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NECESSARY AND SUFFICIENT CONDITIONS FOR SUMMABILITY OF DOUBLE FUNCTION SERIES ALMOST EVERYWHERE

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Dedicated to Professor Ferenc Schipp on his 70th birthday and to Professor Péter Simon on his 60th birthday

Abstract. In this paper we find necessary and sufficient conditions for bounded T-summability of a double functional series almost everywhere. An essential role which plays a part in the proof is due to the generalization (with a few different and simpler proofs) of some results of Nikishin [18] to double series. As an application, the influence of Lebesgue functions on the summability of double function series is considered. In addition, considerable improvements of known results are obtained.

1. Introduction

We shall denote by Q a d-dimensional interval, let μ be a positive measure of sets $E \subseteq \mathbf{R}^d$ with $\mu(E) < \infty$, and let $\mathbf{f} := \{f_{mn}\}$ be a system of functions, which are μ-integrable on Q, in short $\mathbf{f} \subseteq L^1_\mu(Q)$.

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We consider the double function series

$$\sum_{m,n} c_{mn} f_{mn}(x), \tag{1.1}$$

where the double sequence $c := (c_{mn})$ belongs to the Banach space ℓ^2 and $x := (x_1, \ldots, x_d)$ belongs to Q.

In the sequel, unless otherwise indicated, the free indices always run through all values 0, 1, 2,..., and summation is likewise over all 0, 1, 2,...

We assume that the measure μ is absolutely continuous with respect to the Lebesgue measure. We still prefer to retain it in order that the sets of μ-measure zero should at the same time be sets of measure zero also in the sense of Lebesgue (cf. [1], p. 2, and [5], p. 223). We note that μ fulfills this assumptions when, for example, $\mu(x) = \mu_1(x_1), \cdots, \mu_d(x_d)$ for every $x \in E$, where $\mu_1, \cdots, \mu_d$ are positive, bounded and monotone increasing functions in their domains of definition (see [5], p. 224) and the derivatives of $\mu_1, \cdots, \mu_d$ equals zero only in sets with Lebesgue measure zero (see [1], p. 2).

Let T be a triangular summability method with the τ_{mnkl} being the entries of the double series to double sequence transformation matrix.

We say that the double sequence (a_{mn}) is bounded convergent or b-convergent to a and write $b\text{-}\lim\limits_{m,n} a_{mn} = a$, if $a_{mn} = O(1)\ \&\ \exists \lim\limits_{m,n} a_{mn} = a$ (where $m, n \to \infty$). We say that the double series (1.1) is bounded T-summable or T_b-summable μ-a.e. (i.e. μ-almost everywhere) on Q for $c \in \ell^2$ if the double limit

$$b\text{-}\lim_{m,n} \sum_{k,l=0}^{m,n} \tau_{mnkl}\, c_{kl}\, f_{kl}(x) \tag{1.2}$$

exists μ-a.e. on Q.

In this work we find necessary and sufficient conditions in order that the double series (1.1) be T_b-summable μ-a.e. on Q for each $c \in \ell^2$.

2. Lemmas on μ-measurable functions and summability μ-a.e.

Lemma 2.1. *Let the function g be μ-measurable on Q. Then g is finite μ-a.e. on Q if and only if for every $\delta > 0$, a μ-measurable subset $Q_\delta \subset Q$ exists with $\mu Q_\delta > \mu Q - \delta$ such that*

$$\int_{Q_\delta} |\, g(x)|\, d\mu(x) < \infty. \tag{2.1}$$

Proof (cf. [7], pp. 10-11, [21], p. 142).

Lemma 2.2. *Let $l \geq 2$ be an integer, and let $\eta > 0$ be a real number. If the μ-measurable sets $E_i \subset Q$ and $\Phi_i \subset Q$ for each $i = 1, \ldots, l^3$ satisfy the conditions*

$$E_i \subset \Phi_i, \tag{2.2}$$

$$\mu\Phi_i \leq \eta, \tag{2.3}$$

and for all $k = 2, \ldots, l^3$ satisfy the condition

$$E_k \bigcap \bigcup_{j=1}^{k-1} \Phi_j = \emptyset, \tag{2.4}$$

then an l-tuple $[i_1, \ldots, i_l]$ of positive integers with

$$i_1 < i_2 < \ldots < i_l \tag{2.5}$$

exists, such that

$$\mu\left[\left(\bigcup_{\kappa=1}^{l} E_{i_\kappa}\right) \bigcap \bigcup_{k=1}^{l} (\Phi_{i_k} \setminus E_{i_k})\right] \leq \eta/l. \tag{2.6}$$

Proof (cf. [7], pp. 11-13). The one-dimensional case (with $Q = [0, 1]$) of the lemma is due to Nikischin ([18], p. 137, Lemma 1).

A double function sequence (a_{mn}) is called convergent in measure on the set Q to the limit function a, if for any $\varepsilon > 0$

$$\lim_{m,n} \mu\{x \in Q : |a_{mn}(x) - a(x)| \geq \varepsilon\} = 0.$$

A double function series $\sum\limits_{m,n} a_{mn}$ is called convergent in measure on Q to the sum a, if the double sequence of its partial sums converges in measure on Q to the value a.

We denote the partial sums of the double series (1.1) by $S_{mn}c$, that is,

$$S_{mn}c := \sum_{k,l=0}^{m,n} c_{kl}\, f_{kl}. \tag{2.7}$$

Lemma 2.3. *Let (T_{mn}) be a double sequence of continuous linear maps from the B-space X to the F-space Y. In order that*

$$b\text{-}\lim_{m,n}(T_{mn}c)(x)$$

exists μ-a.e. on Q for all $c \in X$, it is necessary and sufficient that

$$1^\circ \qquad \exists \sup_{m,n} |(T_{mn}c)(x)| < \infty \qquad \mu\text{-}a.e. \quad on \quad Q, \qquad \forall c \in X$$

and

$$2^\circ \quad b\text{-}\lim_{M,N} \sup_{m,\kappa \geq M;\, n,\lambda \geq N} |(T_{mn}d)(x) - (T_{\kappa\lambda}d)(x)| = 0$$

exists μ-a.e. on Q for all $d \in \tilde{X}$, where $\tilde{X} \subset X$ is a fundamental set in X.

Proof. See [8], pp. 332-334, Theorem 3.

Lemma 2.4. *Let (σ_{mn}) be a double sequence of functions μ-integrable on the set Q. In order that*

$$\sup_{m,n} |\sigma_{mn}(x)| < \infty$$

μ-a.e. on Q, it is necessary and sufficient that for every $\varepsilon > 0$ there exists a μ-measurable subset $Q_\varepsilon \subset Q$ and a constant $M_\varepsilon > 0$ with $\mu Q_\varepsilon > \mu Q - \varepsilon$ such that

$$\left| \int\limits_{Q_\varepsilon} \sum_{m,n=0}^{K} \chi_{mn}^{K}(x)\sigma_{mn}(x)d\mu(x) \right| \leq M_\varepsilon$$

for each subdivision $\mathfrak{M}$, where

$$\mathfrak{M} := \{\mathfrak{M}_{mn}^{K} : m,n = 0,1,\cdots,K\},$$

$\mathfrak{M}_{mn}^{K} \cap \mathfrak{M}_{\kappa\lambda}^{K} = \emptyset$, if $(m,n) \neq (\kappa,\lambda)$ and

$$\bigcup_{m,n=0}^{K} \mathfrak{M}_{mn}^{K} \subset Q,$$

where χ_{mn}^{K} is the characteristic function of $\mathfrak{M}_{mn}^{K}$.

Proof. By analogy with the proof of Lemma 3 from [21], pp. 142-144.

Having taken in Lemma 2.3

$$(T_{mn}c)(x) := \sum_{k,l=0}^{m,n} \tau_{mnkl}\, c_{kl}\, f_{kl}(x),$$

we obtain necessary and sufficient conditions for T_b-summability μ-a. on Q of the series (1.1) for any $c \in \ell^2$. The condition 2° of Lemma 2.3 is simple to check. To verify condition 1° of Lemma 2.3 we use Lemma 2.4, because this condition 1° is fulfilled iff for each $\varepsilon > 0$ and any $c \in \ell^2$ there exists a μ-measurable set $Q_{\varepsilon c} \subset Q$ with $\mu Q_{\varepsilon c} > \mu Q - \varepsilon$ and a constant $M_{\varepsilon c} > 0$ so that uniformly for the μ-measurable subdivisions $\mathfrak{M}$ of Q, the inequality

$$\left| \int\limits_{Q_{\varepsilon c}} \sum_{m,n=0}^{K} \chi_{mn}^{K}(x)\,(T_{mn}c)(x)\,d\mu(x) \right| \le M_{\varepsilon c} \tag{2.8}$$

is true. The last condition is not linear in the space ℓ^2 and to verify this is complicated. Further we proved some results with the aim to replace condition (2.8) by a linear condition. Our results generalize some lemmas of Nikishin [18], p. 154, the proof of Lemma 6, using ideas of [24], p. 46, the proof of Lemma 5.

Lemma 2.5. *Let the double series* (1.1) *be T_b-summable μ-a.e. on the set Q for any $c \in \ell^2$. Then for every $\varepsilon > 0$ there exists a real number $R_\varepsilon > 0$ such that*

$$\mu\{x \in Q : \, |(Sc)(x)| \ge R_\varepsilon\} \le \varepsilon \tag{2.9}$$

holds on the unit ball $\{c : \, \|c\| \le 1\} \subset \ell^2$, where

$$(Sc)(x) := \sup_{m,n} |(T_{mn}c)(x)|. \tag{2.10}$$

Proof. We consider the F-space $M := M(Q)$, which consists of all μ-a.e. finite μ-measurable functions f on the set Q with the quasi-norm (cf. [8], p. 102 and 104, [25], p. 38, [20], p, 269)

$$\|f\|_M := \inf_{\alpha>0} \{\, \alpha + \mu\{x \in Q : \, |f(x)| \ge \alpha\} \,\}.$$

Hence for any sequence $(f_n) \subset M$ the convergence $f_n \to f$ means that

$$\|f_n - f\|_M = \inf_{\alpha>0} \{\, \alpha + \mu\{x \in Q : \, |f_n(x) - f(x)| \ge \alpha\} \,\} \to 0.$$

Therefore, (f_n) converges to f in M if and only if $f_n \to f$ in measure on Q (see [8], p. 104, Lemma 7, or [25], p. 38), that is, for every $\varepsilon > 0$

$$\lim_n \mu\{x \in Q : \, |f_n(x) - f(x)| \ge \varepsilon\} = 0.$$

Denoting

$$(S_i c)(x) := \max_{m,n \le i} |(T_{mn} c)(x)|,$$

we obtain, that S_i are continuous sublinear (cf. [25], p. 23, or [19], p. 24) operators from ℓ^2 to M, and, therefore,

$$\lim_{i \to \infty} S_i c = Sc,$$

in the space M, where the operator S is defined by (2.10), and this means that for each $\varepsilon > 0$ there exists an $i_\varepsilon > 0$ so that $i > i_\varepsilon$ implies

$$\|S_i c - Sc\|_M < \varepsilon/2.$$

Denote by θ the zero in ℓ^2. By the principle of equi-continuity (see [8], pp. 52–53)

$$\lim_{c \to \theta} S_i c = 0$$

in the space M uniformly in $i \in \mathbf{N}$, i.e. for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ so that $\|c\| < \delta_\varepsilon$ implies

$$\|S_i c\|_M < \varepsilon/2.$$

Therefore, if $\|c\| < \delta_\varepsilon$ and $i > i_\varepsilon$, then

$$\|Sc\|_M \le \|Sc - S_i c\|_M + \|S_i c\|_M < \varepsilon.$$

Hence

$$\lim_{c \to \theta} Sc = 0. \tag{2.11}$$

Therefore, for each $\varepsilon > 0$ exists a $\delta_\varepsilon > 0$ such that $\|c\| < \delta_\varepsilon$ implies $\|Sc\|_M < \varepsilon$.

Denote the unit ball in ℓ^2 by U, that is,

$$U := \{c \in \ell^2 : \ \|c\| \le 1\}.$$

Let $c \in U$ and $\beta \to 0$, then $\beta c \to \theta$ uniformly on U and by (2.11)

$$\lim_{\beta \to 0} \|\beta\, Sc\|_M = 0,$$

(cf. [24], p. 45–46), that is,

$$\lim_{\beta \to 0} \inf_{\alpha > 0} \{\alpha + \mu\{x \in Q : \ |\beta(Sc)(x)| \ge \alpha\}\} = 0.$$

Therefore, for any $\varepsilon > 0$ a $\beta_\varepsilon > 0$ exists such that

$$\gamma_\varepsilon := \inf_{\alpha>0}\{\alpha + \mu\{x \in Q : \, |\beta_\varepsilon(Sc)(x)| \geq \alpha\} < \varepsilon/2.$$

By the definition of the infimum, an $\alpha_\varepsilon > 0$ exists such that

$$\mu\{x \in Q : |\beta_\varepsilon(Sc)(x)| \geq \alpha_\varepsilon\} - \varepsilon/2 < \gamma_\varepsilon < \varepsilon/2,$$

uniformly on U or

$$\mu\{x \in Q : \, |(Sc)(x)| \geq \alpha_\varepsilon/\beta_\varepsilon\} < \varepsilon$$

uniformly on U, and, putting $R_\varepsilon = \alpha_\varepsilon/\beta_\varepsilon$, we obtain (2.9) uniformly on U.

3. The main lemma

Lemma 3.1. *Let $l > 3$ be a positive integer and $\varepsilon, R_\varepsilon, A, D_l$ and C_l be positive real numbers, where $A \geq 1$. Let the double series (1.1) be T_b-summable μ-a.e. on Q for all $c \in \ell^2$. If for a μ-measurable subset $Q_1 \subset Q$ the estimate*

$$\mu\{x \in Q_1 : \, |(Sc)(x)|^2 \geq D_l R_\varepsilon\} \leq C_l\, \varepsilon \tag{3.1}$$

holds uniformly on the unit ball of ℓ^2, then there exists a μ-measurable subset $e \subset Q_1$ with the μ-measure

$$\mu e \leq l^3\, C_l\, \varepsilon \tag{3.2}$$

so that

$$\mu\{x \in Q_1 \setminus e : \, |(Sc)(x)|^2 \geq lAD_l R_\varepsilon\} \leq 3\, C_l\, \varepsilon/l \tag{3.3}$$

uniformly on the unit ball of ℓ^2.

Proof. (Cf. [18], pp. 141-145). We find $c_1 \in U$ such that

$$\mu\{x \in Q_1 : |(Sc_1)(x)|^2 \geq lAD_l R_\varepsilon\} > 3\, C_l\, \varepsilon/l. \tag{3.4}$$

If no such c_1 exists, then (3.3) is true for $e = \emptyset$ and Lemma 3.1 is proved. If (3.4) holds, then we denote

$$E_1 = \{x \in Q_1 : \, |(Sc_1)(x)|^2 \geq lA\, D_l\, R_\varepsilon\},$$

$$\Phi_1 = \{x \in Q_1 : |(Sc_1)(x)|^2 \geq D_l\, R_\varepsilon\}.$$

From (3.4) and (3.1) it follows that

$$E_1 \subset \Phi_1, \quad \mu E_1 > 3C_l\, \varepsilon/l, \quad \mu\Phi_1 \leq C_l\, \varepsilon.$$

We will seek $c_2 \in U$ such that

$$\mu\{x \in Q_1 \setminus \Phi_1 : |(Sc_2)(x)|^2 \geq lA\, D_l\, R_\varepsilon\} > 3\, C_l\, \varepsilon/l. \tag{3.5}$$

If no such c_2 exists, then (3.3) is true for $e = \Phi_1$ and Lemma 3.1 is proved. But if inequality (3.5) is valid, then we denote

$$E_2 = \{x \in Q_1 \setminus \Phi_1 : |(Sc_2)(x)|^2 \geq l\, A\, D_l\, R_\varepsilon\},$$

$$\Phi_2 = \{x \in Q_1 : |(Sc_2)(x)|^2 \geq D_l\, R_\varepsilon\}.$$

From (3.5) and (3.1) we obtain

$$E_2 \subset \Phi_2, \quad E_2 \cap \Phi_1 = \emptyset, \quad \mu E_2 > 3\, C_l\, \varepsilon/l, \quad \mu\Phi_2 \leq C_l\, \varepsilon.$$

Continuing this process until step s we obtain the sets

$$E_i = \left\{ x \in Q_1 \setminus \bigcup_{j=1}^{i-1} \Phi_j : |(Sc_i)(x)|^2 \geq l\, A\, D_l\, R_\varepsilon \right\}, \tag{3.6}$$

$$\Phi_i = \{x \in Q_1 : |(Sc_i)(x)|^2 \geq D_l\, R_\varepsilon\}, \tag{3.7}$$

for every $i = 1, ..., s$; moreover in every step we have two possibilities – either the required point $c_i \in l^2$ exists or the Lemma was proved. If the Lemma was not proved, we obtain sets E_i and Φ_i, satisfying conditions (2.2), (2.4) and

$$\mu\Phi_i \leq C_l\, \varepsilon, \tag{3.8}$$

$$\mu E_i > 3\, C_l\, \varepsilon/l \tag{3.9}$$

for each $i = 1, ..., s$ and $k = 2, ..., s$, and points $c_i \in U$. If we show that in case $s = l^3$ the inequality (3.1) leads to a contradiction, then Lemma 3.1 is proved, because the process described above terminates before $s = l^3$.

For the construction of the contradiction we use Lemma 2.2, having taken there $\eta = C_l\varepsilon$. We consider the set (cf. [18], p. 142)

$$P = \left(\bigcup_{\kappa=1}^{l} E_{i_\kappa}\right) \setminus \Psi_l,$$

where

$$\Psi_l := \left(\bigcup_{\kappa=1}^{l} E_{i_\kappa}\right) \bigcap \bigcup_{k=1}^{l} (\Phi_{i_k} \setminus E_{i_k}).$$

Further, by Lemma 2.2 we will now find an l-tuple $[i_1, \dots, i_l]$ of positive integers satisfying (2.5) such that

$$\mu\Psi_l \le C_l\,\varepsilon/l, \tag{3.10}$$

and also

$$E_i \cap E_j = \emptyset \quad (\forall i \ne j)$$

is satisfied. In view of (3.9) and (3.10)

$$\mu P = \mu\left(\bigcup_{\kappa=1}^{l} E_{i_\kappa}\right) - \mu\Psi_l \ge 3\,C_l\,\varepsilon - C_l\,\varepsilon/l > 2\,C_l\,\varepsilon.$$

If $x \in P$, then $x \in E_{i_m}$ for some m, but $x \notin \Phi_{i_n}$ for $n \ne m$ by (2.4). Therefore, on the set P, in view of (3.6) and (3.7),

$$|(Sc_{i_m})(x)|^2 \ge l\,A\,D_l\,R_\varepsilon \tag{3.11}$$

and for $n \ne m$

$$|(Sc_{i_n})(x)|^2 < D_l\,R_\varepsilon, \tag{3.12}$$

because $x \notin \Phi_{i_n}$. Now we consider the double sequence $I_\beta := (I_{\kappa\lambda}\beta)$, with

$$I_{\kappa\lambda}\beta := (2l)^{-1/2} \sum_{m=1}^{l} r_m(\beta)\, c_{\kappa\lambda}^{i_m},$$

where r_m are the Rademacher functions (see [1], p. 51, [10], p. 42, or [12], p. 19) with $0 \le \beta \le 1$ and $c_{i_m} := (c_{\kappa\lambda}^{i_m})$, We show below that a $\beta_0 \in [0,1]$ exists, such that

$$\sum_{\kappa,\,\lambda} |I_{\kappa\lambda}\beta_0|^2 \le 1 \tag{3.13}$$

and a set $P_1 \subset P$ exists, such that $\mu P_1 > C_l\,\varepsilon$ and

$$\mu\{x \in P_1 : |(SI_{\beta_0})(x)|^2 \geq D_l\,R_\varepsilon\} > C_l\,\varepsilon, \tag{3.14}$$

which is in a contradiction with inequality (3.1), in view of $P_1 \subset Q_1$. Since natural numbers $M(i_m, x)$ and $N(i_m, x)$ exist (cf. [18], pp. 135-136) such that

$$(T_{M(i_m,x),N(i_m,x)}c)(x) := \sum_{k,l=0}^{M(i_m,x),N(i_m,x)} \tau_{M(i_m,x),N(i_m,x),k,l}\, c_{kl}\, f_{kl}(x),$$

and by (2.10)

$$(Sc_{i_m})(x) := \sup_{\kappa,\lambda} |(T_{\kappa\lambda}c_{i_m})(x)| \;=\; |(T_{M(i_m,x),N(i_m,x)}\; c_{i_m})(x)|.$$

For each element $c_{i_m} \in \ell^2$ we define a linear operator $t_{i_m} : \ell^2 \to M$ by the relation

$$(t_{i_m}c)(x) = [\,\mathrm{sgn}\,(T_{M(i_m,x),N(i_m,x)}\; c_{i_m})(x)]\;(T_{M(i_m,x),N(i_m,x)}c)\,(x).$$

By the definition of the operators t_{i_m} we obtain

$$(Sc_{i_m})(x) = (t_{i_m}c_{i_m})(x)$$

and

$$(Sc)(x) \geq |(t_{i_m}c)(x)|.$$

Therefore, for any $m = 1, 2, \cdots, l$, we have μ-a.e. on the set Q

$$(SI_\beta)(x) \geq (2l)^{-1/2} \left| \sum_{k=1}^{l} r_k(\beta)\,(t_{i_m}c_{i_k})(x) \right|. \tag{3.15}$$

Denoting

$$Z_\beta(x) := \begin{cases} r_m(\beta)(t_{i_m}c_{i_m})(x), & \text{if } x \in P \cap E_{i_m}, \\ 0, & \text{if } x \notin P \end{cases}$$

and for $k \neq m$

$$\alpha_k(x) := \begin{cases} (t_{i_m}c_{i_k})(x), & \text{if } x \in E_{i_m}, \\ 0, & \text{if } x \in E_{i_k} \end{cases}$$

from the inequalities (3.11) and (3.12), we obtain

$$|Z_\beta(x)|^2 \ge l\, A\, D_l\, R_\varepsilon, \tag{3.16}$$

$$|\alpha_k(x)|^2 < D_l\, R_\varepsilon, \tag{3.17}$$

but inequality (3.15) yields

$$|(SI_\beta)(x)|^2 \ge |(2l)^{-1/2}|Z_\beta(x)| - (2l)^{-1/2}|Y_\beta(x)|\,|^2, \tag{3.18}$$

where

$$Y_\beta(x) := \sum_{\kappa=1}^{l} r_\kappa(\beta)\, \alpha_\kappa(x).$$

In order to estimate $Y_\beta(x)$ we consider the function Υ, putting

$$\Upsilon(\beta) := (2l)^{-1} \int_P |Y_\beta(x)|^2 d\mu(x).$$

Since

$$\int_0^1 \Upsilon(\beta) d\beta = (2l)^{-1} \int_P d\mu(x) \int_0^1 |Y_\beta(x)|^2 d\beta,$$

by the Hölder inequality, and the orthonormality of Rademacher's system, we obtain

$$\int_0^1 \Upsilon(\beta) d\beta \le (2l)^{-1} \int_P \sum_{\kappa=1}^{l} \alpha_\kappa^2(x) d\mu(x).$$

Therefore, by (3.17)

$$\begin{aligned}\int_0^1 \Upsilon(\beta) d\beta &\le (2l)^{-1} \int_P \sum_{\kappa=1}^{l} D_l R_\varepsilon d\mu(x) = \\ &= (2l)^{-1} D_l R_\varepsilon\, l\ \mu P,\end{aligned}$$

hence

$$\int_0^1 \Upsilon(\beta) d\beta \le (1/2)\, D_l\, R_\varepsilon\ \mu P. \tag{3.19}$$

Denoting

$$\Omega := \{\beta : \beta \in [0,1] \ \&\ \Upsilon(\beta) \le D_l\, R_\varepsilon\ \mu P\},$$

we obtain $\mu\Omega > 1/2$, since otherwise inequality (3.19) is contradicted. Indeed, if $\mu\Omega \le 1/2$, then

$$\mu\{\beta : \Upsilon(\beta) > D_l\, R_\varepsilon\ \mu P\} > 1/2 \quad \& \quad \int\limits_0^1 \Upsilon(\beta) d\beta > D_l\, R_\varepsilon\ \mu P \cdot 1/2.$$

Now we prove that inequality (3.13) holds. We will start with the equality

$$\int\limits_\Omega \sum_{\kappa,\lambda} |I_{\kappa\lambda}\beta|^2 d\beta = (2l)^{-1} \sum_{\kappa,\lambda} \int\limits_\Omega \left| \sum_{m=1}^{l} c_{\kappa\lambda}^{i_m}\, r_m(\beta) \right|^2 d\beta \le$$

$$\le (2l)^{-1} \sum_{\kappa,\lambda} \int\limits_0^1 \left| \sum_{m=1}^{l} c_{\kappa\lambda}^{i_m}\, r_m(\beta) \right|^2 d\beta \le$$

$$\le (2l)^{-1} \sum_{m=1}^{l} \sum_{\kappa,\lambda} (c_{\kappa\lambda}^{i_m})^2 \le 1/2,$$

that is,

$$\int\limits_\Omega \sum_{\kappa,\lambda} |I_{\kappa\lambda}\beta|^2 d\beta \le 1/2. \tag{3.20}$$

Since $\mu\Omega > 1/2$, it follows that there exists a number $\beta_0 \in \Omega$ such that inequality (3.13) holds, because otherwise we have a contradiction to inequality (3.20).

Thus inequality (3.13) is proved. Now we show that inequality (3.14) also holds. Let $\beta_0 \in \Omega$ be such that (3.13) holds. Then from the definition of Ω

$$\Upsilon(\beta_0) \le D_l\, R_\varepsilon\, \mu P,$$

that is,

$$(2l)^{-1} \int\limits_P |Y_{\beta_0}(x)|^2 d\mu(x) \le D_l\, R_\varepsilon\ \mu P. \tag{3.21}$$

Now we define the set

$$P_1 := \{x \in P : (2l)^{-1}|Y_{\beta_0}(x)|^2 \le 2D_l\, R_\varepsilon\}.$$

Let us prove that (3.21) implies

$$\mu P_1 \ge \mu P\,/2. \tag{3.22}$$

In fact, if

$$\mu P_1 < \mu P\,/2,$$

then the μ-measure of the complement of this set

$$\mu\{x \in P : (2l)^{-1}|Y_{\beta_0}(x)|^2 > 2\,D_l\, R_\varepsilon\} \ge \mu P\,/2,$$

and, therefore,

$$\begin{aligned}(2l)^{-1}\int_P |Y_{\beta_0}(x)|^2 d\mu(x) &\ge \\ &\ge 2D_l\, R_\varepsilon\ \mu\{x \in P : (2l)^{-1}|Y_{\beta_0}(x)|^2 > 2D_l\, R_\varepsilon\} > \\ &> 2D_l\, R_\varepsilon\ \mu P\,/2 = D_l\, R_\varepsilon\ \mu P.\end{aligned}$$

This contradicts inequality (3.21). Since, as we calculated above, $\mu P > 2\,C_l\,\varepsilon$, by (3.22) we obtain

$$\mu P_1 > C_l\,\varepsilon. \tag{3.23}$$

If $x \in P_1$, then because of (3.16) and (3.18), in view of the definition of P_1, we have

$$\begin{aligned}|(SI_{\beta_0})(x)|^2 &\ge |(2l)^{-1/2}(l\,A\,D_l\,R_\varepsilon)^{1/2} - (2\,D_l\,R_\varepsilon)^{1/2}|^2 = \\ &= D_l\,R_\varepsilon\ (\sqrt{A/2} - \sqrt{2})^2.\end{aligned}$$

Now we choose the number A as follows:

$$A := 4\,(1 + \sqrt{2})^2 > 1,$$

and hence for any $x \in P_1$ we have

$$|(SI_{\beta_0})(x)|^2 \ge D_l\ R_\varepsilon[\sqrt{2}(1 + \sqrt{2}) - \sqrt{2}]^2 =$$

$$= 4D_l R_\varepsilon > D_l\, R_\varepsilon.$$

Thus the left side of (3.14) is equal to μP_1, and by (3.23) inequality (3.14) holds.

4. A relation between the T-means of a double function series and its coefficients

Theorem 4.1. *Let the double series* (1.1) *be* T_b*-summable* μ*-a.e. on* Q *for each* $c \in \ell^2$. *Then for every* $\varrho \in [1,2)$ *and every* $\eta > 0$, *there exists a* μ*-measurable subset* $E_{\eta,\varrho}$ *with* $\mu E_{\eta,\varrho} > \mu Q - \eta$, *and a constant* $K_{\varrho\,\eta} > 0$ *such that*

$$\left[\int\limits_{E_{\varrho\eta}} \sup_{\kappa,\lambda\le m,n}\left|\sum_{k,l=0}^{\kappa,\lambda}\tau_{\kappa\lambda kl}c_{kl}\,f_{kl}(x)\right|^{\varrho} d\mu(x)\right]^{1/\varrho} \le$$

$$\le K_{\varrho\,\eta}\left[\sum_{\kappa,\lambda=0}^{m,n}|c_{kl}|^2\right]^{1/2}. \tag{4.1}$$

Proof (Cf. Nikishin [18], p. 159). From Lemma 2.5 it follows, that for some number $\bar{R}_\varepsilon > 0$ by (2.9), the inequality

$$\mu\{x \in Q : |(Sc)(x)| \ge \bar{R}_\varepsilon\} \le \varepsilon$$

holds uniformly on the unit ball $U \subset \ell^2$. Put $R_\varepsilon = \bar{R}_\varepsilon^2$. In this case

$$\{x \in Q : |(Sc)(x)|^2 \ge R_\varepsilon\} = \{x \in Q : |(Sc)(x)| \ge \bar{R}_\varepsilon\},$$

so for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that on U we have

$$\mu\{x \in Q : \quad |(Sc)(x)|^2 \ge R_\varepsilon\} \le \varepsilon. \tag{4.2}$$

Later we use Lemma 3.1, taking $Q_1 = Q$ and $D_l = C_l = 1$. Then condition (3.1) of Lemma 3.1 is our condition (4.2). With this, by Lemma 3.1, one can find a μ-measurable set $e_1 \subset Q$, with $\mu e_1 \le l^3\varepsilon$ such that

$$\mu\{x \in Q\setminus e_1 : \quad |(Sc)(x)|^2 \ge lAR_\varepsilon\} \le 3\varepsilon/l \tag{4.3}$$

uniformly on U. Using Lemma 3.1 a second time, taking $Q_1 = Q \setminus e_1$ with $D_l = lA$ and $C_l = 3/l$, condition (3.1) becomes (4.3). Therefore, by Lemma 3.1 there exists a μ-measurable set $e_2 \in Q_1$ such that $\mu e_2 \le 3l^3\varepsilon/l$ and

$$\mu\{x \in Q_1\setminus e_2 : \quad |(Sc)(x)|^2 \ge (lA)^2R_\varepsilon\} \le (3/l)^2\varepsilon$$

uniformly on U. Continuing the above process unlimitedly, we obtain a sequence of μ-measurable disjoint sets (e_k) with $\mu e_k \le l^3(3/l)^{k-1}\varepsilon$ and the inequality

$$\mu\{Q\setminus H:\quad |(Sc)(x)|^2 \ge (lA)^k R_\varepsilon\} \le (3/l)^k\varepsilon \tag{4.4}$$

holds uniformly on U for all $k = 1, 2, \ldots$, where (cf. [18], p. 140, [7], p. 21)

$$H := \bigcup_{k=1}^{\infty} e_k,$$

and since $l > 3$, its measure

$$\mu H \le l^3\varepsilon \sum_{k=1}^{\infty}(3/l)^{k-1} \le 4l^3\varepsilon.$$

We now choose a number $\xi > R_\varepsilon$. There exists a natural number j such that

$$R_\varepsilon(lA)^j \le \xi < R_\varepsilon\,(lA)^{j+1}.$$

Using inequality (4.4) we obtain

$$\mu\{Q\setminus H:\quad |(Sc)(x)|^2 \ge \xi\} \le (3/l)^{j+1}(l/3)\varepsilon. \tag{4.5}$$

Since $\xi/R_\varepsilon < (lA)^{j+1}$ and $lA > 1$, it follows that

$$\ln(\xi/R_\varepsilon) < (j+1)\ln(lA),$$

whence $(3/l)^{j+1} \le \zeta$, where (cf. [18], p. 140)

$$\zeta := (3/l)^{\ln(\xi/R_\varepsilon)/\ln(lA)}.$$

This implies that

$$\zeta = e^{h(l)\ln(R_\varepsilon/\xi)} = R_\varepsilon^{h(l)}\,\xi^{-h(l)},$$

where

$$h(l) := (\ln l - \ln 3)/(\ln l + \ln A).$$

We now fix the numbers $\eta > 0$ and $\delta > 0$. Since $h(l) \to 1$ as $l \to \infty$, there exists l_δ such that $h(l) > 1 - \delta$, for $l \ge l_\delta$. We now find $\varepsilon > 0$ such that

$$\mu H \le 4\,l^3\,\varepsilon < \eta, \quad (l/3)\varepsilon \le 1.$$

Since $H = H(l, \varepsilon)$, the numbers $l = l(\delta)$ and $\varepsilon = \varepsilon(l, \eta)$, it follows that $H = H_1(\delta, \eta)$. Denoting $E_{\delta\eta} := Q \setminus H$, we have $\mu E_{\delta\eta} > \mu Q - \eta$ and the inequality (4.5) in the form

$$\mu\{x \in E_{\delta\eta} : |(Sc)(x)|^2 \geq \xi\} \leq R_\varepsilon^{h(l)}\, \xi^{-h(l)}. \tag{4.6}$$

Since $\varrho < 2$, then $2/\varrho > 1$ and

$$\mu\{x \in E_{\delta\eta} : |(Sc)(x)|^\varrho \geq \xi\} = \mu\{x \in E_{\delta\eta} : |(Sc)(x)|^2 \geq \xi^{2/\varrho}\}.$$

Applying this to (4.6) we obtain that

$$\mu\{x \in E_{\delta\eta} : |(Sc)x)|^\varrho \geq \xi\} \leq R_\varepsilon^{h(l)}\, \xi^{-2h(l)/\varrho}. \tag{4.7}$$

We now find a $\delta = \delta(\varrho)$ such that

$$2(1-\delta)/\varrho := 1 + \alpha, \quad \alpha > 0.$$

As a result, the numbers $l = l(\varrho)$, and $h(l) = h_\varrho$, the set $E_{\delta\eta} = E_{\eta,\,\varrho}$ and by inequality (4.7) uniformly on the unit ball U

$$\mu\{x \in E_{\eta,\,\varrho} : |(Sc)(x)|^\varrho \geq \xi\} \leq R_\varepsilon^{h_\varrho}\, \xi^{-(1+\alpha)}, \tag{4.8}$$

uniformly on the unit ball U for $\xi > \max\{1, R_\varepsilon\}$. Denoting for $k \in \mathbf{N}$

$$E_{\eta,\,\varrho}(k) := \{x \in E_{\eta,\,\varrho} : |(Sc)(x)|^\varrho \geq k\}$$

and

$$G_{\eta,\,\varrho}(k) := \{x \in E_{\eta,\,\varrho} : k \leq |(Sc)(x)|^\varrho < k+1\},$$

we have

$$G_{\eta,\,\varrho}(k) = E_{\eta,\,\varrho}(k) \setminus E_{\eta,\,\varrho}(k+1)$$

and

$$E_{\eta,\,\varrho} = \bigcup_{k=0}^{\infty} G_{\eta,\,\varrho}(k).$$

Denoting further,

$$v_k := \mu\, E_{\eta,\,\varrho}(k),$$

by partial summation (see [10], the first formula on p. 1, or [1], p. 71), for some natural $\kappa > \max\{2,\, R_\varepsilon\}$ by (4.8) we have

$$\sum_{k=\kappa}^{n}(k+1)\,(v_k - v_{k+1}) = \kappa\, v_\kappa - (n+1)\, v_{n+1} + \sum_{k=\kappa}^{n} v_k \le$$
$$\le R_\varepsilon^{h_\varrho}\,[\kappa^{-\alpha} + (n+1)^{-\alpha} + \sum_{k=\kappa}^{n} k^{-1-\alpha}] \le$$
$$\le R_\varepsilon^{h_\varrho}\,[\kappa^{-\alpha} + (n+1)^{-\alpha} + \alpha^{-1}(\kappa-1)^{-\alpha}],$$

from which

$$\sum_{k=\kappa}^{\infty}(k+1)\,(v_k - v_{k+1}) \le (1+\alpha^{-1})(\kappa-1)^{-\alpha}\, R_\varepsilon^{h_\varrho}. \tag{4.9}$$

If $(Sc)(x) = O(1)$, then $v_k = 0$ after some k. Now, by (4.9) we obtain

$$\int\limits_{E_{\eta,\,\varrho}} |(Sc)(x)|^\varrho d\mu(x) \le \sum_k \int\limits_{G_{\eta,\,\varrho}(k)} |(Sc)(x)|^\varrho d\mu(x) \le$$
$$\le \sum_k\,(k+1)\;\mu\, G_{\eta,\,\varrho}(k) =$$
$$= \sum_k\,(k+1)(v_k - v_{k+1}) =$$
$$\le \sum_{k<\kappa} + \sum_{k\ge\kappa} <$$
$$< \kappa(\kappa+1)\,\mu Q +$$
$$+ (1+\alpha^{-1})(\kappa-1)^{-\alpha} R_\varepsilon^{h_\varrho}.$$

Therefore, a constant $K_{\varrho\,\eta} > 0$ exists, such that

$$\int\limits_{E_{\eta,\,\varrho}} |(Sc)(x)|^\varrho d\mu(x) \le K_{\varrho\,\eta}^{\varrho}. \tag{4.10}$$

Now we denote $L^\varrho := L^\varrho_\mu(E_{\eta,\,\varrho})$. This is a Banach space, because $\varrho \ge 1$. In view of (4.10) we proved that for the sublinear operator $S : l^2 \to L^\varrho$, the inequality

$$\|Sc\|_{L^\varrho} \le K_{\varrho\,\eta}$$

holds uniformly on U, that is, it is bounded and its norm

$$\|S\| = \sup\{\|Sc\|_{L^\varrho} : \ c \in U\} \leq K_{\varrho\,\eta},$$

whence (cf. [19], p. 44)

$$\|Sc\|_{L^\varrho} \leq K_{\varrho\,\eta}\|c\|. \tag{4.11}$$

Replacing c in (4.11) with its section, yields (4.1).

From Theorem 4.1, assuming $\varrho = 1$, we deduce:

Corollary 4.2. *Let the double series* (1.1) *be T_b-summable on Q. Then for every $\delta > 0$ there exists a μ-measurable subset $Q_\delta \subset Q$ with $\mu Q_\delta > \mu Q - \delta$ and a constant $K_\delta > 0$ such that for all $c \in \ell^2$*

$$\int\limits_{Q_\delta} \sup_{\kappa,\lambda \leq m,n} \left| \sum_{k,l=0}^{\kappa,\lambda} \tau_{\kappa\lambda kl} c_{kl}\, f_{kl}(x) \right| d\mu(x) \leq K_\delta \left(\sum_{k,l=0}^{m,n} |c_{kl}|^2 \right)^{1/2}. \tag{4.12}$$

The method T is called b-regular, if T transforms every b-convergent double series in a b-convergent double sequence. From a result of Nigam (see [17], p. 265, Theorems 11 and 13, and [11], formula (2.4), cf. Hamilton [9], Theorem 20) it follows that for b-regularity of T are necessary the conditions $\exists \lim_{m,n} \tau_{mnkl} = 1$ and $\sup\limits_{m,n} |\tau_{mnkl}| < \infty$, that is the condition

$$\exists b\text{-}\lim_{m,n} \tau_{mnkl} = 1. \tag{4.13}$$

5. Necessary and sufficient condition for T_b-summability μ-a.e.

We denote $e_{kl} := (\delta_{\kappa k}\, \delta_{\lambda l})$ and we prove that the set $\{e_{kl}\}$ is a fundamental set in the space ℓ^2. In fact, for each $c \in \ell^2$ we have $c = \sum\limits_{k,l} c_{kl}\, e_{kl}$, and if

$$C_{mn} = \sum_{k,l=0}^{m,n} c_{kl}\, e_{kl},$$

then

$$c - C_{mn} = \left(\sum_{k,l=m+1,n+1}^{\infty} + \sum_{k,l=0,n+1}^{\infty} + \sum_{k,l=m+1,0}^{\infty} \right) c_{kl}\, e_{kl},$$

and hence, using twice the triangle inequality, we obtain

$$\|c - C_{mn}\| \leq$$

$$\leq \left(\sum_{k,l=m+1,n+1}^{\infty} c_{kl}^2 \right)^{1/2} + \left(\sum_{k,l=0,n+1}^{\infty} c_{kl}^2 \right)^{1/2} + \left(\sum_{k,l=m+1,0}^{\infty} c_{kl}^2 \right)^{1/2} \to 0$$

as $m, n \to \infty$.

In order that the double series (1.1) be T_b-summable μ-a.e. on Q for any $c \in \ell^2$ by condition $2°$ of Lemma 2.3 it is necessary that

$$b\text{-}\lim_{m,n} (T_{mn} e_{kl})(x) = b\text{-}\lim_{m,n} \tau_{mn\kappa\lambda} f_{\kappa\lambda}(x)$$

exists μ-a. e. on Q. By (4.13) and $|f_{\kappa\lambda}(x)| < \infty$ hold μ-a.e. on Q for any $\kappa, \lambda \in$ $\in \mathbf{N}$, then this is satisfied. Therefore, in view of (2.10) for the T_b-summability μ-a.e. on Q of (1.1) for any $c \in \ell^2$ by Lemma 2.3, it is necessary and sufficient that μ-a.e. on Q is

$$(Sc)(x) := \sup_{m,n} \left| \sum_{k,l=0}^{m,n} \tau_{mnkl}\, c_{kl}\, f_{kl}(x) \right| < \infty$$

for any $c \in \ell^2$. By Lemma 2.1 for the last it is necessary and sufficient that for every $c \in \ell^2$ and $\delta > 0$ there exists a μ-measurable subset $Q_{c\delta} \subset Q$ with $\mu Q_{c\delta} > \mu Q - \delta$ such that

$$\int_{Q_{c\delta}} (Sc)(x)\, d\mu(x) < \infty. \tag{5.1}$$

But by Corollary 4.2 for each $\delta > 0$ there exists a μ-measurable subset $Q_\delta \subset Q$ so that $\mu Q_\delta > \mu Q - \delta$ and a constant $K_\delta > 0$ such that for every $c \in \ell^2$ condition (4.12) is true and hence

$$\int_{Q_\delta} (Sc)(x) d\mu(x) \leq K_\delta \|c\| \tag{5.2}$$

holds. Since (5.2) implies (5.1), then according to Lemma 2.4 we can replace condition (5.2) by the following necessary and sufficient condition for the T_b-summability μ-a.e. on Q of the double series (1.1) for every $c \in \ell^2$:

For each $\delta > 0$ there exists a μ-measurable subset $Q_\delta \subset Q$ with $\mu Q_\delta > > \mu Q - \delta$ and a constant $K_\delta > 0$ such that for every $c \in \ell^2$ that the inequality

$$\left| \int\limits_{Q_\delta} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{k,l=0}^{m,n} \tau_{mnkl}\, c_{kl}\, f_{kl}(x)\, d\mu(x) \right| \le K_\delta \|c\| \tag{5.3}$$

holds uniformly with respect to all μ-measurable subdivisions $\mathfrak{M} = \{\mathfrak{M}_{mn}^K\}$ of μ-measurable disjoint parts of Q_δ, where χ_{mn}^K is the characteristic function of $\mathfrak{M}_{mn}^K$. But the left side of (5.3) is equal to

$$\left| \sum_{k,l=0}^{K} c_{kl}\, \Theta_{kl}^{K\delta} \right|,$$

where

$$\Theta_{kl}^{K\delta} = \int\limits_{Q_\delta} f_{kl}(x) \sum_{m,n \ge k,l}^{K} \chi_{mn}^K(x)\, \tau_{mnkl}\, d\mu(x).$$

Since the left side of the inequality (5.3) is a linear functional in the space l^2 (in contrast with (2.8), that is not linear), then (5.3) is fulfilled iff (see [25], p. 43, [19], p. 44)

$$\sum_{k,l=0}^{K} (\Theta_{kl}^{K\delta})^2 \le K_\delta^2.$$

Replacing here the square of the integral over the set Q_δ by an integral over the set $Q_\delta^2 =: Q_\delta \times Q_\delta$, we obtain

$$\sum_{k,l=0}^{K} (\Theta_{kl}^{K\delta})^2 =$$

$$= \int\limits_{Q_\delta^2} \sum_{k,l=0}^{K} f_{kl}(x)\, f_{kl}(y) \sum_{m,n=k,l}^{K} \chi_{mn}^K(x)\, \tau_{mnkl} \sum_{p,q=k,l}^{K} \chi_{pq}^K(y)\, \tau_{pqkl} d\mu(x)\, d\mu(y).$$

Now replace here the order of summation according to the scheme

$$\sum_{k,l=0}^{K}\sum_{m,n=k,l}^{K}\sum_{p,q=k,l}^{K}=\sum_{m,n=0}^{K}\sum_{k,l=0}^{m,n}\sum_{p,q=k,l}^{K}=$$

$$(5.4)\qquad =\sum_{m,n=0}^{K}\sum_{k,l=0}^{m,n}\sum_{p,q=k,l}^{m,n}+\sum_{m,n=0}^{K}\sum_{k,l=0}^{m,n}\sum_{p,q=m+1,\,n+1}^{K}+$$

$$+\sum_{m,n=0}^{K}\sum_{k,l=0}^{m,n}\sum_{p=m+1}^{K}\sum_{q=l}^{n}+\sum_{m,n=0}^{K}\sum_{k,l=0}^{m,n}\sum_{p=k}^{m}\sum_{q=n+1}^{K}.$$

Replacing once more the order of summation in (5.4), denoting

$$M:=\min\{m,\,p\},\quad N:=\min\{n,\,q\},$$

and

$$(5.5)\qquad \Phi_{\kappa\lambda}(x,y):=\sum_{k,l=0}^{\kappa,\lambda}\tau_{mnkl}\,\tau_{pqkl}\,f_{kl}(x)\,f_{kl}(y),$$

we obtain

$$\sum_{k,l=0}^{K}\sum_{m,n=k,l}^{K}\sum_{p,q=k,l}^{K}=\sum_{m,n=0}^{K}\sum_{p,q=0}^{m,n}\sum_{k,l=0}^{p,q}+\sum_{m,n=0}^{K}\sum_{p,q=m+1,n+1}^{K}\sum_{k,l=0}^{m,n}+$$

$$(5.6)\qquad +\sum_{m,n=0}^{K}\sum_{p=m+1}^{K}\sum_{q=0}^{n}\sum_{k,l=0}^{m,q}+\sum_{m,n=0}^{K}\sum_{p=0}^{m}\sum_{q=n+1}^{K}\sum_{k,l=0}^{p,n}=\sum_{m,n=0}^{K}\sum_{p,q=0}^{K}\sum_{k,l=0}^{M,N}.$$

Thus the following main result is proved:

Theorem 5.1. *Let the method T satisfy* (4.13). *In order that the double series* (1.1) *be T_b-summable μ-a.e. on the set Q for any $c\in\ell^2$, it is necessary and sufficient that for each $\delta>0$ there exists a μ-measurable subset $Q_\delta\subset Q$ with $\mu Q_\delta>\mu Q-\delta$, and a constant $K_\delta>0$ so that uniformly relatively to each μ-measurable disjoint subdivisions $\mathfrak{M}=\{\mathfrak{M}_{mn}^K\}$ of Q_δ, the following inequality is true:*

$$(5.7)\qquad \left|\int\limits_{Q_\delta^2}\sum_{m,n=0}^{K}\chi_{mn}^K(x)\sum_{p,q=0}^{K}\chi_{pq}^K(y)\,\Phi_{MN}(x,y)\,d\mu(x)\,d\mu(y)\right|\le K_\delta^2,$$

where χ^K_{mn} *is the characteristic function of* $\mathfrak{M}^K_{mn}$.

Theorem 5.1 is a generalization of Theorem 1 from [23] to double series.

We conclude some corollaries from Theorem 5.1.

Denote by K_{mn} the T-kernels of the function system $\mathbf{f}$, i.e.

$$K_{mn}(x,y) := \sum_{k,l=0}^{m,n} \tau_{mnkl} f_{kl}(x) f_{kl}(y),$$

and by L_{mn}, L'_{mn} and L''_{mn} its Lebesgue functions, i.e.

$$L_{mn}(x) := \int\limits_Q |K_{mn}(x,y)|\, d\mu(y),$$

$$L'_{mn}(x) := \int\limits_Q \max_{q\le n} |K_{mq}(x,y)|\, d\mu(y), \quad L''_{mn}(x) := \int\limits_Q \max_{p\le m} |K_{pn}(x,y)|\, d\mu(y),$$

The Lebesgue functions play a very important role in the investigation of the convergence and summability of the double series (1.1) (see Móricz [14-16]).

We show that from Theorem 5.1 it follows:

Corollary 5.2. *Let the method* T *satisfy* (4.13). *Let for* (5.5) *exist* $\xi_{ij} \ge 0$ *so that*

$$\Phi_{MN}(x,y) = O(1) \sum_{i,j=0}^{M,N} \xi_{ij}\, |K_{ij}(x,y)| \tag{5.8}$$

for $x, y \in Q$ *and there exist* $\xi'_i = \xi'_i(M)$, $\xi''_j = \xi''_j(N) \ge 0$ *such that*

$$\xi_{ij} = O(1)\, \xi'_i \cdot \xi''_j, \qquad \sum_{i,j=0}^{M,N} \xi'_i \cdot \xi''_j = O(1). \tag{5.9}$$

Suppose, further, that

$$L'_{mn}(x) = O_x(1), \quad L''_{mn}(x) = O_x(1) \tag{5.10}$$

are fulfilled μ*-a.e. on* Q. *Then the series* (1.1) *is* T_b*-summable* μ*-a.e. on* Q *for any* $c \in \ell^2$.

Proof. In order to prove (5.7), we divide Φ_{MN} on 4 parts relatively to the values of M, N, according to (5.6). In fact, by (5.8) and (5.9) the condition

$$L_{mn}(x) = O_x(1) \tag{5.11}$$

implies

$$A := \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=0}^{K} \chi_{pq}^K(y)\, |\Phi_{pq}(x,y)|\, d\mu(x)\, d\mu(y) =$$

$$= O(1) \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=0}^{K} \chi_{pq}^K(y) \sum_{i,j=0}^{p,q} \xi_{ij} |K_{ij}(x,y)|\, d\mu(x)\, d\mu(y) =$$

$$= O(1) \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{i,j=0}^{K} \xi_{ij} |K_{ij}(x,y)| \sum_{p,q=i,j}^{K} \chi_{pq}^K(y)\; d\mu(x)\, d\mu(y) =$$

$$= O(1) \int_{Q_\delta} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{i,j=0}^{K} \xi_{ij} L_{ij}(x)\, d\mu(x) =$$

$$= O(1) \int_{Q_\delta} \sup_{i,j} L_{ij}(x) \sum_{m,n=0}^{K} \chi_{mn}^K(x)\, d\mu(x) = O(1) \int_{Q_\delta} \sup_{i,j} L_{ij}(x)\, d\mu(x),$$

since in view of the disjoint subdivisions of Q_δ, we have

$$\sum_{p,q=0}^{K} \chi_{pq}^K(y) \le 1 \quad \& \quad \sum_{m,n=0}^{K} \chi_{mn}^K(x) \le 1.$$

Analogously, by (5.8) and (5.9) from condition (5.11) implies the estimate of

$$B := \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=m,n}^{K} \chi_{pq}^K(y)\, |\Phi_{mn}(x,y)|\, d\mu(x)\, d\mu(y)$$

(it is more suitable to replace $m+1$ and $n+1$ by m and n, respectively).

Further, by (5.8) the first of the conditions (5.10) and condition (5.9) imply, using (5.6),

$$C := \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=m,0}^{K,n} \chi_{pq}^K(y)\, |\Phi_{mq}(x,y)|\, d\mu(x)\, d\mu(y) =$$

$$= O(1) \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=m,0}^{K,n} \chi_{pq}^K(y) \sum_{i,j=0}^{m,q} \xi_{ij} |K_{ij}(x,y)|\, d\mu(x)\, d\mu(y) =$$

$$= O(1) \int_{Q_\delta} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{i=0}^{m} \xi_i' \int_Q \max_{j\le n} |K_{ij}(x,y)| \times$$

$$\times \sum_{p,q=m,j}^{K,n} \chi_{pq}^K(y) \sum_{j=0}^{q} \xi_j''\, d\mu(y)\, d\mu(x) =$$

$$= O(1) \int_{Q_\delta} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{i=0}^{m} \xi_i' \;\; L_{in}'(x)\, d\mu(x) = O(1) \int_{Q_\delta} \sup_{i,n} L_{in}'(x)\, d\mu(x).$$

Analogously, by the second of the conditions (5.10) with condition (5.9), we estimate

$$D := \int_{Q_\delta^2} \sum_{m,n=0}^{K} \chi_{mn}^K(x) \sum_{p,q=0,n}^{m,K} \chi_{pq}^K(y)\, |\Phi_{pn}(x,y)|\, d\mu(x)\, d\mu(y).$$

Now, if we denote

$$g(x) := 2 \sup_{i,j} L_{ij}(x) + \sup_{i,n} L_{in}'(x) + \sup_{m,j} L_{mj}''(x),$$

then by Lemma 2.1 for every $\delta > 0$ a μ-measurable subset $Q_\delta \subset Q$ exists (we do not change the notation Q_δ) with $\mu Q_\delta > \mu Q - \delta$ and a constant $K_\delta > 0$ such that $\int_{Q_\delta} g(x)\, d\mu(x) \le K_\delta$. Thus,

$$A + B + C + D \le K_\delta^2.$$

Since each of conditions (5.10) implies (5.11), we obtain what was to be proved.

Corollary 5.2 is an improvement of the main results of the papers [4-6] (see [5], Theorem 1, [6], Theorem 4, and cf. Móricz [15, 16]). For single series, Corollary 5.2 is due to Móricz (see [13], pp. 292–293, and cf. [22], Theorem 1, [2], pp. 202 and 297, [3], p. 267).

We can considerably sharpen Corollary 5.2. Namely, one of the conditions (5.10) is sufficient instead of two. In fact, by changing the order of summation and taking into account that $\Phi_{pn}(x,y) = \Phi_{pn}(y,x)$, we obtain

$$D = \int\limits_{Q_\delta^2} \sum_{p,q=0}^{K} \chi_{pq}^K(y) \sum_{m,n=p,0}^{K,q} \chi_{mn}^K(x) \, |\Phi_{pn}(y,x)| \, d\mu(x) \, d\mu(y) = C.$$

Thus we have proved

Corollary 5.3. *Let the method T satisfy* (4.13) *and for* (5.5) *exist $\xi_{kl} \geq 0$ such that conditions* (5.8) *and* (5.9) *hold. Let for T one of conditions* (5.10) *be fulfilled μ-a.e. on Q, then the series* (1.1) *is T_b-summable μ-a.e. on Q for any $c \in \ell^2$.*

If T is the factorable Riesz weighted means method (M_{pq}), where $p = (p_k)$ and $q = (q_l)$ are sequences of complex numbers, then in (1.2)

$$\tau_{mnkl} = (1 - P_{k-1}/P_m)\,(1 - Q_{l-1}/Q_n),$$

where $P_m = p_1 + \cdots + p_m \to \infty$ and $Q_n = q_1 + \cdots + q_n \to \infty$. Hence for (M_{pq}) condition (4.13) is satisfied. Let

$$\sum_{k,l=0}^{m,n} |p_k \, q_l| = O(P_m \, Q_n), \tag{5.12}$$

then condition (5.8) holds with $\xi_i' = |P_M|^{-1}(|p_i| + |p_{i+1}|)$ if $0 \leq i \leq M-1$ and $\xi_j'' = |Q_N|^{-1}(|q_j| + |q_{j+1}|)$ if $0 \leq j \leq N-1$, but $\xi_M' = \xi_N'' = 1$ (see [4], Lemma 2). Hence also condition (5.9) holds. Consequently, from Corollary 5.3 it follows

Corollary 5.4. *If (M_{pq}) satisfies* (5.12) *and for (M_{pq}) one of conditions* (5.10) *is fulfilled μ-a.e. on Q, then the series* (1.1) *is $(M_{pq})_b$-summable μ-a.e. on Q for any $c \in \ell^2$.*

If T is the convergence method E, then $\tau_{mnkl} = 1$ in (1.2) and

$$K_{mn}(x,y) = \Phi_{mn}(x,y) = \sum_{k,l=0}^{m,n} f_{kl}(x) \, f_{kl}(y).$$

Choosing $\xi_{kl} = \delta_{mk}\delta_{nl}$, we obtain that the conditions (5.8) and (5.9) are satisfied. Therefore, from Corollary 5.3 it follows:

Corollary 5.5. *If for E one of conditions* (5.10) *is fulfilled μ-a.e. on Q, then the series* (1.1) *boundedly converges μ-a.e. on Q for any $c \in \ell^2$.*

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