Annales Univ. Sci. Budapest., Sect. Comp. 33 (2010) 101-122

ON THE PAIRS OF MULTIPLICATIVE FUNCTIONS SATISFYING A CONGRUENCE PROPERTY

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Dedicated to Professor Ferenc Schipp on his 70th anniversary

Dedicated to Professor Péter Simon on his 60th anniversary

Abstract. All solutions of the congruence

$$g(An+B) \equiv Cf(n)+D \pmod{n}$$

are given for integer-valued completely multiplicative functions f and g with some integers $A>0$, $B>0$, C and $D \neq 0$. We prove that, except for some special cases, there exist a non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that $n|f(n)$ and $g(m)=m^{\alpha}\chi_A(m)$ are satisfied for all $n,\ m \in \mathbb{N},\ (m,A)=1$. In the case when $C=0$, we also determine all multiplicative functions g of the above congruence.

1. Introduction

Let k be a positive integer and let $\mathbb{N}_k$ denote the set of the natural numbers coprime to k. An arithmetical function $g(n) \not\equiv 0$ is said to be multiplicative function on the set $\mathbb{N}_k$ if $n,\ m \in \mathbb{N}_k,\ (n,m)=1$ implies

$$g(nm)=g(n)g(m)$$

The European Union and the European Social Fund have provided financial support to the project under the grant agreement TÁMOP-4.2.1/B-09/1/KMR-2010-0003.

and it is called completely multiplicative on the set $\mathbb{N}_k$ if this equation holds for all pairs of positive integers $n \in \mathbb{N}_k$ and $m \in \mathbb{N}_k$. In the following let $\mathcal{M}_k$ ($\mathcal{M}_k^*$) be the set of integer-valued multiplicative (completely multiplicative) functions on the set $\mathbb{N}_k$. In the case $k = 1$, we use the following notations:

$$\mathbb{N} := \mathbb{N}_1, \ \mathcal{M} := \mathcal{M}_1 \ \text{ and } \ \mathcal{M}^* := \mathcal{M}^*{}_1.$$

In 1966 M.V. Subbarao [8] proved the following assertion: If $g \in \mathcal{M}$ satisfies

$$(1) \qquad g(n+m) \equiv g(m) \pmod n \quad \text{for all} \quad n, m \in \mathbb{N},$$

then there is a non-negative integer α such that

$$(2) \qquad g(n) = n^\alpha \quad \text{for all} \quad n \in \mathbb{N}.$$

A. Iványi [2] extended this result proving that if $g \in \mathcal{M}^*$ and (1) holds for a fixed $m \in \mathbb{N}$ and for all $n \in \mathbb{N}$, then $g(n)$ has also the same form (2). In the joint paper with J. Fehér, we improved in [6] the results of Subbarao and Iványi mentioned above by proving that if $M \in \mathbb{N}$, $g \in \mathcal{M}$ satisfy the conditions $g(M) \neq 0$ and

$$g(n+M) \equiv g(M) \pmod n \quad \text{for all} \quad n \in \mathbb{N},$$

then (2) holds.

An another characterization of n^α by using congruence property was found by A. Iványi [3], namely he proved that if $g \in \mathcal{M}$ satisfies the relation

$$(3) \qquad g(n+m) \equiv g(n) + g(m) \pmod n \quad \text{for all} \quad n, m \in \mathbb{N},$$

then $g(n)$ is a power of n with positive integer exponent. In [6] we determined all solutions $g \in \mathcal{M}^*$ of (3) under the condition that the congruence (3) holds for a fixed $m \in \mathbb{N}$ and for all $n \in \mathbb{N}$. Later, in the papers [3, 4, 5, 7] we obtained some generalizations of this result, namely we proved the following theorems:

Theorem A. ([3]) *If the integers $A > 0$, $B > 0$, $C \neq 0$, $N > > 0$ with $(A, B) = 1$ and $g \in \mathcal{M}$ satisfy the relation*

$$g(An+B) \equiv C \pmod n \quad \textit{for all} \quad n \geq N,$$

then $g(B) = C$ and there are a non-negative integer α, a real-valued Dirichlet character $\chi_A \pmod A$ such that

$$g(n) = \chi_A(n) n^\alpha \quad \textit{for all} \quad n \in \mathbb{N}_A.$$

Theorem B. ([5]) *Let $A > 0,\ B,\ a > 0,\ b,\ N > 0$ and $D \neq 0$ be fixed positive integers. If a function $g \in \mathcal{M}^*$ satisfies the congruence*

$$g\Big[A(an+b)+B\Big] \equiv D \pmod{an+b} \quad \textit{for all} \quad n \in \mathbb{N}, n > N,$$

then there are a non-negative integer α, a real-valued Dirichlet character χ_{aA} $\pmod{aA}$ such that

$$g(n) = \chi_{aA}(n)n^{\alpha}$$

holds for all $n \in \mathbb{N}_{aA}$.

Theorem C. [7] *Assume that $A > 0,\ B > 0,\ C,\ D \neq 0$ are fixed integers with $(A,\ B) = 1$ and a function $g \in \mathcal{M}^*$ satisfies the congruence*

$$g(An+B) \equiv Cg(n) + D \pmod{n} \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then the following assertions hold:

(A) *If $g(p) = 0$ for some prime p with $(p,\ A) = 1$, then*

$$p = 2,\ -C = D = 1,\ (2, AB) = 1 \ \ \textit{and}\ g(n) = \chi_2(n) \quad \textit{for all} \quad n \in \mathbb{N}_2,$$

(B) *If $g(n) \neq 0$ for all $n \in \mathbb{N}_A$, then either*

$$C + D = 1 \quad \textit{and} \quad g(n) = 1 \quad \textit{for all} \ \ n \in \mathbb{N},$$

or there are a non-negative integer α, a real-valued Dirichlet character χ_{aA} $\pmod{aA}$ such that

$$g(n) = \chi_{aA}(n)n^{\alpha}$$

holds for all $n \in \mathbb{N}_{aA}$.

In [4] we completely solved the equation

$$g(An+B) \equiv g(An) + D \pmod{n}$$

for a multiplicative function g under conditions $A, B \in \mathbb{N}, D \in \mathbb{Z} \setminus \{0\}$ and $(A,\ B) = 1$,$(A,\ 2) = 1$.

The main purpose of this paper is to extend the result of Theorem A and prove a result similar to Theorem C for two completely multiplicative functions. We prove

Theorem 1. *Assume that the integers $A>0, B>0, N>0, D\neq 0, E\neq 0$ and $g\in\mathcal{M}$ satisfy the relation*

$$Eg(An+B)\equiv D \pmod{n} \quad \textit{for all} \quad n\in\mathbb{N},\ n>N.$$

Let $(A,B)=d$ and $A=da$. Then there are a non-negative integer α and a real-valued Dirichlet character $\chi_a \pmod{a}$ such that

$$g(dn)=g(d)\chi_a(n)n^{\alpha} \quad \textit{for all} \quad n\in\mathbb{N}_a.$$

We note that Theorem A is a special case of Theorem 1 when $E=1$ and $(A,B)=1$.

Theorem 2. *Assume that the integers $A>0,\ B>0,\ (A,B)=1,\ C\neq\ \neq 0,\ D\neq 0,\ E\neq 0$ and $f,\ g\in\mathcal{M}^*$ satisfy the relation*

$$Eg(An+B)\equiv Cf(n)+D \pmod{n} \quad \textit{for all} \quad n\in\mathbb{N}.$$

Then the following assertions hold:

(I) *If $f(\pi)=0$ for some prime π, then*

(I. a) *there are a non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that*

$$n|f(n) \quad \textit{and} \quad g(m)=\chi_A(m)m^{\alpha} \quad (n\in\mathbb{N},\ m\in\mathbb{N}_A).$$

(I. b) *If $\pi|A$ and $(\pi,B)=1$, then $C=-2D$ and all further solutions $(f,\ g)$ have the form*

$$\pi=2,\ f(n)=\chi_2(n) \quad \textit{and} \quad g(m)=\chi_{2A}(m)m^{\alpha} \quad (n\in\mathbb{N},\ m\in\mathbb{N}_A),$$

where α is a non-negative integer and χ_{2A} is a real-valued Dirichlet character $\pmod{2A}$ with the condition $\chi_{2A}(A+B)=-\chi_{2A}(B)$.

(I. c) *If $(\pi,AB)=1$, then $C=-D$ and all further solutions $(f,\ g)$ have the form*

$$\pi=2,\ f(n)=\chi_2(n),\ g(2)=0 \quad \textit{and} \quad g(m)=\chi_{2A}(m)m^{\alpha} \quad (n\in\mathbb{N},\ m\in\mathbb{N}_A),$$

where α is a non-negative integer and χ_{2A} is a real-valued Dirichlet character $\pmod{2A}$.

(II) *If* $g(AN+B)=0$ *for some* $N\in\mathbb{N}$ *and* $f(n)\neq 0$ *for all* $n\in\mathbb{N}$, *then*

$$f(n)=1 \quad \textit{for all} \quad n\in\mathbb{N},$$

and either

$$g(An+B)=0 \textit{ for all } \ n\in\mathbb{N} \quad \textit{if} \quad C+D=0,$$

or there are a non-negative integer α *and a real-valued Dirichlet character* χ_A *such that*

$$g(n)=\chi_A(n)n^\alpha \quad \textit{for all} \quad n\in\mathbb{N}_A \quad \textit{if} \quad C+D\neq 0.$$

(III) *If* $g(An+B)f(n)\neq 0$ *for all* $n\in\mathbb{N}$, *then there are a non-negative* α *and a real-valued Dirichlet character* $\chi_A \pmod{A}$ *such that*

$$g(n)=\chi_A(n)n^\alpha \quad \textit{for all} \quad n\in\mathbb{N}_A,$$

furthermore either

$$C+D=Eg(B) \quad \textit{and} \quad f(n)=1 \quad \textit{for all} \quad n\in\mathbb{N}$$

or $n \mid f(n)$ *for all* $n\in\mathbb{N}$.

2. Proof of Theorem 1

Assume that the conditions of Theorem 1 are satisfied, i.e.

$$Eg(An+B)\equiv D \pmod{n} \quad \text{for all} \quad n\in\mathbb{N},\ n>N,$$

where $g\in\mathcal{M},\ A,\ B,\ N\in\mathbb{N}$ and $D\neq 0$ is a nonzero integer.

Let $d=(A,\ B),\ A=da,\ B=db,\ (a,b)=1.$ Then we have

$$Eg\big[d(an+b)\big]\equiv D \pmod{n} \quad \text{for all} \quad n\in\mathbb{N},\ n>N. \tag{4}$$

First we prove that

$$g(d)\neq 0,\quad G\in\mathcal{M}^*{}_a \quad \text{and} \quad Eg(B)=D,$$

where

$$G(n):=\frac{g(dn)}{g(d)} \text{ for all } \ n\in\mathbb{N}.$$

Since $(a,b)=1$, there are infinitely many $n\in\mathbb{N}$ such that $n>N$ and $(an+b,d)=1$. For these n, the relation (4) gives

$$Eg(d)g(an+b)\equiv D\pmod{n},$$

which with $D\neq 0$ shows that $g(d)\neq 0$.

Assume that k and l are fixed positive integers, for which $(kl,a)=1$. Let q be a prime for which

$$q>\max\{k,\ l,\ N,\ |B|,\ |D|,\ |E|,\ |Eg(kB)g(dl)-Dg(dkl)|\}.$$

Since $(kl,\ qa)=1$, $(a,\ b)=1$ and $q>|B|$, one can deduce from the Chinese Remainder Theorem that there are positive integers $x,\ u,\ \ y$ and v such that

$$kx=aqy+1,\quad (x,klbd)=1$$

and

$$lu=aqv+b,\quad (u,kldx)=1.$$

Then by (4), we have

$$klxu=aqT+b,$$

where $T:=by+v+aqyv$. These with (4) and the multiplicativity of g imply that

$$Eg(kB)g(x)=Eg(kBx)=Eg\big(d(aqby+b)\big)\equiv D\pmod{q},$$

$$Eg(dl)g(u)=Eg(dlu)=Eg\big(d(aqv+b)\big)\equiv D\pmod{q}$$

and

$$Eg(dkl)g(x)g(u)=Eg\big(d(klxu)\big)=Eg\big(d(aqT+b)\big)\equiv D\pmod{q}.$$

Hence $q>|D|$ implies that $g(x)g(u)\not\equiv 0\pmod{q}$, consequently

$$Eg(kB)g(dl)\equiv Dg(dkl)\pmod{q}.$$

This relation and the fact $q>|Dg(dkl)-Eg(dkB)g(l)|$ imply that

$$Dg(dkl)=Eg(kB)g(dl)$$

holds for all $k,\ l\in\mathbb{N},\ \ (kl,\ a)=1$.

By applying this relation with $l=1$, we have $Eg(kB)g(d)=Dg(dk)$ for all $k\in\mathbb{N},\ (k,\ a)=1$, therefore we obtain

$$Eg(B)=D\ \text{ and }\ g(dkl)=\frac{Eg(kB)g(dl)}{D}=\frac{Dg(dk)g(dl)}{Dg(d)}=\frac{g(dk)g(dl)}{g(d)}.$$

Consequently

$$G(kl) = \frac{g(dkl)}{g(d)} = \frac{g(dk)}{g(d)} \frac{g(dl)}{g(d)} = G(k)G(l), \quad G \in \mathcal{M}^*{}_a.$$

Hence we infer from (4) and the fact $(a,b)=1$ that

$$Eg\big[d\,(an+b)\big] = Eg(d)G\,(an+b) \equiv D \pmod{n} \ \text{ for all } \ n\in\mathbb{N},\ n>N$$

which, using the fact $G\in\mathcal{M}^*{}_a$, $Eg(d)G(b) = Eg(db) = Eg(B) = D$, gives

(5) $Eg(B)G\,(an+1) = DG(An+1) \equiv D \pmod{n}$ for all $n\in\mathbb{N},\ n>N.$

If $G(aM+1)=0$ for some $M\in\mathbb{N}$, then we get from (5) that

$$0 = DG\Big((aM+1)^t\Big) \equiv D \quad \left(\bmod\ \frac{(aM+1)^t-1}{a}\right)$$

for all $t\in\mathbb{N}$. This is impossible because

$$\frac{(aM+1)^t-1}{a} \to \infty \ \text{ as } \ t\to\infty.$$

Consequently

$$G(an+1)\neq 0 \ \text{ for all } \ n\in\mathbb{N}.$$

On the other hand, if $G(a\ell+1)=-1$ for some $\ell\in\mathbb{N}$, then from (5) we have

$$-D = DG\left[(a\ell+1)^{2t+1}\right] \equiv D \qquad \left(\bmod\ \frac{(a\ell+1)^{2t+1}-1}{a}\right),$$

which is impossible. If $G(an+1)=1$ for all $n\in\mathbb{N}$, then $G(n)=\chi_a(n)$ for all $n\in\mathbb{N}_a$ and so $g(dn)=g(d)G(n)=g(d)\chi_a(n)$. Theorem 1 is proved in this case with $\alpha=0$.

In the following we assume that $G(an+1)\notin\{0,\ -1,\ 1\}$ for all $n\in\mathbb{N}$. Let $M=am+1\in\mathbb{N}$, $N=an+1\in\mathbb{N}$. Then we infer from (5) that

$$DG(N)G(M)^{2t} = DG\big(NM^{2t}\big) \equiv D \ \left(\bmod\ \frac{NM^{2t}-1}{a}\right),$$

and so

$$\big(NM^{2t}-1\big)\ \Big|\ aD\big(G(N)G(M)^{2t}-1\big)$$

hold for all $t \in \mathbb{N}$.

To complete the proof of Theorem 1, we shall use the following result of [5] (see Lemma in [5])

Lemma 1. *Let* $U \geq 1,\ V,\ u \geq 1,\ v,\ \beta > 1,\ \gamma > 1,\ k \geq 1,\ l$ *and* $F \neq 0$ *be fixed integers. If*

$$\left(U\gamma^{kn+l} + V\right) \Bigm| F\Big(u\beta^{kn+l} + v\Big)$$

for all $n \in \mathbb{N}$*, then there is a positive integer* e *such that*

$$\beta = \gamma^e \quad \textit{and} \quad u(-V)^e + vU^e = 0.$$

Since $G(M)^2 > 1$, therefore Lemma 1 shows that there is a non-negative integer α such that

$$G(M)^2 = M^{2\alpha}$$

and

$$G(N) - N^\alpha = G(an+1) - (an+1)^\alpha = 0.$$

Let

$$\mathcal{G}(n) := \frac{G(n)}{n^\alpha}, \quad G(n) = n^\alpha \mathcal{G}(n) \quad (n \in \mathbb{N}).$$

Then we have

$$\mathcal{G}(an+1) = 1 \quad \text{for all} \quad n \in \mathbb{N},$$

which gives that $\mathcal{G}(n) = \chi_a(n)$ for all $n \in \mathbb{N}_a$, where χ_a is a real-valued Dirichlet character $\pmod{a}$. Therefore

$$g(dn) = g(d)G(n) = n^\alpha g(d)\mathcal{G}(n) = n^\alpha g(d)\chi_a(n) \quad \text{for all} \quad n \in \mathbb{N}_a.$$

The proof of Theorem 1 is finished.

3. Proof of (I) of Theorem 2

First we prove the following

Lemma 2. *Assume that* $a,\ b \in \mathbb{N},\ c,\ d \in \mathbb{Z},\ c \neq 0$ *and* $H \in \mathcal{M}^*$ *satisfy the condition*

$$(6) \qquad cH(an+b) + d \equiv 0 \pmod{an+b} \ \textit{for all} \ \ n \in \mathbb{N}.$$

Then we have:

If $d \neq 0$, *then* $H(n) = \chi_a(n)$ *for all* $n \in \mathbb{N}_a$.

If $d = 0$, *then either* $H(an+b) = 0$ *for all* $n \in \mathbb{N}$ *or* $H(n) \equiv 0 \pmod{n}$ *for all* $n \in \mathbb{N}_a$.

Proof. Assume that (6) holds for all $n \in \mathbb{N}$.

First we consider the case when $d \neq 0$. In this case, by (6) we have

$$H(b) \neq 0 \text{ and } cH(b)H(an+1) \equiv -d \pmod{an+1}.$$

Then, for each $N \in \mathbb{N}$ we have

$$-dH(aN+1) = cH(b)H(aN+1)H(an+1) =$$

$$= cH(b)H\Big[(aN+1)(an+1)\Big] \equiv -d \pmod{an+1},$$

consequently

$$-dH(aN+1) = -d, \quad H(aN+1) = 1$$

hold for each $N \in \mathbb{N}$. This shows that $H(n) = \chi_a(n)$ for all $n \in \mathbb{N}_a$.

Assume now that $d = 0$. In this case, by (6) we have

$$cH(an+b) \equiv 0 \pmod{an+b} \text{ for all } n \in \mathbb{N}.$$

We have two possibilities: either $H(an+b) = 0$ for all $n \in \mathbb{N}$ or $H(aN+b) \neq 0$ for some $N \in \mathbb{N}$. Assume that $H(aN+b) \neq 0$. Then

$$cH(aN+b)H(an+1) = cH\Big[(aN+b)(an+1)\Big] \equiv 0 \pmod{an+1}$$

for all $n \in \mathbb{N}$. Thus, for each prime p, $(p,a) = 1$, we have

$$cH(aN+b)H(p)^{\varphi(a)t} = cH(aN+b)H(p^{\varphi(a)t}) \equiv 0 \pmod{p^{\varphi(a)t}} \text{ for all } t \in \mathbb{N}.$$

This with $cH(aN+b) \neq 0$ shows that $p \mid H(p)$. Lemma 2 is proved.

Now assume that the integers $A > 0$, $B > 0$, $(A,B) = 1$, $C \neq 0$, $D \neq 0$, $E \neq 0$ and f, $g \in \mathcal{M}^*$ satisfy the relation

$$Eg(An+B) \equiv Cf(n) + D \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{7}$$

We shall prove the following lemma.

Lemma 3. *Assume that there is a prime π such that $f(\pi) = 0$. Then $Eg(B) = D$ and the following assertions hold:*

(a) *There are a non-negative integer α and a character $\chi_A \pmod{A}$ such that*

$$n|f(n) \quad \textit{and} \quad g(m) = \chi_A(m)m^\alpha \quad (n \in \mathbb{N},\ m \in \mathbb{N}_A).$$

(b) *If $\pi|A$ and $(\pi, B) = 1$, then $C = -2D$ and all further solutions $(f,\ g)$ of (7) have the form*

$$\pi = 2,\ f(n) = \chi_2(n) \quad \textit{and} \quad g(m) = \chi_{2A}(m)m^\alpha \quad (n \in \mathbb{N},\ m \in \mathbb{N}_A),$$

where α is a non-negative integer and χ_{2A} is a character $\pmod{2A}$ with the condition $\chi_{2A}(A + B) = -\chi_{2A}(B)$.

(c) *If $(\pi, AB) = 1$, then $C = -D$ and all further solutions $(f,\ g)$ of (7) have the form*

$$\pi = 2,\ f(n) = \chi_2(n),\ g(2) = 0 \quad \textit{and} \quad g(m) = \chi_{2A}(m)m^\alpha,\ (n \in \mathbb{N},\ m \in \mathbb{N}_A),$$

where α is a non-negative integer and χ_{2A} is a character $\pmod{2A}$.

Proof. Assume that there is a prime π such that $f(\pi) = 0$. Then, by writing $n\pi$ in the place of n in (7), we have

$$Eg(A\pi n + B) \equiv Cf(\pi n) + D = Cf(\pi)f(n) + D = D \pmod{n}$$

for all $n \in \mathbb{N}$. This, by using Theorem 1, implies that there are a non-negative integer α and a real-valued Dirichlet character $\chi_{\pi A} \pmod{\pi A}$ such that

$$g(n) = \chi_{\pi A}(n)n^\alpha \ \text{ holds for all } \ n \in \mathbb{N}_{\pi A}.$$

In the following let

$$G(n) := \frac{g(n)}{n^\alpha} \ \text{ for all } \ n \in \mathbb{N}.$$

Then

$$\text{(8)} \qquad g(n) = n^\alpha G(n) \ \text{ and } \ G(m) = \chi_{\pi A}(m) \ \text{ for all } \ n \in \mathbb{N},\ m \in \mathbb{N}_{\pi A},$$

and we infer from (7) that

$$EB^\alpha G(An + B) \equiv E(An + B)^\alpha G(An + B) = Eg(An + B) \equiv$$
$$\equiv Cf(n) + D \pmod{n}.$$

This gives

$$EB^{\alpha}G(An+B) \equiv Cf(n)+D \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{9}$$

Next we prove that

$$Eg(B)=D. \tag{10}$$

Indeed, by applying (9) with $n = B\pi m$, we obtain from (8)

$$\begin{aligned} Eg(B) &\equiv Eg(B)G(A\pi m+1) = EB^{\alpha}G(B)G(A\pi m+1) = \\ &= EB^{\alpha}G(AB\pi m+B) \equiv Cf(B\pi m)+D = D \pmod{m}, \end{aligned}$$

which proves (10).

We shall use the notation $\varphi_\alpha(n) := n^{\alpha}$, $\alpha \in \mathbb{N}$. Furthermore let $\mathcal{D} \in \mathcal{M}^*$ such that $n|\mathcal{D}(n)$ for all $n \in \mathbb{N}$. It is obvious that $\mathcal{D}(n) = n\mathcal{D}_1(n)$ and $\mathcal{D}_1 \in \mathcal{M}^*$.

We shall prove that the solution $(f,\ g)$ of (7) is $(\mathcal{D},\ \chi_A\varphi_\alpha)$ in the following cases:

(i) $f(p) = 0$ for some prime p, $p \neq \pi$,

(ii) $f(B) = 0$.

Case (i): Assume that there is a prime p, $p \neq \pi$ for which $f(p) = 0$. Then, as we have seen in the proof of (8), we have

$$g(n) = n^{\alpha}G(n) \quad \text{and} \quad G(m) = \chi_{pA}(m) \quad \text{for all} \quad n \in \mathbb{N},\ m \in \mathbb{N}_{pA},$$

which imply that $G(n) = \chi_A(n)$ for all $n \in \mathbb{N}_A$. In this case we infer from (7) and (10) that

$$\begin{aligned} Cf(n) &\equiv Eg(An+B) - D = E(An+B)^{\alpha}G(An+B) - D \equiv \\ &\equiv EB^{\alpha}G(B) - D = Eg(B) - D = 0 \pmod{n}, \end{aligned}$$

which by using Lemma 2 gives $n|f(n)$ for all $n \in \mathbb{N}$. Thus the solution $(f,\ g)$ of (7) is $(\mathcal{D},\ \chi_A\varphi_\alpha)$ in the case (i).

In the following we assume that

$$f(n) \neq 0 \quad \text{if and only if} \quad (n,\ \pi) = 1. \tag{11}$$

Case (ii): Assume that $f(B) = 0$. Then by writing Bn in the place of n in (9), we have

$$\begin{aligned} Eg(B)G(An+1) &= EB^{\alpha}G(B)G(An+1) = \\ &= EB^{\alpha}G(ABn+B) \equiv Cf(B)f(n)+D \equiv D \pmod{n} \end{aligned}$$

for all $n\in\mathbb{N}$. This relation with Theorem 1 implies that $G(n)=\chi_A(n)$ for all $n\in\mathbb{N}_A$. Thus we get from (10) that $EB^\alpha G(An+B)=EB^\alpha G(B)=Eg(B)= = D$, consequently from (9) that $Cf(n)\equiv 0 \pmod n$ for all $n\in\mathbb{N}$. Hence Lemma 2 gives $n\mid f(n)$ for all $n\in\mathbb{N}$ and so the solution of (7) is $(\mathcal{D},\ \chi_A\varphi_\alpha)$ for the case (ii).

In the following we assume that

$$f(B)\neq 0 \ \text{ and } \ (\pi,\ B)=1. \tag{12}$$

For each $\ell\geq 0$ let

$$\mathcal{H}_\ell := \{\ n\in\mathbb{N}\mid \pi^\ell \| An+B\ \}.$$

We note that

$$\mathbb{N} \ = \begin{cases} \mathcal{H}_0 & \text{if } \pi|A,\\ \bigcup\limits_{\ell=0}^{\infty}\mathcal{H}_\ell & \text{if } (\pi,\ A)=1.\end{cases}$$

Let $n_\ell\in\mathcal{H}_\ell$ and $An_\ell+B=\pi^\ell N_\ell$. It is clear to see from $(A\pi,\ B)=1$ that $(n_\ell,\ A\pi)=(N_\ell,\ A\pi)=1$. By writing $\pi^{\ell+1}m+n_\ell$ in place of n in (10), we infer from (8) that

$$\begin{aligned} EB^\alpha G(An_\ell+B) &= EB^\alpha G(\pi^\ell)G(N_\ell)=EB^\alpha G(\pi^\ell)G(A\pi m+N_\ell)=\\ &= EB^\alpha G\Big(A\left(\pi^{\ell+1}m+n_\ell\right)+B\Big)\equiv\\ &\equiv Cf\left(\pi^{\ell+1}m+n_\ell\right)+D \pmod{\pi^{\ell+1}m+n_\ell}\end{aligned}$$

and

$$Cf\left(\pi^{\ell+1}m+n_\ell\right)\equiv EB^\alpha G(An_\ell+B)-D:=K_\ell \pmod{\pi^{\ell+1}m+n_\ell} \tag{13}$$

hold for all $m\in\mathbb{N}$.

We shall consider (13) in two cases, according to $K_\ell=0$ or $K_\ell\neq 0$.

Case I. $K_\ell=0$ for some $\ell\in\mathbb{N},\ 0\leq\ell<\pi$.

In this case, we infer from (11) and the fact $(n_\ell,\ \pi)=1$ that $f\left(\pi^{\ell+1}m+n_\ell\right)\neq 0$, consequently we obtain from Lemma 2 that $n\mid f(n)$ for all $n\in\mathbb{N}_\pi$. Hence by using the fact $f(\pi)=0$, we get from (7) that

$$n\mid f(n) \ \text{ and } \ Eg(An+B)\equiv Cf(n)+D\equiv D \pmod n \ \text{ for all } \ n\in\mathbb{N},$$

which gives that $g(n) = n^\alpha \chi_A(n)$ for all $(n \in \mathbb{N}_A)$, where $\alpha \in \mathbb{N}$ and χ_A is a real-valued Dirichlet character (mod A).

Case II. $K_\ell \neq 0$ for $\ell = 0, 1, \ldots, \pi - 1$.

In this case, by applying (13) for $\ell = 0$, Lemma 2 gives

$$f(n) = \chi_\pi(n) \quad \text{for all} \quad n \in \mathbb{N}_\pi.$$

Since $f(\pi) = 0$, we have

$$f(n) = \chi_\pi(n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{14}$$

Thus, from (13) and (14) we get

$$EB^\alpha G(An_\ell + B) \equiv Cf\left(\pi^{\ell+1}m + n_\ell\right) + D = Cf\left(n_\ell\right) + D \pmod{\pi^{\ell+1}m + n_\ell}$$

and so

$$EB^\alpha G(An_\ell + B) = Cf(n_\ell) + D$$

hold for all $m \in \mathbb{N},\ \ell \in \{0, 1, \ldots, \pi - 1\}$ and $n_\ell \in \mathcal{H}_\ell$. Consequently, we have have

$$EB^\alpha G(An + B) = Cf(n) + D \quad \text{for all} \quad n \in \mathbb{N}. \tag{15}$$

From (10) we have $EB^\alpha G(B) = Eg(B) = D$, and so we get from (15) that

$$DG(An + 1) = Cf(B)f(n) + D \quad \text{for all} \quad n \in \mathbb{N}. \tag{16}$$

We shall deduce from (16) that

$$\pi = 2. \tag{17}$$

Assume that $\pi \geq 3$. Then there is a $\nu \in \mathbb{N}$ such that $(A\nu + 1, A\pi) = (\nu, \pi) = 1$. By (8) and (14) we have $f(\nu) = \pm 1$ and $G(A\nu + 1) = \pm 1$, consequently we infer from (16) that

$$\begin{aligned} D^2 &= \left[DG(A\nu + 1)\right]^2 = (Cf(B)f(\nu) + D)^2 = \\ &= C^2 f(B)^2 f(\nu)^2 + 2CDf(B)f(\nu) + D^2 = \\ &= C^2 f(B)^2 + 2Cf(B)Df(\nu) + D^2, \end{aligned}$$

which implies $2Df(\nu) + Cf(B) = 0$.

Therefore from (16) we have

$$G(An+1) = -2f(\nu)f(n)+1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{18}$$

If $f(\nu) = -1$, then (18) gives that $G(An+1) = 2f(n)+1$ for all $n \in \mathbb{N}$. Hence we have $G(A+1) = 2f(1)+1 = 3$ and $9 = G(A+1)^2 = G\big[A(A+2)+1\big] = 2f(A+2)+1$, which imply $f(A+2) = \chi_\pi(A+2) = 4$. This is impossible.

Thus, we have $f(\nu) = 1$ and so

$$G(An+1) = -2f(n)+1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Since

$$G(An+1)^2 = G\big[An(An+2)+1\big] = -2f(n)f(An+2)+1$$

and

$$G(An+1)^2 = \big[-2f(n)+1\big]^2 = 4f(n)^2 - 4f(n)+1,$$

we have

$$f(n)\big[f(An+2)+2f(n)-2\big] = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

This relation with the fact $f(n) = \chi_\pi(n)$ implies that

$$f(n) = 1 \quad \text{and} \quad f(An+2) = 0 \quad \text{for all} \quad n \in \mathbb{N}_\pi,$$

from which we obtain that

$$\pi | An+2 \quad \text{for all} \quad n \in \mathbb{N}_\pi.$$

Since $1 \in \mathbb{N}_\pi$ and $\pi - 1 \in \mathbb{N}_\pi$, we get from the last relation that $\pi | A+2$, $\pi | A(\pi-1)+2 = A\pi - (A-2)$. These imply $\pi = 2$.

Thus (17) is proved.

Now we prove that $G(A+B) = -G(B)$ in the case $2|A$.

Since $f(n) = \chi_\pi(n) = \chi_2(n)$ for all $n \in \mathbb{N}_2$ and $(B,\ 2) = 1$, we get from (16) that $DG(A+1) = Cf(B)+D = C+D$. It follows from $2|A$ that $(A+1, 2A) = 1$, which implies from (8) that $G(A+1) = -1$ and $C = -2D$. Therefore (10) and (15) imply that

$$EB^\alpha G(A+B) = C+D = -2D+D = -D = -Eg(B) = -EB^\alpha G(B),$$

consequently

$$G(A+B) = -G(B).$$

The part (b) of Lemma 3 is proved.

Assume that $(2,\ AB)=1$. Then for every $\nu\geq 1$ there are $n_\nu\in\mathbb{N}$ and $N_\nu\in\mathbb{N}$ such that $An_\nu+1=2^\nu N_\nu,\ (N_\nu,\ 2)=1$. It is obvious that

$$(n_\nu,\ 2)=(N_\nu,\ 2A)=1,\ f(n_\nu)=\chi_2(n_\nu)=1\ \text{ and }\ G(N_\nu)=\chi_{2A}(N_\nu)=\pm 1.$$

We infer from (16) that

$$DG(An_\nu+1)=DG(2^\nu)G(N_\nu)=Cf(B)f(n_\nu)+D=C+D$$

holds for all $\nu\geq 1$, from which we obtain that

$$G(2)^\nu G(N_\nu)=G(2)G(N_1)=C+D\ \text{ for all }\ \nu\in\mathbb{N},\ \nu\geq 1.$$

This shows that $G(2)\in\{0,\ 1,\ -1\}$. We shall prove that $G(2)=0$. Assume that $G(2)=\pm 1$. Then $G(An+1)=\pm 1$, and so we get from (16) that $DG(A+1)= =Cf(B)f(1)+D=C+D$. Consequently $G(A+1)=-1$ and

$$D=DG(A+1)^2=DG\big[A(A+2)+1\big]=Cf(B)f(A+2)+D=C+D,$$

which is impossible. Thus we have proved that $G(2)=0$. Hence $g(2)=n^\alpha G(2)=0$ and $0=EB^\alpha G(A+B)=Cf(B)+D=C+D$, i.e. $C=-D$.

Lemma 3 and the part (I) of Theorem 2 are proved.

4. Proof of (II) of Theorem 2

We shall prove the part (II) of Theorem 2 by showing

Lemma 4. *Assume that all conditions of Theorem 2 are satisfied and $f(n)\neq 0$ for all $n\in\mathbb{N}$, $g(AN+B)=0$ for some $N\in\mathbb{N}$. Then*

$$f(n)=1\ \textit{ for all }\ n\in\mathbb{N},$$

and either

$$g(An+B)=0\ \textit{ for all }\ n\in\mathbb{N}\ \textit{ if }\ C+D=0,$$

or there are a non-negative integer α and a real-valued character χ_A such that

$$g(n)=\chi_A(n)n^\alpha\ \textit{ for all }\ n\in\mathbb{N}_A\ \textit{ if }\ C+D\neq 0.$$

Proof. We infer from $g(AN+B)=0$ that there is a prime p such that $p|AN+B,\ g(p)=0$ and $(p,A)=1$. By writing $n=pm+N$ in the place of n in

(7), we have $p|An+B = Apm+AN+B,\ g(An+B) = 0$ and $Cf(pm+N)+D \equiv \equiv 0 \pmod{pm+N}$. This with Lemma 2 implies that

$$f(n) = \chi_p(n) \quad \text{for all} \quad n \in \mathbb{N}_p. \tag{19}$$

We shall prove that

$$f(n) = 1 \quad \text{for all} \quad n \in \mathbb{N} \tag{20}$$

By writing nB in the place of n in (7), we have

$$Eg(B)g(An+1) \equiv Cf(B)f(n) + D \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{21}$$

It is obvious that if $g(B) = 0$, then (21) gives $Cf(B)f(n) + D \equiv 0 \pmod{n}$, and so we get from Lemma 2 that $f(n) = 1$ for all $n \in \mathbb{N}$. Thus (20) is true in the case $g(B) = 0$.

In the following we assume that

$$g(B) \neq 0, \ (p, B) = 1. \tag{22}$$

For every $M \in \mathbb{N}$, we have

$$Eg(AMn+B)g(An+1) = Eg\big[An\big(AMn+B+M\big)+B\big] \equiv$$

$$\equiv Cf(AMn+B+M)f(n) + D \pmod{n}$$

and

$$\big(Cf(M)f(n)+D\big)\big(Cf(B)f(n)+D\big) =$$

$$= C^2 f(B)f(M)f(n)^2 + CD\Big(f(B)+f(M)\Big)f(n) + D^2,$$

therefore we get from (7) and (21) that

$$CEg(B)f(AMn+B+M)f(n) \equiv \tag{23}$$

$$\equiv C^2 f(B)f(M)f(n)^2 + CD\Big(f(B)+f(M)\Big)f(n) + D^2 - DEg(B) \pmod{n}.$$

By applying $M = pm, \ (m \in \mathbb{N})$ in (23), using (19), we get

$$f(Apmn+B+pm) = \chi_p(Apmn+B+pm) = \chi_p(B) = f(B)$$

and

$$CEg(B)f(B)f(n) \equiv \tag{24}$$

$$\equiv C^2 f(B)f(pm)f(n)^2 + CD\Big(f(B)+f(pm)\Big)f(n) + D^2 - DEg(B) \pmod{n}.$$

Now, let us write $n(pt+1)$ in the place of n in (24), for every t. From (19) we get $f(pt+1)=1$ and that

$$CEg(B)f(B)f(n) \equiv$$

$$\equiv C^2 f(B)f(pm)f(n)^2 + CD\Big(f(B)+f(pm)\Big)f(n) + D^2 - DEg(B) \pmod{pt+1}$$

hold for all $n,\ m,\ t \in \mathbb{N}$, which gives that

$$C\Big[Eg(B)f(B) - D(f(B)+f(pm))\Big]f(n) = \tag{25}$$

$$= C^2 f(B)f(pm)f(n)^2 + D^2 - DEg(B)$$

hold for all $n,\ m \in \mathbb{N}$.

If there is an $m \in \mathbb{N}$ such that $Eg(B)f(B) - D(f(B)+f(pm)) \neq 0$, then we apply (25) for the case $n \in \mathbb{N}_p$, we have $f(n)^2 = 1$ and

$$f(n) = \frac{C^2 f(B)f(pm) + D^2 - DEg(B)}{CEg(B)f(B) - CD\Big(f(B)+f(pm)\Big)},$$

consequently $f(n)=1$ for all $n \in \mathbb{N}_p$. We apply (25) again for the case when $n = p^\nu,\ n \in \mathbb{N}$. We have $f(p) = \pm 1$ and

$$f(p)^\nu = \frac{C^2 f(B)f(pm) + D^2 - DEg(B)}{CEg(B)f(B) - CD(f(B)+f(pm))}.$$

This shows that $f(p)=1$. Thus (20) is proved in this case.

If $Eg(B)f(B) = D(f(B)+f(pm))$ for all $m \in \mathbb{N}$, then

$$f(m) = \frac{Eg(B)f(B) - Df(B)}{Df(p)} \quad \text{for all } \ m \in \mathbb{N},$$

which implies $f(m)=1$ for all $m \in \mathbb{N}$.

Finally, we infer from (7) and (20) that

$$Eg(An+B) \equiv Cf(n) + D = C + D \pmod{n}.$$

If $C+D \neq 0$, then Theorem 1 implies that there are a non-negative integer α and a real-valued Dirichlet character χ_A such that

$$g(n) = \chi_A(n) n^\alpha \quad \text{for all } \ n \in \mathbb{N}_A.$$

If $C+D=0$, then $Eg(An+B)\equiv 0 \pmod{n}$. Assume that $g(A\ell+B)\neq 0$ for some $\ell\in\mathbb{N}$. Let $q\in\mathcal{P}$, $(q,A)=1$. Then $(A\ell+B)q^{\varphi(A)t}\equiv B \pmod{A}$ and

$$g\Big((A\ell+B)q^{\varphi(A)t}\Big)=g(A\ell+B)g(q)^{\varphi(A)t}\equiv 0 \quad \left(\bmod\ \frac{(A\ell+B)q^{\varphi(A)t}-B}{A}\right)$$

for all $t\in\mathbb{N}$. This implies that $P(t)\ |g(A\ell+B)g(q)$, where

$$P(t)=\text{ the greatest prime factor of } \frac{(A\ell+B)q^{\varphi(A)t}-B}{A}.$$

This is impossible, because well-known that $P(t)$ cannot be bounded.

Lemma 4 and the part (II) of Theorem 2 is proved.

5. Proof of (III) of Theorem 2

Assume that all conditions of Theorem 2 are satisfied and

$$g(An+B)f(n)\neq 0 \ \text{ for all } \ n\in\mathbb{N}. \tag{26}$$

From (7) we have

$$Eg(B)g(An+1)\equiv Cf(B)f(n)+D \pmod{n} \ \text{ for all } \ n\in\mathbb{N},$$

and so

$$\mathcal{E}g(An+1)\equiv \mathcal{C}f(n)+D \pmod{n} \ \text{ for all } \ n\in\mathbb{N}, \tag{27}$$

where $\mathcal{E}:=Eg(B)\neq 0$, $\mathcal{C}:=Cf(B)\neq 0$.

First we infer from (27) that

$$\mathcal{E}^2g(ANn+1)g(AMn+1)\equiv(\mathcal{C}f(N)f(n)+D)(\mathcal{C}f(M)f(n)+D) \pmod{n}$$

and

$$\mathcal{E}^2g(ANn+1)g(AMn+1)=\mathcal{E}^2g\Big[An\big(ANMn+N+M\big)+1\Big]\equiv$$

$$\equiv\mathcal{C}\mathcal{E}f(ANMn+N+M)f(n)+D\mathcal{E} \pmod{n}$$

are satisfied for all $n,\ N,\ M \in \mathbb{N}$. Consequently

$$\mathcal{C}\mathcal{E}f(ANMn+N+M)f(n) \equiv$$

$$\equiv \mathcal{C}^2 f(N)f(M)f(n)^2 + \mathcal{C}D\Big(f(N)+f(M)\Big)f(n) + D^2 - D\mathcal{E} \pmod{n}$$

holds for all $n,\ N,\ M \in \mathbb{N}$. By writing $n(N+M)$ in the place of n in the above congruence, we have

$$a(N,M)f(ANMn+1)f(n) \equiv \tag{28}$$

$$\equiv b(N,M)f(n)^2 + c(N,M)f(n) + d \pmod{n},$$

where

$$a(N,M) := \mathcal{C}\mathcal{E}f(N+M)^2,\ b(N,M) := \mathcal{C}^2 f(N)f(M)f(N+M)^2,$$

and

$$c(N,M) := \mathcal{C}D\Big(f(N)+f(M)\Big)f(N+M), \quad d := D^2 - D\mathcal{E}.$$

By applying (28) with $N = M = 1$, we have

$$a(1,1)f(An+1)f(n) \equiv b(1,1)f(n)^2 + c(1,1)f(n) + d \pmod{n},$$

and if we substitute n by NMn, then

$$a(1,1)f(NM)f(ANMn+1)f(n) \equiv$$

$$\equiv b(1,1)f(NM)^2 f(n)^2 + c(1,1)f(NM)f(n) + d \pmod{n}.$$

Hence, this congruence with (28) implies

$$f(2)^2 f(NM)\big[b(N,M)f(n)^2 + c(N,M)f(n) + d\big] \equiv \tag{29}$$

$$\equiv f(N+M)^2\big[b(1,1)f(NM)^2 f(n)^2 + c(1,1)f(NM)f(n) + d\big] \pmod{n}.$$

Let

$$\lambda(N,M) := f(2)^2 f(NM)c(N,M) - f(N+M)^2 c(1,1)f(NM) =$$

$$= \mathcal{C}Df(N+M)f(2)f(NM)\Big[f(2)\Big(f(N)+f(M)\Big) - 2f(N+M)\Big]$$

and

$$d(N,M) := d\big[f(2)^2 f(NM) - f(N+M)^2\big] =$$

$$= (D^2 - D\mathcal{E})\big[f(2)^2 f(NM) - f(N+M)^2\big].$$

Since

$$f(2)^2 f(NM)b(N,M) - f(N+M)^2 b(1,1)f(NM)^2 = 0$$

hold for all $N,\ M \in \mathbb{N}$, we infer from (29) that

$$\lambda(N,M)f(n) + d(N,M) \equiv 0 \pmod{n}. \tag{30}$$

Next we prove that (30) with (26) implies that

$$\text{either} \quad f(n) = 1 \quad \text{or} \quad n|f(n) \qquad \text{for all} \quad n \in \mathbb{N}. \tag{31}$$

We separate the cases $\lambda(N,M) = 0$ for every $N,\ M$, and $\lambda(N,M) \neq 0$ for some $N,\ N \in \mathbb{N}$.

Case A. $\lambda(N,M) = 0$ *for all* $N, M \in \mathbb{N}$

In this case (26) and (30) imply that

$$f(2)\Big(f(N) + f(M)\Big) - 2f(N+M) = 0 \tag{32}$$

and

$$d(N,M) = (D^2 - D\mathcal{E})\big[f(2)^2 f(NM) - f(N+M)^2\big] = 0 \tag{33}$$

for all $N, M \in \mathbb{N}$.

By (26) we have $f(2) \neq 0$. Then by using (32), we have $2f(3) = 2f(2+1) = = f(2)^2 + f(2)$ and $2f(2)^2 = 2f(3+1) = f(2)f(3) + f(2)$. These with $f(2) \neq 0$ imply that either $f(2) = 1$ or $f(2) = 2$. We apply (32) for the case when $N = 1$ and $M = m,\ m \in \mathbb{N}$. We have

$$2f(m+1) = f(2)f(m) + f(2) \quad \text{for all} \quad m \in \mathbb{N},$$

from which we obtain that $f(n) = 1$ in the case $f(2) = 1$ and $f(n) = n$ in the case $f(2) = 2$.

Case B. $\lambda(N,M) \neq 0$ *for some* $N, M \in \mathbb{N}$

In this case, if $d(N,M) \neq 0$, then we get from Lemma 2 that $f(n) = 1$ for all $n \in \mathbb{N}$. But this is impossible, because

$$d(N,M) = (D^2 - D\mathcal{E})\big[f(2)^2 f(NM) - f(N+M)^2\big] \neq 0.$$

Thus, in this case we have $d(N,M) = 0$, which by using (26) and Lemma 2 gives that $n|f(n)$ for all $n \in \mathbb{N}$. Consequently we proved (31) for all cases.

In the following we assume that (7), (26) and (31) hold.

First we assume that $f(n) = 1$ for all $n \in \mathbb{N}$. Then we infer from (7) that

$$Eg(An+B) \equiv Cf(n) + D = C + D \pmod{n} \quad \text{for all} \quad n \in \mathbb{N}. \tag{34}$$

We prove that in this case $C+D = \mathcal{E} = Eg(B)$ and $g(n) = n^\alpha \chi_A(n)$ is satisfied for all $n \in \mathbb{N}_A$.

Suppose that $C + D = 0$. Then $Eg(An+B) \equiv 0 \pmod{n}$ for all n. By (26) we have $g(AN+B) \neq 0$ is true for all $N \in \mathbb{N}$. Then $(AN+B)^{\varphi(A)t+1} \equiv B \pmod{A}$ and

$$Eg(AN+B)^{\varphi(A)t+1} =$$
$$= Eg\big[(AN+B)^{\varphi(A)t+1}\big] \equiv 0 \quad \left(\mathrm{mod}\ \frac{(AN+B)^{\varphi(A)t+1} - B}{A}\right)$$

hold for all $t \in \mathbb{N}$. This is impossible, because well-known that $P(t)$ cannot be bounded, where

$$P(t) \ := \ \text{the largest prime divisor of } \ \frac{(AN+B)^{\varphi(A)t+1} - B}{A}.$$

Thus, we proved that in the case $f(n) = 1$ for all n, we have $C + D \neq 0$. This with (34) shows that $g(n) = n^\alpha \chi_A(n)$ for all $n \in \mathbb{N}_A$, where $\alpha \geq 0$ is an integer. Finally we infer from (34) that

$$C + D \equiv Eg(An+B) = E(An+B)^\alpha \chi_A(An+B) \equiv$$
$$\equiv EB^\alpha \chi_A(B) = Eg(B) \pmod{n},$$

and so $C + D = Eg(B)$.

Now assume that $n|f(n)$ for all $n \in \mathbb{N}$. In this case we get from (7), (33) and Theorem 1 that $D = \mathcal{E} = Eg(B)$ and $g(n) = n^\alpha \chi_A(n)$ ($n \in \mathbb{N}_A$) with some a non-negative integer α.

The proof of the part (III) is completed and Theorem 2 is proved.

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