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ABSOLUTE SUMMABILITY FACTORS OVER BANACH ALGEBRAS

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Dedicated to Professors Ferenc Móricz, Ferenc Schipp and Péter Simon on their birthdays

Abstract. Let α be a nonnegative integer, A a unital Banach algebra, X a unital Banach A-algebra, $|T_A^\alpha|$ a method of absolute summability for X, defined by a normal series-to-series matrix over A, the inverse matrix of which has exactly $\alpha + 1$ non-zero diagonals, and B_A a method of summability defined by an infinite matrix over A. The cases, where T_A^α is a) a Riesz weighted means summability method over A and b) the product of Riesz weighted means summability method P_A and Q_A over A, are considered as application.

1. Introduction

1.1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, A an associative (not necessarily commutative) Banach algebra over $\mathbb{K}$ (for

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short, a Banach algebra) with norm $\|\cdot\|_A$ and X a left Banach A-module, i.e. a Banach space over $\mathbb{K}$ with norm $\|\cdot\|_X$ for which there has been defined a bilinear map (the multiplication over A) $(a,x)\to ax$ from $A\times X$ into X such that (cf. [6], p. 49, or [7], pp. 51 and 238)

1° $a(bx)=(ab)x$ for each $a,b\in A$ and $x\in X$;

2° $\|ax\|_X\leq\|a\|_A\|x\|_X$ for each $a\in A$ and each $x\in X$;

3° if A has the unit element e_A, then $e_Ax=x=xe_A$ for each $x\in X$.

A left Banach A-module X is called a *left Banach A-algebra* if its underlying Banach space is a Banach algebra (see [1], p. 238, or [2]). In this case

$$\|x_1x_2\|_X\leq\|x_1\|_X\|x_2\|_X$$

for each $x_1,x_2\in X$ and $ae_X=e_Xa$ for each $a\in A$ if X has the unit element e_X.

1.2. Let $\mathbb{N}_0=\{0,1,\ldots\}$, A a Banach algebra, X a left Banach A-module, $x=(x_n)$, where $x_n\in X$ for each $n\in\mathbb{N}_0$,

$$c(X):=\{x:\exists\lim_{n\to\infty}x_n\in X\}$$

and

$$l(X):=\left\{x:\sum_{n=0}^{\infty}\|x_n\|_X<\infty\right\}.$$

The addition and the multiplication over $\mathbb{K}$ in $l(X)$ we define coordinatewise, the multiplication over A by $ax=(ax_n)$ for each $a\in A$ and each $x\in l(X)$ and the norm $\|x\|_{l(X)}$ of $x\in l(X)$ by

$$\|x\|_{l(X)}=\sum_{n=0}^{\infty}\|x_n\|_X.$$

Then

$$\|ax\|_{l(X)}\leq\|a\|_A\|x\|_{l(X)}$$

for each $a\in A$ and each $x\in l(X)$. Taking this into account, we obtain that $l(X)$ is a left Banach A-module.

1.3. Let A be a Banach algebra with unit element e_A and (a_{nk}) a *normal matrix* over A that is, an infinite matrix over A for which $a_{nk}=\theta_A$ (the null element in A) if $k>n$ for each $k,n\in\mathbb{N}_0$ and a_{nn} is invertible in A for each

$n \in \mathbb{N}_0$. Hence every normal matrix over A has the inverse matrix (ξ_{nk}) such that

$$\sum_{k=\nu}^{n} a_{n\nu}\xi_{\nu k} = \delta_{nk},$$

where

$$\delta_{nk} = \begin{cases} e_A & \text{if } k = n, \\ \theta_A & \text{if } k \neq n \end{cases}$$

for each $k, n \in \mathbb{N}_0$.

1.4. Let A be a Banach algebra, X a left Banach A-module, (τ_{nk}) an infinite matrix over A, which defines a matrix transformation

$$T_n x = \sum_{k=0}^{\infty} \tau_{nk} x_k \tag{1}$$

of a series $\sum_k x_k$ (with terms x_k from X) to a sequence $(T_n x)$, and $(\overline{\tau}_{nk})$ is an infinite matrix over A, which defines a matrix transformation

$$\overline{T}_n x = \sum_{k=0}^{\infty} \overline{\tau}_{nk} x_k \tag{2}$$

of a series $\sum_k x_k$ to a series $\sum_n \overline{T}_n x$. We shall say that a normal series-to-series matrix $(\overline{\tau}_{nk})$ over A is a *T_A^{α}–matrix* if the inverse matrix $(\overline{\eta}_{nk})$ of $(\overline{\tau}_{nk})$, given by (2), has exactly $\alpha + 1$ (α is a nonnegative integer) non-zero diagonals that is, $\overline{\eta}_{nk} = \theta_A$ for $n < k$ and $n > k + \alpha$. The inverse matrix (η_{nk}) of the corresponding series-to-sequence matrix (τ_{nk}) has then $\alpha + 2$ nonzero diagonals because $\eta_{nk} = \overline{\eta}_{nk} - \overline{\eta}_{n,k+1}$ for each $n, k \in \mathbb{N}_0$ with $k \geq n$ (see [5], p. 56, formula (9.7)).

A series $\sum_k x_k$ is called to be

a) *summable by the method* T_A, defined by a matrix (τ_{nk}) over A (for short, T_A*-summable*) if $(T_n x) \in c(X)$ and

b) *absolutely summable by the method* $\overline{T}_A$, defined by a matrix $(\overline{\tau}_{nk})$ over A, (for short $|T_A|$-summable) if $(\overline{T}_n x) \in l(X)$.

Let $\varepsilon = (\varepsilon_n)$, where $\varepsilon_n \in A$ for each $\mathbb{N}_0$. If B_A is a method of summability over A, then the sequence ε *is called to be a summability factor of*

a) $(|T_A^\alpha|, B_A)$-*type* for X (for short, $\varepsilon \in (|T_A^\alpha|, B_A)$) if the series $\sum_k \varepsilon_k x_k$ is B_A-summable for each $|T_A|$-summable series $\sum_k x_k$ in X and

b) $(|T_A^\alpha|, |B_A|)$-*type* for X (for short, $\varepsilon \in (|T_A^\alpha|, |B_A|)$) if the series $\sum_k \varepsilon_k x_k$ is $|B_A|$-summable for each $|T_A|$-summable series $\sum_k x_k$ in X (see [3], p. 147).

In [3] have been proved the following generalizations of a classical Knopp-Lorentz theorem (see [8], p. 12, or [5], p. 34) and Hahn theorem (see [5], p. 25):

Proposition 1. *Let A be a Banach algebra, X a left Banach A-algebra with unit element e_X and $(\overline{\tau}_{nk})$ an infinite matrix over A. The matrix transformation* (2), *defined by $(\overline{\tau}_{nk})$, maps $l(X)$ into $l(X)$ if and only if*

$$\sum_{n=0}^{\infty} \|\overline{\tau}_{nk} e_X\|_X = O(1). \tag{3}$$

Proof. See [3], pp. 149–150.

Proposition 2. *Let A be a Banach algebra, X a left Banach A-algebra with unit e_X and (τ_{nk}) an infinite matrix over A. The matrix transformation* (1), *defined by (τ_{nk}), maps $l(X)$ into $c(X)$ if and only if*

1) $\|\tau_{nk} e_X\| = O(1)$;

2) $(\tau_{nk} e_X)$ *converges in X* for each $k \in \mathbb{N}_0$.

Proof. See [3], pp. 151–152.

1.5. In the paper [3], Theorems 4 and 5, have been found the necessary and sufficient conditions for elements ε_n of a Banach algebra A to be $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for a left Banach A-algebra X, where (see [3], pp. 147–148) P_A denotes the Riesz weighted means summability method over A (it is a T_A^1-method of summability) and B_A is a described summability method over A.

In the present paper we generalize these results giving the necessary and sufficient conditions for elements ε_n of a Banach algebra A to be $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors of summability for a left Banach A-algebra X in case, when an integer $a \geq 1$ and B_A is a described summability method over A. As an application, the $(|P_A|, B_A)$-factors, $(|P_A|, |B_A|)$-factors, $(|Q_A P_A|, B_A)$-factors and $(|Q_A P_A|, |B_A|)$-factors of summability for a left Banach A-algebra X are described in case when the matrix method B_A satisfies other conditions than in [3].

2. Main result

Before to describe the main result of this paper we give the necessary notations. For a given unital Banach algebra A method T_A^α of summability, defined by the matrix (τ_{nk}) over A, and a sequence $\varepsilon := (\varepsilon_n)$ in A let

$$D_n = \sup_k \|\tau_{n+k,n+k}\eta_{n+k,k}\|_A,$$

$$K\varepsilon_n = \sum_{k=n}^{n+\alpha}\sum_{\nu=k}^{n+\alpha}\varepsilon_\nu\eta_{\nu k},$$

where (η_{nk}) is the inverse matrix of (τ_{nk}).

If (η_{nk}) is the inverse matrix of (τ_{nk}) over A and $(\overline{\eta}_{nk})$ is the corresponding matrix (on the series to series form), then elements of these matrices are connected by

$$\overline{\eta}_{nk} = \sum_{\nu=k}^{n}\eta_{n\nu} \tag{4}$$

(see [5], formula (9.6)) for each $n \geq k$. By means of (4) we obtain

$$K\varepsilon_n = \sum_{\nu=n}^{n+\alpha}\varepsilon_\nu\overline{\eta}_{\nu n}. \tag{5}$$

Proposition 3. *Let α be a nonnegative integer, A a unital Banach algebra and X a left Banach A-algebra with unit element e_X. Let $|T_A^\alpha|$ be a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$, and B_A a series-to-sequence method of summability defined by a matrix (β_{nk}) over A. Let $\overline{\beta}_{nk} = \beta_{nk} - \beta_{n-1,k}$. Then we have*

a) *If (β_{nk}) satisfies the condition*

$$\lim_{n\to\infty}\beta_{nk} = e_A \tag{6}$$

and elements ε_k of A are $(|T_A^\alpha|, B_A)$-factors for X, then

$$\|\beta_{nn}\varepsilon_n\tau_{nn}^{-1}e_X\|_X = O(1) \tag{7}$$

and

$$\|(K\varepsilon_n)e_X\|_X = O(1). \tag{8}$$

If, in addition $|T_A^\alpha|$ *preserves the absolute convergence*[1], *then also*

$$\|\varepsilon_n e_X\|_X = O(1). \tag{9}$$

b) *If* D_n *is finite for each* $n \in \{0, 1, \ldots, \alpha+1\}$ *and* B_A *is a normal method of summability which satisfies the conditions* (6),

$$\sum_{n=\nu}^{\infty} \|(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1}\|_A = O(1), \tag{10}$$

$$\|\beta_{kk}\beta_{k+1,k+1}^{-1}\|_A = O(1) \tag{11}$$

and

$$\sum_{n=k}^{\infty} \|\overline{\beta}_{nk}\|_A = O(1), \tag{12}$$

then the elements ε_k *of* A *are* $(|T_A^\alpha|, |B_A|)$*-factors for* X *if conditions* (7) *and* (8) *hold.*

Proof. a) Since every T_A^α-matrix is normal, there exists the inverse transformation

$$x_k = \sum_{\nu=0}^{k} \overline{\eta}_{k\nu}\overline{T}_\nu x \tag{13}$$

of (2). Therefore,

$$B_n(\varepsilon x) := \sum_{\nu=0}^{n} \beta_{n\nu}\varepsilon_\nu x_\nu = \sum_{\nu=0}^{n} \beta_{n\nu}\varepsilon_v \sum_{k=0}^{\nu} \overline{\eta}_{vk}\overline{T}_k x = \sum_{k=0}^{n} \left(\sum_{v=k}^{n} \beta_{nv}\varepsilon_v\overline{\eta}_{vk}\right)\overline{T}_k x$$

or

$$B_n(\varepsilon x) = \sum_{k=0}^{n} \gamma_{nk}\overline{T}_k x,$$

[1] That is, every sequence $x \in l(X)$ is $|T_A^\alpha|$-summable.

where

$$\gamma_{nk} = \sum_{\nu=k}^{k+\alpha} \beta_{n\nu}\varepsilon_\nu\overline{\eta}_{\nu k}$$

for $k \le n$ because $\overline{\eta}_{nk} = \theta_A$ if $n > k+\alpha$. The matrix (γ_{nk}) transforms $(\overline{T}_n x) \in l(X)$ into $(B_n(\varepsilon x)) \in c(X)$ if and only if

$$\|\gamma_{nk}e_X\|_X = O(1) \tag{14}$$

and there exists

$$\lim_{n\to\infty} \gamma_{nk}e_X = \gamma_k \tag{15}$$

in X for each $k \in \mathbb{N}_0$ by Proposition 2. Hence,

$$\|\beta_{kk}\varepsilon_k\overline{\eta}_{kk}e_X\|_X = \|\gamma_{kk}e_X\|_X = O(1),$$

from which follows the condition (7) because $\overline{\eta}_{kk} = \tau_{kk}^{-1}$ for each $k \in \mathbb{N}_0$ (see [5], p. 57).

Since $\gamma_k = (K\varepsilon_k)e_X$ by the condition (6) in view of (5), then the condition (8) holds. In the particular case, when $|T_A^\alpha|$ preserves absolute convergence in X, then holds also (9) by Lemma 1 from [3], p. 152.

b) Let the elements $\varepsilon_n \in A$ satisfy the conditions (7) and (8). To show that $\varepsilon \in (|T_A^\alpha|, |B_A|)$, we have to show that the series $\sum_n \varepsilon_n x_n$ is $|B_A|$-summable for each $|T_A^\alpha|$-summable series $\sum_n x_n$ in X. For it we assume that $\sum_n x_n$ is a $|T_A^\alpha|$-summable series in X. Since every T_A^α-matrix is normal, there exists the inverse transformation (13) of (2). Then

$$\overline{B}_n(\varepsilon x) := \sum_{\nu=0}^{n} \overline{\beta}_{n\nu}\varepsilon_\nu x_\nu = \sum_{\nu=0}^{n} \overline{\beta}_{n\nu}\varepsilon_v \sum_{k=0}^{\nu} \overline{\eta}_{vk}\overline{T}_k x = \sum_{k=0}^{n}\left(\sum_{v=k}^{n} \overline{\beta}_{nv}\varepsilon_v\overline{\eta}_{vk}\right)\overline{T}_k x$$

or

$$\overline{B}_n(\varepsilon x) = \sum_{k=0}^{n} \overline{\gamma}_{nk}\overline{T}_k x,$$

where

$$\overline{\gamma}_{nk} = \sum_{\nu=k}^{n} \overline{\beta}_{n\nu}\varepsilon_\nu\overline{\eta}_{\nu k} = \sum_{\nu=k}^{k+\alpha} \overline{\beta}_{n\nu}\varepsilon_\nu\overline{\eta}_{\nu k}$$

for $k \geq n$ because again $\overline{\eta}_{nk} = \theta_A$ if $n > k + \alpha$. The matrix $(\overline{\gamma}_{nk})$ transforms $(\overline{T}_n x) \in l(X)$ into $(\overline{B}_n(\varepsilon x)) \in l(X)$ if and only if

$$\sum_{n=k}^{\infty} \|\overline{\gamma}_{nk} e_X\|_X = O(1) \tag{16}$$

by Proposition 1.

By partial summation we have

$$\overline{\gamma}_{nk} = \overline{\beta}_{n,n+1} \sum_{\nu=k}^{n} \varepsilon_\nu \overline{\eta}_{\nu k} + \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \sum_{l=k}^{\nu} \varepsilon_l \overline{\eta}_{lk} = \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \left(\sum_{l=k}^{k+\alpha} - \sum_{l=\nu+1}^{k+\alpha} \right) \varepsilon_l \overline{\eta}_{lk}.$$

Since $\overline{\beta}_{n,n+1} = 0$, then

$$\overline{\gamma}_{nk} = \left(\sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \right) K\varepsilon_k - \overline{\lambda}_{nk} = \overline{\beta}_{nk} K\varepsilon_k - \overline{\lambda}_{nk}$$

by (5), where

$$\overline{\lambda}_{nk} = \sum_{\nu=k}^{n} \Delta\overline{\beta}_{n\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk}.$$

We show first that

$$\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\| = O(1). \tag{17}$$

Indeed,

$$\begin{aligned}
\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\|_X &= \sum_{n=k}^{\infty} \left\| \sum_{\nu=k}^{n} (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X \leq \\
&\leq \sum_{n=k}^{\infty} \sum_{\nu=k}^{n} \left\| (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \right\|_A \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X = \\
&= \sum_{\nu=k}^{\infty} \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X \sum_{n=\nu}^{\infty} \left\| (\Delta\overline{\beta}_{n\nu}) \beta_{\nu\nu}^{-1} \right\|_A = \\
&= O(1) \sum_{\nu=k}^{\infty} \left\| \beta_{\nu\nu} \sum_{l=\nu+1}^{k+\alpha} \varepsilon_l \overline{\eta}_{lk} e_X \right\|_X
\end{aligned}$$

by the assumption (10). Since the method B_A defined by (β_{nk}) satisfies the condition (11), there is a number $M > 1$ (which does not depend on k) such that $\|\beta_{kk}\beta^{-1}_{k+1k+1}\|_A \leq M$ for each $k \in \mathbb{N}$. Therefore

$$\sum_{\nu=k}^{l} \|\beta_{\upsilon\upsilon}\beta_{ll}^{-1}\|_A \leq \sum_{r=0}^{l-k} M^r \leq (l-k+1)M^{l-k} \tag{18}$$

for each $k \leq l \leq k+a$. Taking this into account, we have

$$\sum_{n=k}^{\infty} \|\overline{\lambda}_{nk}e_X\|_X = O(1)\sum_{\nu=k}^{\infty}\left\|\beta_{\nu\nu}\sum_{l=\nu+1}^{k+\alpha}\beta_{ll}^{-1}(\beta_{ll}\varepsilon_l\tau_{ll}^{-1})(\tau_{ll}\overline{\eta}_{lk})e_X\right\|_X =$$

$$= O(1)\sum_{\nu=k}^{\infty}\sum_{l=\nu}^{k+\alpha}\|\beta_{\nu\nu}\beta_{ll}^{-1}\|_A\,\|\beta_{ll}\varepsilon_l\tau_{ll}^{-1}e_X\|_X\,\|\tau_{ll}\overline{\eta}_{lk}e_X\|_X =$$

$$= O(1)\sum_{l=k}^{k+\alpha}\|\beta_{ll}\varepsilon_l\tau_{ll}^{-1}e_X\|_X\,\|\tau_{ll}\sum_{s=k}^{l}\eta_{ls}e_X\|_X\sum_{\nu=k}^{l}\|\beta_{\nu\nu}\beta_{ll}^{-1}\|_A =$$

$$= O(1)\sum_{l=k}^{k+\alpha}\|\beta_{ll}\varepsilon_l\tau_{ll}^{-1}e_X\|_X\sum_{s=k}^{l}\|\tau_{ll}\eta_{ls}e_X\|_X$$

by (4) and (18). Now, by the condition (7), we have

$$\sum_{n=k}^{\infty}\|\overline{\lambda}_{nk}e_X\|_X = O(1)\sum_{s=k}^{k+\alpha}\sum_{l=s}^{k+\alpha}\|\tau_{ll}\eta_{ls}\|_A = O(1)\sum_{s=k}^{k+\alpha}\sum_{l=0}^{k+\alpha-s}\|\tau_{l+s,l+s}\eta_{l+s,s}\|_A =$$

$$= O(1)\sum_{s=k}^{k+\alpha}\sum_{l=0}^{k+\alpha-s}D_l = O(1)\sum_{l=0}^{\alpha}(\alpha+1-l)D_l = O(1)$$

because $D_0, D_1, \ldots, D_{\alpha-1}$ and D_α are finite.

Next we show that

$$\sum_{n=k}^{\infty}\|\overline{\beta}_{nk}(K\varepsilon_k)e_X\|_X = O(1). \tag{19}$$

Since B_A satisfies the condition (12), then by the condition (8) we have

$$\sum_{n=k}^{\infty}\|\overline{\beta}_{nk}(K\varepsilon_k)e_X\|_X \leq \sum_{n=k}^{\infty}\|\overline{\beta}_{nk}\|_A\|(K\varepsilon_k)e_X\|_X = O(1)\sum_{n=k}^{\infty}\|\overline{\beta}_{nk}\|_A = O(1).$$

Hence,

$$\sum_{n=k}^{\infty} \|\overline{\gamma}_{nk} e_X\|_X \leq \sum_{n=k}^{\infty} \|\beta_{nk}(K\varepsilon_k) e_X\|_X + \sum_{n=k}^{\infty} \|\overline{\lambda}_{nk} e_X\|_X = O(1)$$

by (16) and (17). Consequently, ε_n are $(|T_A^\alpha|, |B_A|)$-factors.

Theorem 1. *Let α be a nonnegative integer, A a unital Banach algebra, X a left Banach A-algebra with unit element e_X, $|T_A^\alpha|$ a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$ over A, and B_A a method of summability defined by a matrix (β_{nk}) over A. If (β_{nk}) satisfies the conditions* (6), (10), (11) *and* (12) *and D_n is finite for each $n \in \{0, 1, \ldots, \alpha\}$, then elements ε_n of A are $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors of summability for X if and only if the conditions* (7) *and* (8) *hold.*

Proof. Since $B_A \supset |B_A|$ (that is, every $|B_A|$-summable series is also B_A-summable), then conditions, necessary for $(|T_A^\alpha|, B_A)$-factors of summability for X, are necessary for $(|T_A^\alpha|, |B_A|)$-factors of summability for X also and conditions, sufficient for $(|T_A^\alpha|, |B_A|)$-factors of summability for X, are sufficient for $(|T_A^\alpha|, B_A)$-factors of summability for X also. Therefore, Theorem 1 holds by Proposition 3.

In particular case, when $A = \mathbb{R}$ or $A = \mathbb{C}$, Theorem 1 has been proved in [4], Theorem 3, and when T_A is the Riesz weighted means summability method over Banach algebra A, the Theorem 1 is proved in [3], Theorems 3 and 5.

Corollary 1. *Let α be a nonnegative integer, A a unital Banach algebra, $|T_A^\alpha|$ a series-to-series method of summability, defined by a T_A^α-matrix $(\overline{\tau}_{nk})$ over A, and B_A a method of summability defined by a matrix (β_{nk}) over A. If D_n is finite for each $n \in \{0, 1, \ldots, \alpha\}$ and (β_{nk}) is normal and satisfies the conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|T_A^\alpha|, B_A)$-factors and $(|T_A^\alpha|, |B_A|)$-factors for A if and only if*

$$\|\beta_{nn} \varepsilon_n \tau_{nn}^{-1}\|_A = O(1)$$

and

$$\|K\varepsilon_n\|_A = O(1).$$

3. Applications to the Riesz weighted means summability method over Banach algebras

1. Let A be a Banach algebra with unit element e_A and (p_n) be such a sequence of elements of A for which

$$P_n = p_0 + \ldots + p_n$$

is invertible in A for each $n \in \mathbb{N}_0$. The *Riesz weighted means summability method* P_A (which transforms a series to sequence) is defined by the matrix (τ_{nk}), where

$$\tau_{nk} = \begin{cases} e_A - P_n^{-1}P_{k-1} & \text{if } k \leq n, \\ \theta_A & \text{if } k > n, \end{cases}$$

and the Riesz weighted means summability method $|P_A|$ (which transforms a series to a series) is defined by the matrix $(\overline{\tau}_{nk})$, where

$$\overline{\tau}_{nk} = P_{n-1}^{-1} p_n P_n^{-1} P_{k-1}$$

for each $k, n \in \mathbb{N}_0$ with $k \leq n$ (see [3], p. 147–148). Moreover, if all elements p_n are also invertible in A, then we can speak about the inverse matrix $(\overline{\eta}_{nk})$ of $(\overline{\tau}_{nk})$, where

$$\overline{\eta}_{nk} = \begin{cases} p_n^{-1}P_n & \text{if } k = n, \\ -p_{n-1}^{-1}P_{n-2} & \text{if } k = n-1, \\ \theta_A & \text{if } k < n-1 \text{ or } k > n \end{cases}$$

for each $k, n \in \mathbb{N}_0$. Taking this and the equality $\eta_{nk} = \overline{\eta}_{nk} - \overline{\eta}_{n,k+1}$ into account, we see that elements of the inverse matrix (η_{nk}) of (τ_{nk}) have the form

$$\eta_{nk} = \begin{cases} p_n^{-1}P_n & \text{if } k = n, \\ -(p_{n-1}^{-1} + p_n^{-1})P_{n-1} & \text{if } k = n-1, \\ -p_{n-1}^{-1}P_{n-2} & \text{if } k = n-2, \\ \theta_A & \text{if } k < n-2 \text{ or } k > n \end{cases}$$

for each $n, k \in \mathbb{N}_0$. Therefore, in the present case

$$D_0 = \sup_k \|\tau_{kk}\eta_{kk}\|_A = \sup_k \|(e_A - P_k^{-1}P_{k-1})p_k^{-1}P_k\|_A =$$
$$= \sup_k \|P_k^{-1}(P_k - P_{k-1})p_k^{-1}P_k\|_A = \sup_k \|P_k^{-1}p_k p_k^{-1}P_k\|_A = 1,$$

$$D_1 = \sup_k \|\tau_{k+1,k+1}\eta_{k+1,k}\|_A = \sup_k \|(e_A - P_{k+1}^{-1}P_k)[-(p_k^{-1} + p_{k+1}^{-1})P_k]\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}(P_{k+1} - P_k)[(p_k^{-1} + p_{k+1}^{-1})P_k]\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}p_{k+1}(p_k^{-1} + p_{k+1}^{-1})P_k)\|_A =$$
$$= \sup_k \|P_{k+1}^{-1}p_{k+1}p_k^{-1}P_k + P_{k+1}^{-1}P_k)\|_A \leq$$
$$\leq \sup_k \|P_{k+1}^{-1}p_{k+1}p_k^{-1}P_k\|_A + 1 + \sup_k \|P_{k+1}^{-1}p_{k+1}\|_A,$$

$$D_2 = \sup_k \|\tau_{k+2,k+2}\eta_{k+2,k}\|_A = \sup_k \|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_k\|_A \leq$$
$$\leq \sup_k \|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}\|_A \left(1 + \sup_k \|P_{k+1}^{-1}p_{k+1}\|_A\right)$$

and if $n \geq 3$ then $D_n = 0$. Hence, if

$$\|P_n^{-1}p_n\|_A = O(1) \tag{20}$$

and

$$\|P_{n+1}^{-1}p_{n+1}p_n^{-1}P_n\|_A = O(1), \tag{21}$$

then D_1 and D_2 are finite. Moreover,

$$K\varepsilon_n = \sum_{\nu=n}^{n+1} \varepsilon_\nu \overline{\eta}_{\nu n} = \varepsilon_n p_n^{-1}P_n - \varepsilon_{n+1}p_n^{-1}P_{n-1} = (\Delta\varepsilon_n)p_n^{-1}P_n + \varepsilon_{n+1}. \tag{22}$$

Taking this into account, we have

Theorem 2. *Let A be a unital Banach algebra, (p_n) a sequence in A such that p_n and P_n are invertible in A for each $n \in \mathbb{N}_0$ and X a left Banach A-algebra with unit element e_X. Let P_A the Riesz weighted means summability method over A. Let B_A be a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21) *and*

$$\sum_{n=k}^{\infty} \|P_{n-1}^{-1}p_n P_n^{-1}P_{k-1}\|_A = O(1) \tag{23}$$

have been satisfied and B_A is normal and satisfies conditions (6), (10), (11) *and* (12), *then elements ε_k of A are $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for X if and only if hold* (9),

$$\|\beta_{nn}\varepsilon_n P_n^{-1} p_n e_X\|_X = O(1) \tag{24}$$

and

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X = O(1). \tag{25}$$

Remark 1. By Proposition 1 the method B_A (resp. P_A) preserves the absolute convergence if and only if (12) (resp. (23)) is satisfied.

Proof of Theorem 2. If $\varepsilon \in (|P_A|, B_A)$ and $\varepsilon \in (|P_A|, |B_A|)$, then condition (9) holds by Proposition 3 (because $|P_A|$ preserves the absolute convergence by Proposition 1) and

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X + \varepsilon_{n+1} e_X\|_X = O(1) \tag{26}$$

hold by Theorem 1 and the equality (22). Since

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X \le \|(\Delta\varepsilon_n) p_n^{-1} P_n e_X + \varepsilon_{n+1} e_X\|_X + \|\varepsilon_{n+1} e_X\|_X,$$

the condition (25) holds by (9) and (26).

Let now the elements ε_n of A satisfy the conditions (9), (24) and (25). Then the condition (7) of Theorem 1 holds. Since

$$\|(K\varepsilon_n) e_X\|_X \le \|(\Delta\varepsilon_n) p_n^{-1} P_n e_X\|_X + \|\varepsilon_{n+1} e_X\|_X$$

by the equality (22), the condition (8) of Theorem 1 holds by the conditions (9) and (25). Consequently, the elements $\varepsilon \in (|P_A|, B_A)$ and $\varepsilon \in (|P_A|, |B_A|)$ by Theorem 1.

Corollary 2. *Let A be a unital Banach algebra, (p_n) a sequence in A such that p_n and P_n are invertible in A for each $n \in \mathbb{N}_0$. Let P_A be the Riesz weighted means summability method over A and B_A a method of summability, defined by a matrix (β_{nk}) over A. If the conditions* (20), (21) *and* (23) *have been satisfied and (β_{nk}) is a normal matrix which satisfies the conditions* (6), (10), (11) *and* (12), *then elements ε_k of A are $(|P_A|, B_A)$-factors and $(|P_A|, |B_A|)$-factors of summability for A if and only if*

$$\|\varepsilon_n\|_A = O(1), \tag{27}$$

$$\|\beta_{nn}\varepsilon_n P_n^{-1} p_n\|_A = O(1) \tag{28}$$

and

$$\|(\Delta\varepsilon_n) p_n^{-1} P_n\|_A = O(1). \tag{29}$$

2. Let A be a Banach algebra with unit element e_A. Let (p_n) and (q_n) be two sequences of elements of A for which

$$P_n = p_0 + \ldots + p_n \quad \text{and} \quad Q_n = q_0 + \ldots + q_n$$

are invertible in A for each $n \in \mathbb{N}_0$. The method $(QP)_A$ of summability over A (which first transforms the sequence $x = (x_n)$ to the sequence $y = (y_n)$ and then the sequence (y_n) to the sequence (z_n)) we define by the matrix transformations

$$z_n = \sum_{k=0}^{n} t_{nk} x_k, \tag{30}$$

where

$$t_{nk} = \begin{cases} Q_n^{-1}\left(\sum\limits_{i=k}^{n} q_i P_i^{-1}\right) p_k & \text{if } k \le n, \\ \theta_A & \text{if } k > n. \end{cases}$$

Then (by the formula (8.5) from [5], p. 51) elements of the corresponding matrix (τ_{nk}) of this method of summability (which transforms series to sequence) has the form

$$\tau_{nk} = \begin{cases} Q_n^{-1} \sum\limits_{i=k}^{n} q_i (e_A - P_i^{-1} P_{k-1}) & \text{if } k \le n, \\ \theta_A & \text{if } k > n, \end{cases}$$

and the summability method $|(QP)_A|$ (which transforms a series to a series) we define by the matrix $(\overline{\tau}_{nk})$, where[2]

$$\overline{\tau}_{nk} = \begin{cases} Q_{n-1}^{-1} q_n Q_n^{-1} Q_{k-1} - \overline{\Delta}\left(Q_n^{-1} \sum\limits_{i=k}^{n} q_i P_i^{-1} P_{k-1}\right) & \text{if } k \le n, \\ \theta_A & \text{if } k > n, \end{cases}$$

[2] Here and later on $\overline{\Delta} a_{nk} = a_{nk} - a_{n-1,k}$, where $a_{nk} \in A$ for each $k, n \in \mathbb{N}$, and $\overline{\Delta} a_n = a_n - a_{n-1}$, where $a_n \in A$ for each $n \in \mathbb{N}$.

because $\overline{\tau}_{nk} = \overline{\Delta}\tau_{nk}$ for each $n, k \in \mathbb{N}_0$ (see [5], p. 50, formula (8.2)).

Transforming the equation (30), we have

$$\overline{\Delta}(Q_n z_n) = \overline{\Delta}\left[\sum_{k=0}^{n} q_k P_k^{-1}\left(\sum_{i=0}^{k} p_i x_i\right)\right] = q_n P_n^{-1}\left(\sum_{i=0}^{n} p_i x_i\right)$$

and

$$\overline{\Delta}\left(P_n q_n^{-1}\overline{\Delta}(Q_n z_n)\right) = p_n x_n.$$

Therefore

$$\begin{aligned} x_n &= p_n^{-1}\left[P_n q_n^{-1}\overline{\Delta}(Q_n z_n) - P_{n-1}q_{n-1}^{-1}\overline{\Delta}(Q_{n-1}z_{n-1})\right] = \\ &= p_n^{-1}\left[P_n q_n^{-1}(Q_n z_n - Q_{n-1}z_{n-1}) - P_{n-1}q_{n-1}^{-1}(Q_{n-1}z_{n-1} - Q_{n-2}z_{n-2})\right] = \\ &= \left[p_n^{-1}P_n q_n^{-1}Q_n\right] z_n - \left[p_n^{-1}P_n q_n^{-1}Q_{n-1} + p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right] z_{n-1} + \\ &+ \left[p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2}\right] z_{n-2}. \end{aligned}$$

Hence, elements of the inverse matrix (ξ_{nk}) of (t_{nk}) we

$$\xi_{nk} = \begin{cases} p_n^{-1}P_n q_n^{-1}Q_n & \text{if } k = n, \\ -\left[p_n^{-1}P_n q_n^{-1}Q_{n-1} + p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right] & \text{if } k = n-1, \\ p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2} & \text{if } k = n-2, \\ \theta_A & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

For finding η_{nk}, we calculate

$$\begin{aligned} &\overline{\Delta}x_n = \\ &= \left[p_n^{-1}P_n q_n^{-1}Q_n\right] z_n - \\ &- \left[p_n^{-1}P_n q_n^{-1}Q_{n-1} + p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1} + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right] z_{n-1} + \\ &+ \left[p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2} + p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2} + p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2}\right] z_{n-2} - \\ &- \left[p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-3}\right] z_{n-3}. \end{aligned}$$

Consequently, the elements of the inverse matrix (η_{nk}) of (τ_{nk}) are

$$\eta_{nk}=\begin{cases} p_n^{-1}P_nq_n^{-1}Q_n & \text{if } k=n,\\ -\left[p_n^{-1}P_nq_n^{-1}Q_{n-1}+p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}+\right.\\ \left.+\,p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right] & \text{if } k=n-1,\\ p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2}+p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-2}+\\ +p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2} & \text{if } k=n-2,\\ -p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-3} & \text{if } k=n-3,\\ \theta_A & \text{if } k<n-3 \text{ or } k>n. \end{cases}$$

Now by (4) we have that

$$\overline{\eta}_{nn}=p_n^{-1}P_nq_n^{-1}Q_n,$$

$$\overline{\eta}_{n,n-1}=$$

$$\begin{aligned}&=\left(p_n^{-1}P_nq_n^{-1}Q_n-p_n^{-1}P_nq_n^{-1}Q_{n-1}\right)-\\ &\qquad-\left(p_n^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}+p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right)=\\ &=p_n^{-1}P_n-\left(p_n^{-1}P_nq_{n-1}^{-1}Q_{n-1}-q_{n-1}^{-1}Q_{n-1}+p_{n-1}^{-1}P_{n-1}q_{n-1}^{-1}Q_{n-1}\right)=\\ &=p_n^{-1}P_n\left(e_A-q_{n-1}^{-1}Q_{n-1}\right)+\left(e_A-p_{n-1}^{-1}P_{n-1}\right)q_{n-1}^{-1}Q_{n-1}=\\ &=-p_n^{-1}P_nq_{n-1}^{-1}Q_{n-2}-p_{n-1}^{-1}P_{n-2}q_{n-1}^{-1}Q_{n-1},\end{aligned}$$

$$\overline{\eta}_{n,n-2}=$$

$$\begin{aligned}&=\left[-p_n^{-1}(P_{n-1}+p_n)q_{n-1}^{-1}Q_{n-2}\right]+\left[-p_{n-1}^{-1}P_{n-2}q_{n-1}^{-1}Q_{n-2}-q_{n-1}+\right.\\ &\left.+p_{n-1}^{-1}(P_{n-2}+p_{N-1})q_{n-1}^{-1}Q_{n-2}\right]+p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2}=\\ &=-q_{n-1}^{-1}Q_{n-2}-p_{n-1}^{-1}P_{n-2}+q_{n-1}^{-1}Q_{n-2}+p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-2}=\\ &=p_{n-1}^{-1}P_{n-2}(q_{n-2}^{-1}Q_{n-2}-e_A)=p_{n-1}^{-1}P_{n-2}q_{n-2}^{-1}Q_{n-3}\end{aligned}$$

and $\overline{\eta}_{n,n-2}=\theta_A$ for each $k\le n-3$. Then $(QP)_A$ is a T_A^2-matrix.

Hence,

$$K\varepsilon_n=\sum_{\nu=n}^{n+2}\varepsilon_\nu\overline{\eta}_{\nu n}=$$

$$= \varepsilon_n p_n^{-1} P_n q_n^{-1} Q_n - \varepsilon_{n+1}[p_{n+1}^{-1} P_n q_n^{-1} Q_n + q_n^{-1}(Q_n - q_n) - p_{n+1}^{-1} P_n +$$
$$+ p_n^{-1} P_n q_n^{-1} Q_n - q_n^{-1} Q_n] + \varepsilon_{n+2}[p_{n+1}^{-1} P_n q_n^{-1} Q_n - p_{n+1}^{-1} P_n] =$$
$$= (\Delta\varepsilon_n) p_n^{-1} P_n q_n^{-1} Q_n - (\Delta\varepsilon_{n+1}) p_{n+1}^{-1} P_n q_n^{-1} Q_n + (\Delta\varepsilon_{n+1})(p_{n+1}^{-1} P_{n+1} - e_A) +$$
$$+ \varepsilon_{n+1} =$$
$$= \Delta(\Delta\varepsilon_n p_n^{-1}) P_n q_n^{-1} Q_n + (\Delta\varepsilon_{n+1}) p_{n+1}^{-1} P_{n+1} + \varepsilon_{n+2}.$$

To find D_1, D_2 and D_3, we calculate

$$\tau_{k+1,k+1}\eta_{k+1,k} =$$
$$= Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} \left[p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_k + p_{k+1}^{-1} P_k q_k^{-1} Q_k + p_k^{-1} P_k q_k^{-1} Q_k\right] =$$
$$= -Q_{k+1}^{-1}(Q_{k+1} - q_{k+1}) - Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1}(P_{k+1} - p_{k+1}) q_k^{-1} Q_k -$$
$$- Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} p_k^{-1} P_k q_k^{-1} Q_k =$$
$$= -e_A + Q_{k+1}^{-1} q_{k+1} + Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k + Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} q_k^{-1} Q_k -$$
$$- Q_{k+1}^{-1} q_{k+1} P_{k+1}^{-1} p_{k+1} p_k^{-1} P_k q_k^{-1} Q_k,$$
$$\tau_{k+2,k+2}\eta_{k+2,k} =$$
$$= Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} \left[p_{k+2}^{-1} P_{k+1} q_{k+1}^{-1} Q_k + p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_k + \right.$$
$$\left. + p_{k+1}^{-1} P_k q_k^{-1} Q_k\right] =$$
$$= Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1}(P_{k+2} - p_{k+2}) q_{k+1}^{-1}(Q_{k+1} - q_{k+1}) +$$
$$+ Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{q+1}^{-1}(Q_{k+1} - q_{k+1}) +$$
$$+ Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1}(P_{k+1} - p_{k+1}) q_k^{-1} Q_k =$$
$$= Q_{k+2}^{-1} q_{k+2} q_{k+1}^{-1} Q_{k+1} - Q_{k+2}^{-1} q_{k+2} - Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} q_{k+1}^{-1} Q_{k+1} +$$
$$+ (Q_{k+2}^{-1} q_{k+2})(P_{k+2}^{-1} p_{k+2}) + Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_{k+1} -$$
$$- (Q_{k+2}^{-1} q_{k+2})(P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1}) +$$
$$+ (Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} p_{k+1}^{-1} P_{k+1} q_{k+1}^{-1} Q_{k+1})(Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k) -$$
$$- (Q_{k+2}^{-1} q_{k+2} P_{k+2}^{-1} p_{k+2} q_{k+1}^{-1} Q_{k+1})(Q_{k+1}^{-1} q_{k+1} q_k^{-1} Q_k)$$

and

$$\tau_{k+3,k+3}\eta_{k+3,k} =$$
$$= -Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} \left[p_{k+2}^{-1}(P_{k+2} - p_{k+2}) q_{k+1}^{-1}(Q_{k+1} - q_{k+1})\right] =$$
$$= -\left(Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} p_{k+2}^{-1} P_{k+2} q_{k+2}^{-1} Q_{k+2}\right)\left(Q_{k+2}^{-1} q_{k+2} q_{k+1}^{-1} Q_{k+1}\right) +$$
$$+ (Q_{k+3}^{-1} q_{k+3})(P_{k+3}^{-1} p_{k+3} p_{k+2}^{-1} P_{k+2}) +$$
$$+ Q_{k+3}^{-1} q_{k+3} P_{k+3}^{-1} p_{k+3} Q_{k+2}^{-1} q_{k+2} q_k^{-1} Q_k q_{k+1}^{-1} Q_{k+1} - (Q_{k+3}^{-1} q_{k+3})(P_{k+3}^{-1} p_{k+3}).$$

Therefore

$$\|\tau_{k+1,k+1}\eta_{k+1,k}\|_A \leq$$

$$\leq 1 + \|Q_{k+1}^{-1}q_{k+1}\|_A + \|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A + \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}p_k^{-1}P_kq_k^{-1}Q_k\|_A,$$

$$\|\tau_{k+2,k+2}\eta_{k+2,k}\|_A \leq$$

$$\leq \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \|Q_{k+2}^{-1}q_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}\|_A + \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1})\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A$$

and

$$\|\tau_{k+3,k+3}\eta_{k+3,k}\|_A \leq$$

$$\leq \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}q_{k+2}^{-1}Q_{k+2}\|_A\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}\|_A\|P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}q_{k+2}^{-1}Q_{k+2}\|_A\|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+3}^{-1}q_{k+3}\|_A\|P_{k+3}^{-1}p_{k+3}\|_A.$$

Hence

$$D_1 \leq$$

$$\leq 1 + \|Q_{k+1}^{-1}q_{k+1}\|_A + \|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A + \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+1}^{-1}q_{k+1}P_{k+1}^{-1}p_{k+1}p_k^{-1}P_kq_k^{-1}Q_k\|_A,$$

$$D_2 \leq$$

$$\leq \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \|Q_{k+2}^{-1}q_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}\|_A\|P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}p_{k+1}^{-1}P_{k+1}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A +$$
$$+ \|Q_{k+2}^{-1}q_{k+2}P_{k+2}^{-1}p_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A\|Q_{k+1}^{-1}q_{k+1}q_k^{-1}Q_k\|_A$$

and

$$\begin{aligned} D_3 \leq & \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}q_{k+2}^{-1}Q_{k+2}\|_A \, \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \\ & + \|Q_{k+3}^{-1}q_{k+3}\|_A \, \|P_{k+3}^{-1}p_{k+3}p_{k+2}^{-1}P_{k+2}\|_A + \\ & + \|Q_{k+3}^{-1}q_{k+3}P_{k+3}^{-1}p_{k+3}q_{k+2}^{-1}Q_{k+2}\|_A \, \|Q_{k+2}^{-1}q_{k+2}q_{k+1}^{-1}Q_{k+1}\|_A + \\ & + \|Q_{k+3}^{-1}q_{k+3}\|_A \|P_{k+3}^{-1}p_{k+3}\|_A. \end{aligned}$$

Consequently, D_k is finite for each $k \leq 3$ if the methods P_A and Q_A satisfy the following conditions:

$$\|Q_n^{-1}q_n\| = O(1), \tag{31}$$

$$\|Q_{n+1}^{-1}q_{n+1}q_n^{-1}Q_n\|_A = O(1), \tag{32}$$

$$\|Q_{n+1}^{-1}q_{n+1}P_{n+1}^{-1}p_{n+1}q_n^{-1}Q_n\|_A = O(1) \tag{33}$$

and

$$\|Q_{n+1}^{-1}q_{n+1}P_{n+1}^{-1}p_{n+1}p_n^{-1}P_nq_n^{-1}Q_n\|_A = O(1). \tag{34}$$

In the particular case, when A is a commutative Banach algebra, then conditions (33) and (34) are superfluous, because they hold by conditions (20), (31) and (32). Taking this into account, we have

Theorem 3. *Let A be a unital Banach algebra, (p_n) and (q_n) sequences in A such that p_n, q_n, P_n and Q_n are invertible in A for each $n \in \mathbb{N}_0$, X a left Banach A-algebra with unit element e_X. Let P_A and Q_A two Riesz weighted means summability methods over A such that $|(QP)_A| \supset |P_A|$ and B_A a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21), (23), (31)–(34),

$$\sum_{n=k}^{\infty} \|Q_{n-1}^{-1}q_nQ_n^{-1}Q_{k-1}\|_A = O(1) \tag{35}$$

and

$$\sum_{n=k}^{\infty} \left\| \overline{\Delta} \left(Q_n^{-1} \sum_{i=k}^{n} q_i P_i^{-1} P_{k-1} \right) \right\|_A = O(1) \tag{36}$$

have been satisfied (in case, when A is commutative, then (20), (21), (23), (31), (32), (35) *and* (36)*) and B_A is normal and satisfies conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|(QP)_A|, B_A)$-factors and $(|(QP)_A|, |B_A|)$-factors of summability for X if and only if* (9), (24), (25) *and*

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1}Q_n e_X\|_X = O(1) \tag{37}$$

have been satisfied.

Proof. If $\varepsilon \in (|(QP)_A|, B_A)$ and $\varepsilon \in (|(QP)_A|, |B_A|)$, then conditions (8) and (9) hold by Proposition 3 because $|(QP)_A|$ preserves the absolute convergence by (35) and (36) (see Proposition 1). Since every $|P_A|$-summable series in X is $|Q_A P_A|$-summable also by $|(QP)_A| \supset |P_A|$, the method $|P_A|$ preserves the absolute convergence by the condition (23) (see [3], Corollary 3), then the condition (24) and (25) hold. Moreover,

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1}Q_n e_X\|_X \le$$

$$\le \|(K\varepsilon_n)e_X\|_X + \|(\Delta\varepsilon_{n+1})p_{n+1}^{-1}P_{n+1}e_X\| + \|\varepsilon_{n+2}e_X\|.$$

Therefore, the condition (37) holds by conditions (8), (9) and (25).

Let now elements ε_n of A satisfy the conditions (9), (25) and (37). Since

$$\begin{aligned}\|(K\varepsilon_n)e_X\|_X \le & \|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1}Q_n e_X\|_X + \|(\Delta\varepsilon_{n+1})p_{n+1}^{-1}P_{n+1}e_X\|_X + \\ & + \|\varepsilon_{n+2}e_X\|_X,\end{aligned}$$

then the condition (8) has been satisfied by (9), (25) and (37). Hence, we have $\varepsilon \in (|(QP)_A|, B_A)$ and $\varepsilon \in (|(QP)_A|, |B_A|)$ by Theorem 1.

Corollary 3. *Let A be a unital Banach algebra, (p_n) and (q_n) sequences in A such that p_n, q_n, P_n and Q_n are invertible in A for each $n \in \mathbb{N}_0$, P_A and Q_A two Riesz weighted means summability methods over A and B_A a method of summability defined by a matrix (β_{nk}) over A. If conditions* (20), (21), (23), (31)–(34), (35) *and* (36) *(in case, when A is commutative, then conditions* (20), (21), (23), (31), (32), (35) *and* (36)*) have been satisfied and B_A is normal and satisfies conditions* (6), (10), (11) *and* (12)*, then elements ε_k of A are $(|(QP)_A|, B_A)$-factors and $(|(QP)_A|, |B_A|)$-factors of summability for X if and only if* (27), (28), (29) *and*

$$\|\Delta(\Delta\varepsilon_n p_n^{-1})P_n q_n^{-1}Q_n\|_A = O(1)$$

are fulfilled.

Remark 2. In the particular case, when A is the field of real or complex numbers, Corollary 2 (see [3], Corollary 6) and Corollary 3 (see [4], p. 177) are known.

Remark 3. The condition (10) is satisfied, for example, for $B_A = Q_A$, if Q_A conserves the absolute convergence, that is iff (35) is satisfied.

Indeed, $(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1} = -Q_{n-1}^{-1}q_n Q_n^{-1} q_\nu Q_\nu q_\nu^{-1} = -Q_{n-1}^{-1}q_n Q_n^{-1}Q_\nu$, and

$$\sum_{n=\nu}^{\infty} \|(\Delta\overline{\beta}_{n\nu})\beta_{\nu\nu}^{-1}\|_A \le \|e_A\|_A + \sum_{n=\nu+1}^{\infty} \|Q_{n-1}^{-1}q_n Q_n^{-1}Q_\nu\|_A = O(1).$$

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