1}{|I|}\int\limits_I |f_O| =$$

$$\begin{aligned} &= 2\left(\frac{|a|}{b-a}\left(\frac{1}{|a|}\int\limits_0^{|a|}|f|\right) + \frac{b}{b-a}\left(\frac{1}{b}\int\limits_0^{b}|f|\right)\right) \le \\ &\le 2\sup_{\delta>0}\frac{1}{\delta}\int\limits_0^{\delta}|f|. \end{aligned} \tag{7}$$

In particular, if $I = [-a, a]$ $(a > 0)$ then

$$\frac{1}{|I|}\int\limits_I \left|f_O - \frac{1}{|I|}\int\limits_I f_O\right| = \frac{1}{a}\int\limits_0^{a}|f|. \tag{8}$$

By (6) and (8) we have that $\|f_O\|_{BMO_{\mathbb{R}}} \ge \frac{1}{2}\|f\|_{BMO_{\mathbb{R}+}}$. On the other hand it follows from (6) and (7) that $\|f_O\|_{BMO_{\mathbb{R}}} \le 2\|f\|_{\mathbf{BMO}_{\mathbb{R}+}}$. The isomorphism is proved. Consequently, any $f \in BMO_{\mathbb{R}+}$ defines a bounded linear functional on $H_{\mathbb{R}}$ by

$$L(g) = \int\limits_{\mathbb{R}} g f_O \qquad (g \in H_{\mathbb{R}}),$$

and $\|L\| \approx \|f_O\|_{BMO_{\mathbb{R}}} \approx \|f\|_{BMO_{\mathbb{R}+}}$. Let g_+ and g_- denote the even and odd parts of g respectively. Obviously $L(g) = L(g_-)$ $(g \in H_{\mathbb{R}})$. Recall that $H_{\mathbb{R}+}$ is isomorphic to $H_{\mathbb{R}}^-$ the subspace of odd functions of $H_{\mathbb{R}}$. Hence

$$F(h) = \frac{1}{2}\int\limits_{\mathbb{R}} h_O f_O \qquad (h \in H_{\mathbb{R}+})$$

is a bounded linear functional on $H_{\mathbb{R}^+}$. Moreover,

$$\|F\| = \|L\| \approx \|f\|_{BMO_{\mathbb{R}^+}},$$

and F can be written in the form

$$F(h) = \int\limits_{\mathbb{R}^+} hf \qquad (h \in H_{\mathbb{R}^+}).$$

Suppose now that F is a bounded linear functional on $H_{\mathbb{R}^+}$. Then one can define a bounded linear functional L on $H_{\mathbb{R}}^-$ by $L(g) = 2F(f)$ $(g \in H_{\mathbb{R}}^-)$, where $f \in H_{\mathbb{R}^+}$ for which $f_O = g$. Then $\|L\| = \|F\|$. Let us take the norm preserving extension of L onto $H_{\mathbb{R}}$ be defined as

$$L(g) = L(g_-) \qquad (g \in H_{\mathbb{R}}).$$

We note that if $g \in H_{\mathbb{R}}$ then $g_+, g_- \in H_{\mathbb{R}}$, and $\|g_-\|_{H_{\mathbb{R}}} \leq \|g\|_{H_{\mathbb{R}}}$. The same applies to $BMO_{\mathbb{R}}$.

By (5) there exists a unique $h \in BMO_{\mathbb{R}}$ such that

$$L(g) = \int\limits_{\mathbb{R}} gh, \quad \text{and} \quad \|L\| \approx \|h\|_{BMO_{\mathbb{R}}}.$$

Since

$$L(g) = \int\limits_{\mathbb{R}} g_- h_- + \int\limits_{\mathbb{R}} g_+ h_+ = L(g_-) + L(g_+)$$

we have by the definition of L that $\int\limits_{\mathbb{R}} g_+ h_+ = L(g_+) = 0$. Consequently, we may suppose that $h \in BMO_{\mathbb{R}}^-$. Let $f \in BMO_{\mathbb{R}^+}$ for which $f_O = h$. Then $\|f\|_{BMO_{\mathbb{R}^+}} \approx \|h\|_{BMO_{\mathbb{R}}}$. Using f, the functional F can be written in the following form.

$$F(g) = \frac{1}{2} L(g_O) = \frac{1}{2} \int\limits_{\mathbb{R}} g_O h_- = \int\limits_{\mathbb{R}^+} gf \qquad (g \in H_{\mathbb{R}^+}),$$

and $\|f\|_{BMO_{\mathbb{R}^+}} \approx \|F\|$.

Proof of Theorem 2. Let us start with the proof of (ii). Recall, see *Remark 2*, that the statement of (ii) is equivalent to the isomorphism of

$BMO^{\bullet}_{\mathbb{R}+}$ and $BMO_{\mathbb{N}}$. By definition if $f \in BMO^{\bullet}_{\mathbb{R}+}$ then there exists a unique sequence $\mathbf{a}$ for which $\mathcal{P}\mathbf{a} = f$. Moreover

$$\|\mathbf{a}\|_{BMO_{\mathbb{N}}} = \sup_{\ell\in\mathbb{N}} \frac{1}{\ell} \int_0^{\ell} |f| + \sup_{k,n\in\mathbb{N}} \frac{1}{k} \int_n^{n+k} \left| f - \frac{1}{k} \int_n^{n+k} f \right| \le \|f\|_{BMO_{\mathbb{R}+}}.$$

Let us now suppose that $\mathbf{a} \in BMO_{\mathbb{N}}$, and consider $\|\mathcal{P}\mathbf{a}\|_{BMO_{\mathbb{R}+}}$. For any finite interval $I \subset \mathbb{R}^+$ and $f \in L^1(I)$ define $\sigma_I f$ as

$$\sigma_I f = \frac{1}{|I|} \int_I f.$$

Since $\mathcal{P}\mathbf{a}$ is constant on the intervals $[n, n+1)$ $(n \in \mathbb{N})$ we have that $\sigma_{[x,c)}\mathcal{P}\mathbf{a}$, and $\sigma_{[c,x)}\mathcal{P}\mathbf{a}$ $(c, x \in \mathbb{R}^+)$ are monotonic in x on any such interval. Therefore,

$$\sup_{\delta>0} \frac{1}{\delta} \int_0^{\delta} |\mathcal{P}\mathbf{a}| = \sup_{\ell\in\mathbb{N}} \frac{1}{\ell} \sum_{j=0}^{\ell-1} |a_j|. \tag{9}$$

Let us consider

$$\frac{1}{|I|} \int_I \left| \mathcal{P}\mathbf{a} - \frac{1}{|I|} \int_I \mathcal{P}\mathbf{a} \right|,$$

where $I \subset \mathbb{R}^+$ is a finite interval. Then, with proper $n, k \in \mathbb{N}$ and $0 \le \delta_i < 1$ $(i = 1, 2)$, I can be written in the form $I = [n - \delta_1, n + k + \delta_2]$. We may suppose that k is at least 1. Indeed, if $I = [n - \delta_1, n + \delta_2]$ then define I' by

$$I' = \begin{cases} \left[n - 1, n + \frac{\delta_2}{\delta_1}\right] & \text{if } \delta_1 \ge \delta_2, \\[2ex] \left[n - \frac{\delta_1}{\delta_2}, n + 1\right] & \text{if } \delta_1 < \delta_2. \end{cases}$$

Then a simple calculation shows that

$$\frac{1}{|I|} \int_I \left| \mathcal{P}\mathbf{a} - \frac{1}{|I|} \int_I \mathcal{P}\mathbf{a} \right| = \frac{1}{|I'|} \int_{I'} \left| \mathcal{P}\mathbf{a} - \frac{1}{|I'|} \int_{I'} \mathcal{P}\mathbf{a} \right|.$$

Let $I = [n-\delta_1, n+k+\delta_2]$ with $k \geq 1$. Then

$$\begin{aligned}
\frac{1}{|I|}\int_I \Big|\mathcal{P}\mathbf{a} - \frac{1}{|I|}\int_I \mathcal{P}\mathbf{a}\Big| =& \\
=&\frac{1}{k+\delta_1+\delta_2}\int_{n-\delta_1}^{n+k+\delta_2} \big|\mathcal{P}\mathbf{a} - \sigma_{[n-\delta_1,n+k+\delta_2]}\mathcal{P}\mathbf{a}\big| \leq \\
\leq&\frac{1}{k+\delta_1+\delta_2}\int_{n-\delta_1}^{n+k+\delta_2} \big|\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k+1]}\mathcal{P}\mathbf{a}\big| + \\
&+ \Big(\Big|\sigma_{[n-1,n+k+1]}\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k+\delta_2]}\mathcal{P}\mathbf{a}\Big| + \\
&+ \Big|\sigma_{[n-1,n+k+\delta_2]}\mathcal{P}\mathbf{a} - \sigma_{[n-\delta_1,n+k+\delta_2]}\mathcal{P}\mathbf{a}\Big|\Big) = \\
=&J_1 + J_2.
\end{aligned}$$

For J_1 we have

$$\begin{aligned}
I_1 &\leq \frac{2}{k+1}\int_{n-1}^{n+k+1} \big|\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k+1]}\mathcal{P}\mathbf{a}\big| = \\
&= \frac{2}{k+1}\sum_{j=0}^{k}\left|a_{n-1+j} - \frac{1}{k+1}\sum_{\ell=0}^{k} a_{n-1+\ell}\right| \leq \\
&\leq 2\|\mathbf{a}\|_{BMO_{\mathbb{N}}}.
\end{aligned}$$

In order to get a similar estimate for I_2 we need to replace the non-integer intervals in σ by integer intervals. Therefore, we use the monotonicity of $\sigma_{[c,x]}$, and $\sigma_{[x,c]}$ to obtain

$$\big|\sigma_{[n-1,n+k+1]}\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k+\delta_2]}\mathcal{P}\mathbf{a}\big| \leq \big|\sigma_{[n-1,n+k+1]}\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k]}\mathcal{P}\mathbf{a}\big|,$$

and

$$\begin{aligned}
\big|\sigma_{[n-1,n+k+\delta_2]}\mathcal{P}\mathbf{a} - \sigma_{[n-\delta_1,n+k+\delta_2]}\mathcal{P}\mathbf{a}\big| &\leq \\
&\leq \big|\sigma_{[n-1,n+k+\delta_2]}\mathcal{P}\mathbf{a} - \sigma_{[n,n+k+\delta_2]}\mathcal{P}\mathbf{a}\big| \leq \\
&\leq \max_{i,j=0 \text{ or } 1} \big|\sigma_{[n-1,n+k+i]}\mathcal{P}\mathbf{a} - \sigma_{[n,n+k+j]}\mathcal{P}\mathbf{a}\big| \leq \\
&\leq \max_{i,j=0 \text{ or } 1} \Big(\big|\sigma_{[n-1,n+k+i]}\mathcal{P}\mathbf{a} - \sigma_{[n-1,n+k+j]}\mathcal{P}\mathbf{a}\big| + \\
&\qquad + \big|\sigma_{[n-1,n+k+j]}\mathcal{P}\mathbf{a} - \sigma_{[n,n+k+j]}\mathcal{P}\mathbf{a}\big|\Big).
\end{aligned}$$

Observe that every term is of the form

$$\left|\sigma_{[N-1,M]}\mathcal{P}\mathbf{a}-\sigma_{[N,M]}\mathcal{P}\mathbf{a}\right| \quad \text{or} \quad \left|\sigma_{[N,M-1]}\mathcal{P}\mathbf{a}-\sigma_{[N,M]}\mathcal{P}\mathbf{a}\right|$$

$$(N,M\in\mathbb{N},\ N<M).$$

It follows from the definition of σ, $\mathcal{P}$ and the $BMO_{\mathbb{N}}$ norm that

$$\begin{aligned}\left|\sigma_{[N-1,M]}\mathcal{P}\mathbf{a}-\sigma_{[N,M]}\mathcal{P}\mathbf{a}\right| &= \left|\frac{1}{M-N}\sum_{j=N+1}^{M}a_j-\sigma_{[N-1,M]}\mathcal{P}\mathbf{a}\right| \le \\ &\le \frac{1}{M-N}\sum_{j=N+1}^{M}\left|a_j-\frac{1}{M-N+1}\sum_{\ell=N}^{M}a_\ell\right| \le \\ &\le 2\frac{1}{M-N+1}\sum_{j=N}^{M}\left|a_j-\frac{1}{M-N+1}\sum_{\ell=N}^{M}a_\ell\right| \le \\ &\le 2\|\mathbf{a}\|_{\mathbf{BMO}_{\mathbb{N}}}.\end{aligned}$$

Obviously, the same estimate holds for $\left|\sigma_{[N,M-1]}\mathcal{P}\mathbf{a}-\sigma_{[N,M]}\mathcal{P}\mathbf{a}\right|$. Then we have

$$J_2\le 6\|\mathbf{a}\|_{\mathbf{BMO}_{\mathbb{N}}}.$$

Consequently,

$$\|\mathcal{P}\mathbf{a}\|_{BMO_{\mathbb{R}+}}\le 8\|\mathbf{a}\|_{BMO_{\mathbb{N}}},$$

and (ii) of *Theorem 2* is proved.

The proof of (i) will be based on the fact that $\mathcal{P}$ is an isomorphism between $BMO_{\mathbb{N}}$ and $BMO^{\bullet}_{\mathbb{R}+}$, and between $H_{\mathbb{N}}$ and $H^{\bullet}_{\mathbb{R}+}$. $H^{\bullet}_{\mathbb{R}+}$ is defined similarly to $BMO^{\bullet}_{\mathbb{R}+}$. Namely it is the subspace of $H_{\mathbb{N}}$ formed by those elements that are constant on intervals $[n,n+1)$ $(n\in\mathbb{N})$. For the isomorphism between $H_{\mathbb{N}}$ and $H^{\bullet}_{\mathbb{R}+}$ we refer to [3].

Let L be a bounded linear functional on $H_{\mathbb{N}}$. Then by the isomorphism between $H_{\mathbb{N}}$ and $H^{\bullet}_{\mathbb{R}+}$ we have that

$$N(\mathcal{P}\mathbf{a})=L\mathbf{a} \qquad (\mathbf{a}\in H_{\mathbb{N}})$$

defines a bounded linear functional N on $H^{\bullet}_{\mathbb{R}+}$, and $\|L\|\approx\|N\|$. A norm preserving extension M of N onto $H_{\mathbb{R}+}$ can be given by

$$Mf=N(\mathcal{E}f) \qquad (f\in H_{\mathbb{R}+}).$$

It was shown in [3] that there exists a unique $g \in BMO_{\mathbb{R}^+}$ such that

$$Mf = \int_{\mathbb{R}^+} fg, \qquad \text{and} \qquad \|M\| \approx \|g\|_{BMO_{\mathbb{R}^+}}.$$

By the definitions of M and $\mathcal{E}$ we have

$$Mf = M(\mathcal{E}f) = \int_{\mathbb{R}^+} (\mathcal{E}f)g = \int_{\mathbb{R}^+} f\mathcal{E}g \qquad (f \in H_{\mathbb{R}^+}).$$

Hence, $g = \mathcal{E}g$, i.e. $g \in BMO^\bullet_{\mathbb{R}^+}$. Then $g = \mathcal{P}\mathbf{b}$ holds with a proper sequence $\mathbf{b}$. It follows from (ii) that $\mathbf{b} \in BMO_{\mathbb{N}}$, and $\|g\|_{BMO_{\mathbb{R}^+}} \approx \|\mathbf{b}\|_{\mathbf{BMO}_{\mathbb{N}}}$. Consequently,

$$L\mathbf{a} = N(\mathcal{P}\mathbf{a}) = \int_{\mathbb{R}^+} \mathcal{P}\mathbf{a}g = \int_{\mathbb{R}^+} \mathcal{P}\mathbf{a}\mathcal{P}\mathbf{b} = \sum_{k=0}^{\infty} a_k b_k \qquad (\mathbf{a} \in \mathbf{H}_{\mathbb{N}}),$$

and $\|L\| \approx \|\mathbf{b}\|_{BMO_{\mathbb{N}}}$.

If, on the other hand, $\mathbf{b} \in BMO_{\mathbb{N}}$ then $\mathcal{P}\mathbf{b} \in BMO^\bullet_{\mathbb{R}^+} \subset BMO_{\mathbb{R}^+}$. Hence by *Theorem 1* we have that $Nf = \int_{\mathbb{R}^+} f\mathcal{P}\mathbf{b}$ $(f \in H_{\mathbb{R}^+})$ is a bounded linear functional on $H_{\mathbb{R}^+}$. Moreover, $\|N\| \approx \|\mathcal{P}\mathbf{b}\|_{BMO_{\mathbb{R}^+}} \approx \|\mathbf{b}\|_{BMO_{\mathbb{N}}}$. Since $\left\| N\big|_{H^\bullet_{\mathbb{R}^+}} \right\| = \|N\|$ we have by (ii) that

$$L\mathbf{a} = N(\mathcal{P}\mathbf{a}) = \int_{\mathbb{R}^+} \mathcal{P}\mathbf{a}\mathcal{P}\mathbf{b} = \sum_{k=0}^{\infty} a_k b_k \qquad (a \in H_{\mathbb{N}})$$

is a bounded linear functional on $H_{\mathbb{N}}$, and $\|L\| \approx \|\mathbf{b}\|_{BMO_{\mathbb{N}}}$.

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