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ON SUMS OF POWERS OF PRIME FACTORS OF AN INTEGER

J.-M. DeKoninck (Québec, Canada)

F. Luca (Morelia, Mexico)

Abstract. In this paper, we look at the positive integers n such that the equality

$$\sum_{p^{\alpha_p}||n} p^{\alpha_p} = \left(\sum_{p|n} p\right)^2$$

holds.

1. Introduction

For every positive integer

$$n = \prod_{p^{\alpha_p}||n} p^{\alpha_p}$$

let

$$B(n) = \sum_{p^{\alpha_p}||n} p^{\alpha_p} \qquad \text{and} \qquad \beta(n) = \sum_{p|n} p.$$

Plainly, $B(n) = \beta(n)$ if and only if n is squarefree. In this paper, we look at the positive integers n such that $B(n) = \beta(n)^2$. Let $\mathcal{A}$ be the set of such n. For a positive real number x we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. Since $\mathcal{A}$ contains all squares of primes, we get that $\#\mathcal{A}(x) \geq \pi(x^{1/2}) \gg x^{1/2}/\log x$. In this note, we show that $\mathcal{A}$ contains a lot more numbers, and in fact our main result is the following.

Theorem 1. *The estimates*

$$\frac{x}{\exp\left((2\sqrt{34/3}+o(1))\sqrt{\log x \log\log x}\right)} \le$$

$$\le \#\mathcal{A}(x) \le \frac{x}{\exp\left(\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log x \log\log x}\right)}$$

hold as $x \to \infty$.

Throughout, we use the Vinogradov symbols $\gg$ and $\ll$ and the Landau symbols O and o with their regular meanings. We use log for the natural logarithm and p, q and r with or without subscripts for prime numbers.

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2. The upper bound

For a positive integer n we write $P(n)$ for the largest prime factor of n. Let us consider that following sets:

$$\mathcal{A}_1(x) = \{n \le x \mid n = p^2\},$$

$$\mathcal{A}_2(x) = \{n \le x \mid P(n) < y\}$$

and

$$\mathcal{A}_3(x) = \left\{n \le x \mid n \notin \mathcal{A}_2(x),\ P(n)^2 \mid n\right\},$$

where y is a parameter which depends on x to be chosen later and which satisfies $\exp((\log\log x)^2) \le y \le x$, and $P(n)$ denotes the largest prime factor of n.

Plainly,

$$\#\mathcal{A}_1(x) = \pi(x^{1/2}) \ll \frac{x^{1/2}}{\log x}. \tag{1}$$

From standard estimates on smooth numbers [1], we know that if we set $u = \log x/\log y$, then

$$\#\mathcal{A}_2(x) \ll \frac{x}{\exp((1+o(1))u\log u)} \qquad (x \to \infty) \tag{2}$$

in our range for y versus x, while

$$\#\mathcal{A}_3(x) \le \sum_{\substack{p \text{ prime} \\ p \ge y}} \left\lfloor \frac{x}{p^2} \right\rfloor \le x \sum_{n \ge y} \frac{1}{n^2} \ll \frac{x}{y}. \tag{3}$$

Let $\mathcal{A}_4(x) = \mathcal{A}(x) \setminus (\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{A}_3(x))$. If $n \in \mathcal{A}_4(x)$, then we can write $n = P(n)m$, where $m > 1$ (note that $\omega(n) > 1$ since n belongs to $\mathcal{A}(x)$ but not to $\mathcal{A}_1(x)$). Furthermore, since $n \notin \mathcal{A}_3(x)$, we have $P(n) \nmid m$. Since $n \in \mathcal{A}(x)$, we can write

$$P(n) + B(m) = B(n) = \beta(n)^2 = (\beta(m) + P(n))^2,$$

so that

$$P(n)^2 + (2\beta(m) - 1)P(n) + (\beta(m)^2 - B(m)) = 0.$$

Hence, $P(n)$ is determined in at most two ways by m. Furthermore, note that for the positive integers n under consideration, we have that $P(n) \ge y$, implying that $m \le x/y$, so that

$$\#\mathcal{A}_4(x) \le 2 \sum_{m \le x/y} \ll \frac{x}{y}. \tag{4}$$

From estimates (1), (2), (3) and (4), we immediately deduce that

$$\#\mathcal{A}(x) \le \#\mathcal{A}_1(x) + \#\mathcal{A}_2(x) + \#\mathcal{A}_3(x) + \#\mathcal{A}_4(x) \ll$$
$$\ll \frac{x^{1/2}}{\log x} + \frac{x}{y} + \frac{x}{\exp((1+o(1))u\log u)}.$$

To minimize the right hand side above we choose $y = \exp(u \log u)$, which amounts to

$$\log^2 y = \log x \log\left(\frac{\log x}{\log y}\right).$$

Thus, we get that $\log y = (1+o(1))\sqrt{\log x \log\log x}$ as $x \to \infty$, and with this choice of y versus x we obtain

$$\#\mathcal{A}(x) \ll \frac{x}{\exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log x \log\log x}\right)}$$

as $x \to \infty$, thus establishing the upper bound in Theorem 1.

3. The lower bound

Let N be a large odd integer which is not a multiple of 3. In particular, $N^2 \equiv 1 \pmod{24}$. Let M and k be some positive functions of N to be specified more precisely later on, where for now $M = N^{1+o(1)}$ and $k = o(\log N)$ as $N \to \infty$. We assume that k is an integer congruent to 17 modulo 24. Let $c_1 = (1-1/1.1)^{1/3}$, $c_2 = (1-1/6.9)^{1/3}$, and let $p \in \mathcal{I} = [c_1 N^{2/3}, c_2 N^{2/3}]$ and $q_1 < \cdots < q_k \in \mathcal{J} = [M/2, M]$ be all primes congruent to 1 modulo 24. Let

$$N_1 = N - p - \sum_{i=1}^{k} q_i \qquad \text{and} \qquad N_2 = N^2 - p^3 - \sum_{i=1}^{k} q_i. \tag{5}$$

Note that N_1 is odd and $N_2 \equiv 7 \pmod{24}$. Furthermore, assuming that

$$kM = o(N), \tag{6}$$

then

$$N_1 = N(1 - p/N + O(kM/N)) = N(1+o(1))$$

and

$$N_2 = N^2(1 - p^3/N^2 + O(kM/N^2)),$$

so that

$$\frac{N_1^2}{N_2} = \frac{N^2(1+o(1))}{N^2(1-p^3/N^2+o(1))} \in [1.1+o(1), 6.9+o(1)]$$

as $N \to \infty$, because $p \in \mathcal{I}$. Thus, from the application of Theorem 16 on page 139 in Hua's book [2] mentioned on page 156 of the same book, we have that the number of solutions $(p_1, \ldots, p_7)$ of the system of equations

$$p_1 + \cdots + p_7 = N_1 \qquad \text{and} \qquad p_1^2 + \cdots + p_7^2 = N_2 \tag{7}$$

is

$$\geq c_3 \frac{N^4}{(\log N)^7}, \tag{8}$$

where c_3 is some positive absolute constant. We now show that for large N, most of such solutions have $p_i \neq p_j$ for $i \neq j$ in $\{1, \dots, 7\}$ and also that $p_i \notin \{p, q_1, \dots, q_k\}$ for any $i \in \{1, \dots, 7\}$.

Let us count the solutions to the system of equations (7) having $p_i = p_j$ for some $i \neq j$. We assume that $p_6 = p_7$. Since $p_i \leq N$ for all $i = 1, \dots, 7$, it follows that the triplet (p_4, p_5, p_6) can be chosen in at most $O(N^3/(\log N)^3)$ ways. Assume now that p_4, p_5 and p_6 have been chosen and that $p_6 = p_7$. Then

$$p_1 + p_2 + p_3 = A \qquad \text{and} \qquad p_1^2 + p_2^2 + p_3^2 = B, \tag{9}$$

where $A = N_1 - (p_4 + p_5 + 2p_6)$ and $B = N_2 - (p_4^2 + p_5^2 + 2p_6^2)$. Expressing p_3 versus p_1 and p_2 from the left equation above and inserting the answer into the right equation above we get

$$p_1^2 + p_2^2 + (A - p_1 - p_2)^2 = B,$$

or

$$p_1^2 + p_2^2 + p_1 p_2 - A p_1 - A p_2 + A^2/2 = B/2.$$

This last equation above can be rewritten as

$$\left(p_1 + \frac{p_2}{2} - \frac{A}{2}\right)^2 + \frac{3}{4}\left(p_2 - \frac{A}{3}\right)^2 = \frac{3B - A^2}{6},$$

or

$$3U^2 + V^2 = 2(3B - A^2), \quad \text{where} \quad U = 2p_1 + p_2 - A \text{ and } V = 3p_2 - A.$$

Note that p_1 and p_2 determine U and V uniquely and vice-versa. It is well known that if m is any fixed positive integer then the number of integer solutions (x, y) of the equation $3x^2 + y^2 = m$ is $O(2^{\omega_1(m)})$, where $\omega_1(m)$ is the number of prime divisors of m which are congruent to 1 modulo 3. It is well known that $\omega_1(m) \leq \omega(m) \leq c_4 \log m / \log\log m$, where $\omega(m)$ is the total number of distinct prime factors of m and c_4 is some absolute constant. Since $2(B - A^2) < 2N^2$, it follows that the number of possibilities for (p_1, p_2, p_3) once p_4, p_5 and p_6 are fixed is $\leq \exp(3c_4 \log N / \log\log N)$ provided N is sufficiently large. Hence, the total number of solutions of the system of equations (7) with $p_i = p_j$ for some $i \neq j$ is

$$\ll \frac{N^3}{(\log N)^3} \exp\left(3c_4 \frac{\log N}{\log\log N}\right) = N^{3+o(1)}.$$

Comparing this with estimate (8), it follows that for large N, at least

$$\frac{c_3}{2} \frac{N^4}{(\log N)^7} \tag{10}$$

solutions $(p_1, \ldots, p_7)$ exist with all the components distinct.

We now find an upper bound for the number of solutions for which $p_i \in$ $\in \{p, q_1, \ldots, q_k\}$. Since $p \ll N^{2/3}$, $q_i \le M$ and $M = N^{1+o(1)} > N^{2/3}$ for large N, it follows that $p_i \le M$ for some $i = 1, \ldots, 7$. Assume that $p_7 \le M$. Then the quadruplet (p_4, p_5, p_6, p_7) can be chosen in at most $O(N^3 M/(\log N)^4)$ ways. Now for each one of these choices we are led again to a system of equations (9) with suitable A and B depending on p_4, p_5, p_6 and p_7, which, by the previous argument, admits at most $\exp(3c_4 \log N/\log\log N)$ solutions provided N is sufficiently large. Thus, the number of such possibilities does not exceed

$$c_5 \frac{N^3 M}{(\log N)^4} \exp\left(3c_4 \frac{\log N}{\log\log N}\right)$$

for some positive constant c_5. We require that the above upper bound is smaller than a half of the expression shown at (10). This will be the case if

$$M \le \frac{c_3}{4c_5} \frac{N}{(\log N)^3} \exp\left(-3c_4 \frac{\log N}{\log\log N}\right),$$

and at least

$$\frac{c_3}{4} \frac{N^4}{(\log N)^7}$$

solutions $(p_1, \ldots, p_7)$ exist where all the primes p_i are distinct and do not belong to $\{p, q_1, \ldots, q_k\}$. Hence, we see that it suffices to choose M smaller than

$$N \exp\left(-4c_4 \frac{\log N}{\log\log N}\right),$$

and then in light of the fact that we have assumed that $k = o(\log N)$, it follows that inequality (6) also holds.

With such choices $(p_1, \dots, p_7, p, q_1, \dots, q_k)$, we note that the number

$$n = p^3 \prod_{i=1}^{k} q_i \prod_{j=1}^{7} p_j^2$$

satisfies, recalling relations (5) and (7),

$$B(n) = p^3 + \sum_{i=1}^{k} q_i + \sum_{j=1}^{7} p_j^2 = N^2 = \left(p + \sum_{i=1}^{k} q_i + \sum_{j=1}^{7} p_j \right)^2 = \beta(n)^2.$$

Further, note that

$$n \le (c_2 N^{2/3})^3 M^k N^{14} := x. \tag{11}$$

The number of such n is, by the above argument and unique factorization,

$$\ge \frac{c_3}{4} \frac{N^4}{(\log N)^7} \pi'(\mathcal{I}) \binom{\pi'(\mathcal{J})}{k},$$

where $\pi'(\mathcal{I})$ and $\pi'(\mathcal{J})$ denote the number of primes congruent to 1 modulo 24 in the intervals $\mathcal{I}$ and $\mathcal{J}$, respectively. Since certainly

$$\pi'(\mathcal{I}) \ge \frac{(c_2 - c_1 + o(1))}{\phi(24)} \frac{N^{2/3}}{\log(N^{2/3})}$$

as $N \to \infty$, where ϕ stands for Euler's function, we get that $\pi'(\mathcal{I}) \ge$ $\ge c_5 N^{2/3}/\log N$ for large N, where c_5 is some appropriate positive constant. Furthermore, by a similar argument, we get

$$\pi'(\mathcal{J}) \ge c_6 \frac{M}{\log N},$$

where c_6 is also some appropriate constant. Hence, the number of such n is at least

$$\ge c_7 \frac{N^{4+2/3}}{(\log N)^8} \binom{\lfloor c_6 M/\log N \rfloor}{k} \ge c_7 \frac{N^{14/3}}{(\log N)^8} \frac{(c_8 M)^k}{(\log N)^k k!},$$

where $c_8 = c_6/2$, provided that N is large, because $M = N^{1+o(1)}$ and $k = o(\log N)$. We thus get, using estimate (11), that

$$\#\mathcal{A}(x) \geq \frac{c_7 c_8^k N^{14/3} M^k}{(\log N)^{k+8} k^k} = \frac{x}{\exp\left(34/3 \log N + (k+8)\log\log N + k\log k + O(k)\right)}. \tag{12}$$

In light of (11) we get

$$\log x = k \log M + 16 \log N + O(1),$$

and since $\log M = (1+o(1))\log N$, we arrive at $\log x = (1+o(1))k\log N$. In order to optimize the lower bound of $\#\mathcal{A}(x)$ shown at (12), we choose k and N such that $\log N$, $k\log\log N$ and $k\log k$ all have the same order of magnitude. This suggests choosing

$$k = \left\lfloor c_9 \sqrt{\frac{\log x}{\log\log x}} + O(1) \right\rfloor,$$

where c_9 is a constant to be optimally chosen later on, and $O(1)$ accounts for the fact that $k \equiv 17 \pmod{24}$. Thus, since

$$k = c_9(1+o(1)) \left(\frac{\log x}{\log\log x}\right)^{1/2},$$

and $k \log N = (1+o(1))\log x$, we get

$$\log N = c_9^{-1}(1+o(1))(\log x \log\log x)^{1/2},$$

$$k \log k = \frac{c_9}{2}(1+o(1))(\log x \log\log x)^{1/2},$$

and

$$k \log\log N = \frac{c_9}{2}(1+o(1))(\log x \log\log x)^{1/2},$$

showing that

$$\frac{34}{3}\log N + (k+8)\log\log N + k\log k + O(k) = \left(\frac{34}{3c_9} + c_9 + o(1)\right)(\log x \log\log x)^{1/2}.$$

Thus, the optimal constant c_9 is the one which minimizes the function $h(t) = 34/(3t) + t$. Since $h'(t) = -34/(3t^2) + 1$, we get that the minimum of this function is achieved at $t_0 = \sqrt{34/3}$, for which $h(t_0) = 2\sqrt{34/3}$. Thus, choosing $c_9 = t_0$, we get that

$$\#\mathcal{A}(x) \geq \frac{x}{\exp\left((2\sqrt{34/3} + o(1))\sqrt{\log x \log\log x}\right)},$$

thus completing the proof of the theorem.

References

[1] **Hildebrand A.,** On the number of positive integers $\leq x$ and free of prime factors $> y$, *J. Number Theory,* **22** (1986), 289-307.

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J.-M. De Koninck
Département de mathématiques
Université Laval
Québeck G1K 7P4
Canada
`jmdk@mat.ulaval.ca`

F. Luca
Instituto de Matemáticas
Universidad Autónoma de México
C.P. 58089 Morelia
Michoacán, México
`fluca@matmor.unam.mx`