Annales Univ. Sci. Budapest., Sect. Comp. 32 (2010) 143-149

ARITHMETICAL FUNCTIONS INVOLVING EXPONENTIAL DIVISORS: NOTE ON TWO PAPERS BY L. TÓTH

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Abstract. Asymptotic estimates of L. Tóth [5, 6] on the summatory functions of three arithmetical functions involving exponential divisors are improved. For two of them the improvement is on the upper bound of the size of the remainder term (O-estimate), and is reached by appealing to lattice points estimates using exponent pairs due to Krätzel [1], and by having as well a closer look at the first terms of the generating Dirichlet series. For the third one, a lower bound on the size of the remainder term (Ω-estimate) is replaced by two-sided oscillation ($\Omega_{\pm}$-estimate), by appealing to a method of Pétermann and Wu [2].

1. Notation and definitions

An *exponential divisor* (e-divisor) $d = p_1^{b_1} \cdots p_r^{b_r}$ of $n = p_1^{a_1} \cdots p_r^{a_r}$, satisfies by definition $b_i \mid a_i \quad (i = 1, \ldots, r)$. The integer n is thus called *exponentially squarefree* (e-squarefree) if all the a_i are squarefree. These two notions were introduced by M.V. Subbarao [4]. Other authors further extended the analogies with notions related to usual divisors. For instance, if n and m have the same prime divisors, we call $\kappa(n)(= \kappa(m)) := p_1 \cdots p_r$ their *kernel*, and then their *greatest common exponential divisor* (e-gcd) is defined as $(n, m)_{(e)} := \prod_{1 \le i \le r} p_i^{(a_i, b_i)}$. And if $(n, m)_{(e)} = \kappa(n) = \kappa(m)$ we say that n and m are *exponentially-coprime* (e-coprime).

Mathematics Subject Classification: 11N37

Several functions related to exponential divisors, as the number $\tau^{(e)}(n)$ and the sum $\sigma^{(e)}(n)$ of e-divisors of n to begin with, were studied by Subbarao and then by several other authors: see [5] for references.

In [5] and [6], L. Tóth studied some such functions, three of which are the subjects of this note. These are: (i) the number $t^{(e)}(n)$ of e-squarefree e-divisors of n, (ii) the number $\phi^{(e)}(n)$ of divisors d of n which are e-coprime with n (the e-analogue of the Euler function ϕ), and (iii) $\tilde{P}(n) := \sum_{1\le j\le n, \kappa(j)=\kappa(n)} (j,n)_{(e)}$ (the e-analogue of the Pillai function $P(n) := \sum_{1\le j\le n} (j,n)$ [3]).

Let ζ denote the Riemann zeta function. Let ϕ and μ be the Euler and Möbius functions. For a positive integer n put as usual $\omega(n)$ for the number of distinct prime divisors of n. For a positive integer k let $\mathbf{1}_k$ be the characteristic function of the integers n of the form $n = m^k$ (where m is an integer), and similarly let $\mu_k(n) = \mu(m)$ if $n = m^k$ and $\mu_k(n) = 0$ otherwise.

2. Results

Tóth proved the following estimates for the summatory functions of $t^{(e)}(n)$, $\phi^{(e)}(n)$ and $\tilde{P}(n)$.

Theorem A. *We have*

$$E_t(x) := \sum_{n\le x} t^{(e)}(n) - C_1 x - C_2 x^{1/2} = O(x^{1/4+\epsilon})$$

for every $\epsilon > 0$, where C_1 and C_2 are constants given by

$$C_1 = \prod_p \left(1 + \frac{1}{p^2} + \sum_{a\ge 6} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 = \zeta\left(\frac{1}{2}\right) \prod_p \left(1 + \sum_{a\ge 4} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}}\right).$$

Theorem B. *We have*

$$E_\phi(x) := \sum_{n\le x} \phi^{(e)}(n) - C_3 x - C_4 x^{1/3} = O(x^{1/5+\epsilon})$$

for every $\epsilon > 0$, where C_3 and C_4 are constants given by

$$C_3 = \prod_p \left(1 + \sum_{a \geq 3} \frac{\phi(a) - \phi(a-1)}{p^a}\right),$$

$$C_4 = \zeta\left(\frac{1}{3}\right) \prod_p \left(1 + \sum_{a \geq 5} \frac{\phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)}{p^{a/3}}\right).$$

Theorem C. *We have*

$$E_p(x) := \sum_{n \leq x} \tilde{P}(n) - C_5 x^2 = \begin{cases} O(x(\log x)^{5/3}), \\ \Omega(x \log\log x), \end{cases}$$

where the constant C_5 is given by

$$C_5 := \frac{1}{2} \prod_p \left(1 + \sum_{a \geq 2} \frac{\tilde{P}(p^a) - p\tilde{P}(p^{a-1})}{p^{2a}}\right).$$

Notes.

(1) Theorem A is Theorem 4 in [6], which however contains two misprints: the term $1/p^2$ is missing in the factor defining C_1, and the rightmost exponent in the factors defining C_2 is incorrect ($\omega(a-4)$ instead of $\omega(a-3)$; the same mistake is repeated in the proof on p.164). Theorem B is Theorem 1 in [5], which also contains misprints: the rightmost term in the factor defining C_4 is incorrect ($-\phi(a-4)$ instead of $+\phi(a-4)$), and the product symbol $\prod$ is missing. The O-estimate in Theorem C is Theorem 3 in [5], and the Ω-estimate is a direct consequence of Theorem 4 in [5], which states that $\limsup_{n\to\infty} \tilde{P}(n)/(n \log\log n) = 6e^\gamma/\pi^2$.

(2) The proofs of Theorems A and B make use of estimates due to Krätzel for

$$\Delta(a, b; x) := \sum_{n_1^a n_2^b \leq x} 1 - \zeta(b/a)x^{1/a} - \zeta(a/b)x^{1/b}$$

in the case where a and b are integers with $1 \leq a < b$. The elementary Theorem 5.3 in [1] yields $\Delta(a, b; x) = O(x^{1/(2a+b)})$, and is applied to the case $a = 1,\ b = 2$ for the proof of Theorem A, and to the case $a = 1,\ b = 3$ for the proof of Theorem B.

But, from more elaborate arguments involving exponent pairs in this same Chapter 5 of [1], we see that $\Delta(1,2;x) = O(x^{\tau})$ with $\tau < 1/4$ and $\Delta(1,3;x) = O(x^{\varphi})$ with $\varphi < 1/5$. This will be used in the Proof of Theorem 1 below.

[For the best known values of τ and φ: Theorem 5.11 p.223 yields $\Delta(1,2;x) = O(x^{37/167+\epsilon})$ $(x \to \infty)$ for every $\epsilon > 0$ (see the Note on Section 5.3 on p.230), and Theorem 5.12 p.227 yields $\Delta(1,3;x) = O(x^{0.175}(\log x)^2)$ $(x \to \infty)$ with the exponent pair (1/14, 11/14) (as indicated in the small table at the bottom of page 227)].

There are two objects to this note. The first one is to refine the argument yielding Theorems A and B, and to prove

Theorem 1. *We have* $E_t(x) = O(x^{1/4})$ *and* $E_\phi(x) = O(x^{1/5}\log x)$.

The other object is to replace the Ω-estimate in Theorem C by an oscillation estimate.

Theorem 2. *We have*

$$E_P(x) = \Omega_\pm(x \log\log x).$$

3. Proofs

Proof of Theorem 1. We begin with E_t. The proof of Theorem A in [6] exploits the expression

$$T(s) := \sum_{n\geq 1} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s) \quad (\sigma > 1),$$

where, for $v(p^a) := 2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}$ $(a \geq 4)$ and $v(p^a) = 0$ $(1 \leq a \leq 3)$, the series

$$V(s) := \sum_{n\geq 1} \frac{v(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^{4s}} + \sum_{a\geq 5} \frac{v(p^a)}{p^{as}}\right)$$

is absolutely convergent for $\sigma > 1/4$.

A closer look thus easily shows that $V(s) = H(s)/\zeta(4s)$, with

$$H(s) := \sum_{n\ge 1} \frac{h(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^{6s}} + \sum_{a\ge 7} \frac{h(p^a)}{p^{as}}\right).$$

Since $|h(p^a)| = |(\mathbf{1}_4 * v)(p^a)| \le \sum_{i\le a} 2^{\omega(i)} = O(a^2)$, we see that $H(s)$ converges absolutely for $\sigma > 1/6$.

Now if $H_0(s) := \zeta(s)\zeta(2s)/\zeta(4s) =: \sum_{n\ge 1} h_0(n)n^{-s}$, we have $h_0 = \mathbf{1} * \mathbf{1}_2 * \mu_4$, whence by using the fact that $\Delta(1,2;x) = O(x^\tau)$ for some $\tau < 1/4$ (see Note (2) above) we have

$$\sum_{n\le x} h_0(n) = \sum_{n=n_1 n_2^2 N^4 \le x} \mu(N) =$$

$$= \sum_{N\le x^{1/4}} \mu(N) \left(\zeta(2)\frac{x}{N^4} + \zeta\left(\frac{1}{2}\right)\frac{x^{1/2}}{N^2} + O\left(\frac{x^\tau}{N^{4\tau}}\right)\right).$$

From the prime number theorem under the form $\sum_{n\ge y} \mu(n)/n = o(1)$ $(y \to \infty)$ it follows that, if $\ell > 1$, $\sum_{n<y} \mu(n)/n^\ell = 1/\zeta(\ell) + o(y^{1-\ell})$, whence

$$\sum_{n\le x} h_0(n) = \frac{\zeta(2)}{\zeta(4)}x + \frac{\zeta(1/2)}{\zeta(2)}x^{1/2} + O(x^{1/4}).$$

Finally, with $t^{(e)} = h * h_0$, we see that

$$\sum_{n\le x} t^{(e)}(n) = \frac{\zeta(2)}{\zeta(4)}H(1)x + \frac{\zeta(1/2)}{\zeta(2)}H(1/2)x^{1/2} + O(x^{1/4}).$$

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The proof of $E_\phi(x) = O(x^{1/5})$ is similar. Instead of considering as in [5] the expression

$$\Phi(s) := \sum_{n\ge 1} \frac{\phi^{(e)}(n)}{n^s} = \zeta(s)\zeta(3s)U(s) =$$

$$= \zeta(s)\zeta(3s)\prod_p \left(1 + \frac{2}{p^{5s}} + \sum_{a\ge 6} \frac{u(p^a)}{p^{as}}\right) \quad (\sigma > 1),$$

where the Dirichlet series for $U(s)$ converges absolutely for $\sigma > 1/5$, we note that

$$U(s) = (\zeta(5s))^2 J(s) = (\zeta(5s))^2 \prod_p \left(1 - \frac{3}{p^{6s}} + \sum_{a \geq 7} \frac{j(p^a)}{p^{as}}\right),$$

where the Dirichlet series for $J(s)$ converges absolutely for $\sigma > 1/6$. Indeed $j = \mu_5 * \mu_5 * u$ where $u(p^a) = \phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)$ $(a \geq 5)$ and $u(p^a) = 0$ $(1 \leq a \leq 4)$, whence $j(p^a) = O(a)$ (more precisely, $|j(p^a)| \leq 8a$). Thus by using the fact that $\Delta(1,3;x) = O(x^{\varphi})$ for some $\varphi < 1/5$ we obtain, similarly as before,

$$\sum_{n \leq x} \phi^{(e)}(n) = \zeta(3)(\zeta(5))^2 J(1)x + \zeta(1/3)(\zeta(5/3))^2 J(1/3)x^{1/3} + O(x^{1/5}\log x).$$

Proof of Theorem 2. We leave this proof to the reader, whom we refer to the proof of Theorem 3 in [2], since the argument there may be very closely followed with only minor adaptations. Indeed the latter theorem establishes that $\sum_{n \leq x} \sigma^{(e)}(n) = Dx^2 + \Omega_{\pm}(x \log\log x)$ for some constant D by exploiting the expression $\sum_{n \geq 1} \sigma^{(e)}(n)n^{-s} = \zeta(s-1)\zeta(2s-1)(\zeta(3s-2)^{-1}K(s)$, where the Dirichlet series for $K(s)$ absolutely converges for $\sigma > 3/4$; and similarly we have (see Lemma 3 of [5]) $\sum_{n \geq 1} \tilde{P}(n)n^{-s} = \zeta(s-1)\zeta(2s-1)(\zeta(3s-2)^{-1}W(s)$, where the Dirichlet series for $W(s)$ absolutely converges for $\sigma > 3/4$.

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(Received February 11, 2009)

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