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A SIMPLY–OBTAINED UPPER BOUND FOR OVERPARTITIONS

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Abstract. An *overpartition* of the natural number n is a partition such that at most one part of each size may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. Using elementary techniques, we obtain an upper bound for $\overline{p}(n)$.

1. Introduction

In [1, p.317], an upper bound for $p(n)$ (the partition function) was obtained using simple analytic techniques. In [3], we used similar methods to obtain an upper bound for $q(n)$ (the number of partitions of n into odd parts, or into distinct parts). In this note, we obtain an upper bound for $\overline{p}(n)$, the number of overpartitions of the natural number n.

2. Preliminaries

Let $0 < x < 1$.

Definitions.

$$F(x) = \prod_{n=1}^{\infty} (1 + x^n),$$

$$G(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1}.$$

Identities.

$$(1) \qquad \sum_{n=1}^{\infty} \overline{p}(n)x^n = \prod_{n=1}^{\infty} \left(\frac{1+x^n}{1-x^n}\right) = F(x)G(x).$$

Proposition 1.

$$\text{Log } G(x) < \frac{\pi^2}{6}\left(\frac{x}{1-x}\right).$$

Remarks. Equation (1) is the generating function identity for overpartitions (see [2]). Proposition 1 appears in [1, p.317].

3. The main results

Before presenting our upper bound for $\overline{p}(n)$, we need the following

Lemma 1. *If* $0 < x < 1$, *then* $\text{Log } F(x) < \frac{x}{1-x}$.

Proof.

$$\text{Log } F(x) = \sum_{n=1}^{\infty} \log(1+x^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}x^{mn}}{m} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{n=1}^{\infty} x^{mn} =$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\left(\frac{x^m}{1-x^m}\right).$$

Since this is an alternating series, the sum is less than the initial positive term, so we have

$$\text{Log } F(x) < \frac{x}{1-x}.$$

Theorem 1.

$$\overline{p}(n) < \sqrt{\frac{\pi^2+6}{6(n-1)}} \exp\left(\sqrt{\frac{2}{3}(\pi^2+6)(n-1)}\right).$$

Proof. Since $\overline{p}(n)$ is a strictly increasing function of n, we have

$$\frac{\overline{p}(n)x^n}{1-x} = \sum_{k=n}^{\infty} \overline{p}(n)x^k < \sum_{k=n}^{\infty} \overline{p}(k)x^k < \prod_{n=1}^{\infty} \left(\frac{1+x^n}{1-x^n}\right)$$

as implied by (1). Therefore

$$\log \overline{p}(n) + n \log x - \log(1-x) < \log F(x)G(x) = \log F(x) + \log G(x).$$

Invoking Proposition 1 and Lemma 1, we have

$$\log \overline{p}(n) + n \log x - \log(1-x) < \left(\frac{\pi^2+6}{6}\right)\left(\frac{x}{1-x}\right),$$

so that

$$\log \overline{p}(n) < \left(\frac{\pi^2+6}{6}\right)\left(\frac{x}{1-x}\right) + n \log \frac{1}{x} + \log(1-x).$$

Now let $t = \frac{1-x}{x}$, so that $0 < t < \infty$, $1+t = \frac{1}{x}$ and $\log(1-x) = \log t - \log(1+t)$. Thus we have

$$\log \overline{p}(n) < \left(\frac{\pi^2+6}{6}\right)\left(\frac{1}{t}\right) + (n-1)\log(1+t) - \log t,$$

hence

$$\log \overline{p}(n) < \left(\frac{\pi^2+6}{6}\right)\left(\frac{1}{t}\right) + (n-1)t - \log t.$$

Now let

$$f(t) = \left(\frac{\pi^2+6}{6}\right)\left(\frac{1}{t}\right) + (n-1)t - \log t,$$

so that

$$f'(t) = -\left(\frac{\pi^2+6}{6}\right)\left(\frac{1}{t^2}\right) + (n-1) - \frac{1}{t}.$$

It is easily seen that $f(t)$ has an absolute minimum when

$$t = \frac{-1 + \sqrt{1 + 4(\frac{\pi^2+6}{6})(n-1)}}{2(n-1)}.$$

Let

$$t^* = \sqrt{\frac{\pi^2+6}{6(n-1)}},$$

so that

$$f(t^*) = \sqrt{\frac{2}{3}(\pi^2+6)(n-1)} + \log\sqrt{\frac{\pi^2+6}{6(n-1)}}.$$

We have $\log \overline{p}(n) < f(t^*)$, from which the conclusion follows.

References

[1] **Apostol T.,** *Introduction to analytic number theory,* Springer Verlag, 1976.

[2] **Robbins N.,** Some properties of overpartitions, *JP Journal of Algebra, Number Theory and Applications,* **3** (2003), 395-404.

[3] **Robbins N.,** A simply-obtained upper bound for $q(n)$, *Annales Univ. Sci. Budapest. Sect. Comp.,* **27** (2007), 39-43.

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