STRONG MARCINKIEWICZ SUMMABILITY OF MULTI–DIMENSIONAL FOURIER SERIES

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. Strong summability results are proved for the Marcinkiewicz means of d-dimensional Fourier series of $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ and $f \in \in L_p(\mathbb{T}^d)$.

1. Introduction

It was proved by Lebesgue [6] that the Fejér means of the trigonometric Fourier series of an integrable function converge a.e. to the function, i.e.

$$\frac{1}{n+1}\sum_{k=0}^{n}\Big(s_k f(x) - f(x)\Big) \to 0 \quad \text{as} \quad n \to \infty$$

for a.e. $x \in \mathbb{T}$, where $s_k f$ denotes the kth partial sum of the Fourier series of f.

Hardy and Littlewood [5] considered the so called strong summability and verified that the strong means

$$\left(\frac{1}{n+1}\sum_{k=0}^{n}|s_k f(x) - f(x)|^q\right)^{1/q}$$

This research was supported by the Hungarian National Foundation for Scientific Research under grant OTKA K067642.

Mathematics Subject Classification: Primary 42B08; Secondary 42A24, 42A38

tend a.e. to 0 as $n \to \infty$, whenever $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$. This result was generalized for $L_1(\mathbb{T})$ functions and for $q = 2$ by Marcinkiewicz [7] and for all $q > 0$ by Zygmund [21].

In the two-dimensional case Marcinkiewicz [8] verified that

$$\frac{1}{n+1} \sum_{k=0}^{n} \Big(s_{k,k} f(x) - f(x) \Big) \to 0 \quad \text{a.e.as} \quad n \to \infty$$

for all functions $f \in L \log L(\mathbb{T}^2)$. Here we take the Fejér means of the two-dimensional Fourier series over the diagonal. Later Zhizhiashvili [19, 20] extended this convergence to all $f \in L_1(\mathbb{T}^2)$ and to Cesàro means.

We generalized this result and, for a given function θ, we investigated in [18, 16] the so called Marcinkiewicz-θ-summability. Under some conditions on θ we proved that

$$\sum_{k=0}^{\infty} \Delta_1 \theta \left(\frac{k}{n+1} \right) \Big(s_{k,k} f(x) - f(x) \Big) \to 0 \quad \text{a.e.as} \quad n \to \infty$$

for all $f \in L_1(\mathbb{T}^2)$, where $\Delta_1 \theta\Big(\frac{k}{n+1}\Big) := \theta\Big(\frac{k}{n+1}\Big) - \theta\Big(\frac{k+1}{n+1}\Big)$. Of course, if $\theta(x) = \max(0, 1 - |x|)$ $(x \in \mathbb{R})$ then we obtain the Marcinkiewicz-Fejér means above. The θ-summation was considered in the one-dimensional case in a great number of papers and books, such as Butzer and Nessel [1], Bokor, Schipp, Szili and Vértesi [10, 12, 11], Natanson and Zuk [9], Trigub and Belinsky [13], Weisz [17] and Feichtinger and Weisz [3, 4].

In this paper we will generalize these results for all dimensions and will consider the strong Marcinkiewicz-θ-summability. We will prove that under some conditions on θ the strong summability result

$$\lim_{n \to \infty} \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}} f(x) - f(x)|^q \right)^{1/q} = 0$$

holds for all strong p-Lebesgue points of f (hence a.e.) and all $0 < q < \infty$, whenever $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ or $f \in L_r(\mathbb{T}^d)$ for some $1 < p < r < \infty$ and $\mathbf{k} := (k, \ldots, k)$. If in addition $f \in L_\infty(\mathbb{T}^d)$ is continuous on an open set $G \subset \mathbb{T}^d$ (resp. on $\mathbb{T}^d$) then the convergence holds uniformly on every $K \subset G$ compact set (resp. on $\mathbb{T}^d$). Eight special cases of the θ-summation are listed, such as Fejér, Riesz, Weierstrass, Abel, de La Vallée-Poussin, Rogosinski and Cesàro summation. Finally, the analogous results will be described for Fourier transforms.

2. Preliminaries and notations

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. We shall prove results for $\mathbb{R}^d$ and $\mathbb{T}^d$, therefore it is convenient to use sometimes the symbol $\mathbb{X}$ for either $\mathbb{R}$ or $\mathbb{T}$, where $\mathbb{T} := [-1/2, 1/2)$ is the torus. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k \qquad \text{and} \qquad |x| := \left(\sum_{k=1}^{d} |x_k|^2 \right)^{1/2}.$$

We briefly write $L_p(\mathbb{X}^d)$ instead of $L_p(\mathbb{X}^d, \lambda)$ space equipped with the norm (or quasi-norm)

$$\|f\|_p := \left(\int_{\mathbb{X}^d} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \leq \infty),$$

where λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I.

Set $\log^+ u = 1_{\{u>1\}} \log u$. The space $L_p(\log L)^k(\mathbb{X}^d)$ $(1 \leq p < \infty)$ is consisting of all functions f for which

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{X}^d} |f|^p (\log^+ |f|)^k \, d\lambda \right)^{1/p} < \infty.$$

If $p = \infty$ then set $L_\infty(\log L)^k(\mathbb{X}^d) = L_\infty(\mathbb{X}^d)$.

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{X}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity.

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p, α and β.

3. Lebesgue points

The *strong maximal function* of $f \in L_1(\mathbb{T}^d)$ is introduced by

$$M_s f(x) := \sup_{x\in I} \frac{1}{|I|} \int_I |f|\, d\lambda \qquad (x \in \mathbb{T}^d),$$

where the supremum is taken over all rectangles $I \subset \mathbb{T}^d$ with sides parallel to the axes. It is known that there is a function $f \in L_1(\mathbb{T}^d)$ such that $M_s f = \infty$ a.e. Thus M_s cannot be of weak type $(1,1)$, but we have

$$\sup_{\rho>0} \rho\lambda(x : M_s f(x) > \rho) \leq C_d + C_d \|f\|_{L_1(\log L)^{d-1}} \tag{1}$$

and

$$\|M_s f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d),\ \ 1 < p \leq \infty). \tag{2}$$

For these results see Chang and Fefferman [2], Zygmund [22] or Weisz [17, p.71]. For $1 \leq p < \infty$ let

$$M_{s,p} f(x) := \sup_{x\in I} \left(\frac{1}{|I|} \int_I |f|^p\, d\lambda \right)^{1/p} \qquad (x \in \mathbb{T}^d),$$

where the supremum is taken over all rectangles with sides parallel to the axes. Since $M_{s,p}^p f = M_s(|f|^p)$, we have

$$\sup_{\rho>0} \rho\lambda(x : M_{s,p} f(x) > \rho)^{1/p} \leq C_{d,p} + C_{d,p} \|f\|_{L_p(\log L)^{d-1}}$$

$(f \in L_p(\log L)^{d-1}(\mathbb{T}^d))$ and

$$\|M_{s,p} f\|_r \leq C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d),\ \ p < r \leq \infty).$$

Inequalities (1) and (2) imply

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \dots \int_{-h_d}^{h_d} f(x+u)\, du = f(x)$$

for a.e. $x \in \mathbb{T}^d$, where $f \in L_1(\mathbb{T}^d)$. A point $x \in \mathbb{T}^d$ is called a *strong* p-*Lebesgue point* of f if $M_{s,p}f(x)$ is *finite* and

$$\lim_{h\to 0}\left(\frac{1}{\prod_{j=1}^{d}(2h_j)}\int_{-h_1}^{h_1}\dots\int_{-h_d}^{h_d}|f(x+u)-f(x)|^p\,du\right)^{1/p}=0$$

$(1 \le p < \infty)$. One can show that almost every point $x \in \mathbb{T}^d$ is a strong p-Lebesgue point of $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ and $f \in L_r(\mathbb{T}^d)$ $(p < r \le \infty)$. Moreover, if $p < r$ and x is a strong r-Lebesgue point of f then it is also a strong p-Lebesgue point. If f is bounded and continuous at x then x is a strong p-Lebesgue point of f $(1 \le p < \infty)$.

4. Summability

The n-th *Fourier coefficient* of $f \in L_1(\mathbb{T}^d)$ is

$$\hat{f}(k) := \int_{\mathbb{T}^d} f(u)e^{-2\pi\imath k\cdot u}\,du \qquad (k \in \mathbb{Z}^d),$$

where $\imath = \sqrt{-1}$. Denote by s_nf $(n \in \mathbb{N}^d)$ the n-th partial sum of the Fourier series of $f \in L_1(\mathbb{T}^d)$, namely

$$s_nf(x) := \sum_{j=1}^{d}\sum_{k_j=-n_j}^{n_j}\hat{f}(k)e^{2\pi\imath k\cdot x}.$$

Under $\sum_{j=1}^{d}\sum_{k_j=-n_j}^{n_j}$ we mean the sum $\sum_{k_1=-n_1}^{n_1}\dots\sum_{k_d=-n_d}^{n_d}$. It is known that

$$s_nf(x) = \int_{\mathbb{T}^d} f(x-t)(D_{n_1}(t_1)\otimes\dots\otimes D_{n_d}(t_d))\,dt,$$

where

$$D_m(y) = \sum_{l=-m}^{m} e^{2\pi\imath ly} = \frac{\sin(\pi(2m+1)y)}{\sin(\pi y)} \qquad (y \in \mathbb{T}, m \in \mathbb{N})$$

is the m-th *Dirichlet kernel.*

For $f \in L_1(\mathbb{T}^d)$ the *Marcinkiewicz-Fejér means* are defined by

$$\sigma_n f(x) := \sum_{j=1}^{d} \sum_{l_j=-n}^{n} \left(1 - \frac{|l_1| \vee \ldots \vee |l_d|}{n+1}\right) \hat{f}(l) e^{2\pi \imath l \cdot x} \qquad (n \in \mathbb{N}).$$

Let $\mathbf{n} := (n, \ldots, n) \in \mathbb{N}^d$ for $n \in \mathbb{N}$. It is easy to see that

$$\sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_{\mathbf{k}} f(x) \qquad (n \in \mathbb{N}).$$

In other words this is the *Fejér summation* over the diagonal.

It is known that for all $f \in L_1(\mathbb{T}^d)$,

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{a.e.} \tag{3}$$

(see Lebesgue [6] or Zygmund [22] in the one-dimensional case and Marcinkievicz [8], Zhizhiashvili [19, 20] and Weisz [15] in the two-dimensional case). We say that the Fourier series of a function f is *strongly Marcinkiewicz summable* with the exponent q at $x \in \mathbb{T}^d$ if

$$\lim_{n\to\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |s_{\mathbf{k}} f(x) - f(x)|^q \right)^{1/q} = 0. \tag{4}$$

Obviously, if (4) is true for some $1 \leq q < \infty$ then (3) holds as well. As we mentioned in the introduction Marcinkiewicz [7] and Zygmund [21] proved in the one-dimensional case that (4) holds a.e. for all $f \in L_1(\mathbb{T}^d)$. We will extend this result to the multi-dimensional case.

First we introduce the so called *Marcinkiewicz-θ-summability* of Fourier series. We will assume that $\theta \in C_0(\mathbb{R})$ is even, $\theta(0) = 1$ and

$$\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| (1+k)^d \leq C$$

for all $n \in \mathbb{N}$, where

$$\Delta_1 \theta \left(\frac{k}{n+1} \right) := \theta \left(\frac{k}{n+1} \right) - \theta \left(\frac{k+1}{n+1} \right).$$

The *Marcinkiewicz-θ-means* of $f \in L_1(\mathbb{T}^d)$ are defined by

$$\begin{aligned}\sigma_n^\theta f(x) &:= \sum_{j=1}^{d} \sum_{l_j=-\infty}^{\infty} \theta\left(\frac{|l_1| \vee \ldots \vee |l_d|}{n+1}\right) \hat{f}(l) e^{2\pi \imath l \cdot x} = \\ &= \sum_{k=0}^{\infty} \Delta_1 \theta\left(\frac{k}{n+1}\right) \sum_{l_1=-k}^{k} \cdots \sum_{l_d=-k}^{k} \hat{f}(l) e^{2\pi \imath l \cdot x} = \\ &= \sum_{k=0}^{\infty} \Delta_1 \theta\left(\frac{k}{n+1}\right) s_{\mathbf{k}} f(x) \qquad (n \in \mathbb{N}).\end{aligned}$$

Of course, if $\theta(x) = \max(0, 1-|x|)$ $(x \in \mathbb{R})$ then we obtain the Marcinkiewicz-Fejér means. Under some conditions on θ we ([18]) have generalized (3) by

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x) \qquad \text{a.e.} \tag{5}$$

for all $f \in L_1(\mathbb{T}^d)$.

In what follows we will always suppose the following conditions:

(6)
$$\begin{cases} \theta \in C_0(\mathbb{R}) \text{ is even and } \theta(0) = 1, \\ \sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right| (1+k)^\alpha \leq C(n+1)^\alpha \text{ for all } 0 \leq \alpha < \infty \text{ and } n \in \mathbb{N}, \\ \text{there exists } 1 < q_0 < \infty \text{ such that for all } 0 \leq \alpha < \infty \text{ and } n \in \mathbb{N} \\ \sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right|^{q_0} (1+k)^\alpha \leq C(n+1)^{\alpha+1-q_0}. \end{cases}$$

Obviously, the Fejér function $\theta(x) = \max(0, 1-|x|)$ $(x \in \mathbb{R})$ satisfies these conditions. We give another sufficient condition:

(7)
$$\begin{cases} \theta \in C_0(\mathbb{R}) \text{ is even and } \theta(0) = 1, \\ \theta \text{ is continuously differentiable on } \mathbb{R} \text{ except of finitely many points}, \\ \theta' \neq 0 \text{ except of finitely many points and finitely many intervals}, \\ \text{there exists } 1 < q_0 < \infty \text{ such that} \\ \theta'(x)^{q_0}(1+x)^\alpha \text{ is integrable over } \mathbb{R}_+ \text{ for all } 0 \leq \alpha < \infty. \end{cases}$$

It follows from this condition that $\theta'(x)(1+x)^\alpha$ is integrable over $\mathbb{R}_+$ for all $0 \leq \alpha < \infty$. Indeed,

$$\int_0^\infty |\theta'(x)|(1+x)^\alpha dx \leq$$

$$\text{(8)} \qquad \begin{aligned} &\leq \left(\int_0^\infty |\theta'(x)|^{q_0}(1+x)^{(\alpha+1)q_0} dx\right)^{q_0} \left(\int_0^\infty (1+x)^{-q_0'}\, dx\right)^{q_0'} \leq \\ &\leq C \left(\int_0^\infty |\theta'(x)|^{q_0}(1+x)^{(\alpha+1)q_0}\, dx\right)^{q_0} < \infty. \end{aligned}$$

If θ has compact support then the last condition of (7) is equivalent to $\theta' \in$ $\in L_{q_0}(\mathbb{R})$. Using standard arguments we can show that (7) implies (6).

The Fourier series of a function f is called *strongly Marcinkiewicz-θ-summable* with the exponent q at $x \in \mathbb{T}^d$ if

$$\text{(9)} \qquad \lim_{n\to\infty} \left(\sum_{k=0}^\infty \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}f(x) - f(x)|^q\right)^{1/q} = 0.$$

This convergence with some exponent $1 \leq q < \infty$ implies (5). For $q = 1$ it is clear, for $1 < q < \infty$ it follows from the next lemma.

Lemma 1. *Suppose that (6) is satisfied. If f is strongly Marcinkiewicz-θ-summable with the exponent q at $x \in \mathbb{T}^d$ then f is strongly Marcinkiewicz-θ-summable with the exponent p at the same point for all* $0 < p < q$.

Proof. By Hölder's inequality

$$\left(\sum_{k=0}^\infty \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}f(x) - f(x)|^p\right)^{1/p} =$$

$$= \left(\sum_{k=0}^\infty \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right|^{p/q} |s_{\mathbf{k}}f(x) - f(x)|^p \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right|^{1-p/q}\right)^{1/p} \leq$$

$$\leq \left(\sum_{k=0}^\infty \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}f(x) - f(x)|^q\right)^{1/q} \left(\sum_{k=0}^\infty \left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right|\right)^{1/p-1/q}.$$

The second condition of (6) proves the Lemma.

5. Strong Marcinkiewicz-θ-summability

Theorem 1. *Suppose that (6) is satisfied and $1 < p \le 2$, $p < q_0 < \infty$. If $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ or $f \in L_r(\mathbb{T}^d)$ for some $p < r < \infty$ then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}} f(x) - f(x)|^{p_0} \right)^{1/p_0} = 0 \tag{10}$$

at each strong p-Lebesgue point of f, where $p_0 := p'/q_0'$ and p' denotes the dual index to p. If in addition $f \in L_\infty(\mathbb{T}^d)$ is continuous on an open set $G \subset \mathbb{T}^d$ (resp. on $\mathbb{T}^d$) then (10) holds uniformly on every $K \subset G$ compact set (resp. on $\mathbb{T}^d$).

Proof. For simplicity, we will prove the theorem in the two-dimensional case, only. The proof is similar for higher dimensions. Suppose that x is a p-Lebesgue point of f. Since

$$\int_{\mathbb{T}} D_k(t)\, dt = 1 \qquad (k \in \mathbb{N}),$$

we have

$$s_{\mathbf{k}} f(x) - f(x) = \int_{\mathbb{T}^2} (f(x-t) - f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} \frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)}\, dt$$

and

$$\left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}} f(x) - f(x)|^{p_0} \right)^{1/p_0} \le$$

$$\le \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| \left| \int_{\mathbb{T}^2} (f(x-t) - f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} \frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)}\, dt \right|^{p_0} \right)^{1/p_0}.$$

The integral can be decomposed into the sum of the integrals

$$\int_H \int_H + \int_{H^c} \int_H + \int_H \int_{H^c} + \int_{H^c} \int_{H^c},$$

where $H := [-1/(n+1), 1/(n+1)]$. Denote the corresponding integrals by $s_{\mathbf{k}}^{(i)} f(x)$ $(i = 1,2,3,4)$. Thus we have to investigate the terms

$$\left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}}^{(i)} f(x)|^{p_0} \right)^{1/p_0} \qquad (i = 1,2,3,4). \tag{11}$$

By the definition of the strong p-Lebesgue points, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left(4(n+1)^2 \int_{-1/(n+1)}^{1/(n+1)} \int_{-1/(n+1)}^{1/(n+1)} |f(x+u) - f(x)|^p \, du \right)^{1/p} < \epsilon, \tag{12}$$

whenever $1/(n+1) < \delta$. From now on we assume that $n+1 > 1/\delta$. Since $\sin t/t$ is bounded, we get for the first term that

$$|s_{\mathbf{k}}^{(1)} f(x)| =$$

$$= \left| \int_{-1/(n+1)}^{1/(n+1)} \int_{-1/(n+1)}^{1/(n+1)} (f(x-t) - f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} \frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)} dt \right| \leq$$

$$\leq C(k+1)^2 \int_{-1/(n+1)}^{1/(n+1)} \int_{-1/(n+1)}^{1/(n+1)} |f(x-t) - f(x)| dt \leq$$

$$\leq C(k+1)^2 (n+1)^{-2} \left((n+1)^2 \int_{-1/(n+1)}^{1/(n+1)} \int_{-1/(n+1)}^{1/(n+1)} |f(x-t) - f(x)|^p dt \right)^{1/p} \leq$$

$$\leq C\epsilon (k+1)^2 (n+1)^{-2}.$$

Hence

$$\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}}^{(1)} f(x)|^{p_0} \leq \tag{13}$$

$$\leq C\epsilon^{p_0} (n+1)^{-2p_0} \sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| (k+1)^{2p_0} \leq C\epsilon^{p_0},$$

because of (6).

We write the integral $\int\limits_{[-1/(n+1),1/(n+1)]^c} \int\limits_{-1/(n+1)}^{1/(n+1)}$ in the second term $s_{\mathbf{k}}^{(2)}f(x)$ as the sum of the integrals

$$\int\limits_{1/(n+1)}^{1/2} \int\limits_{0}^{1/(n+1)} + \int\limits_{1/(n+1)}^{1/2} \int\limits_{-1/(n+1)}^{0} + \int\limits_{-1/2}^{-1/(n+1)} \int\limits_{0}^{1/(n+1)} + \int\limits_{-1/2}^{-1/(n+1)} \int\limits_{-1/(n+1)}^{0},$$

and denote the corresponding terms by $s_{\mathbf{k}}^{(2,i)}f(x)$, $i = 1, 2, 3, 4$. Then, by (11) and Minkowski's inequality,

$$\left(\sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}^{(2,1)}f(x)|^{p_0}\right)^{1/p_0} =$$

$$= \left(\sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right|\right.$$

$$\left.\left|\int\limits_{1/(n+1)}^{1/2} \int\limits_{0}^{1/(n+1)} (f(x-t) - f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} \frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)} dt\right|^{p_0}\right)^{1/p_0}$$

$$= \int\limits_{0}^{1/(n+1)} \left(\sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right|\right.$$

$$\left.\left|\int\limits_{1/(n+1)}^{1/2} (f(x-t) - f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} \frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)} dt_1\right|^{p_0}\right)^{1/p_0} dt_2.$$

Since

$$\left|\frac{\sin(\pi(2k+1)t_2)}{\sin(\pi t_2)}\right| \le C(k+1)$$

we obtain

$$\left(\sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}^{(2,1)}f(x)|^{p_0}\right)^{1/p_0} \le$$

$$\le C \int\limits_{0}^{1/(n+1)} \left(\sum_{k=0}^{\infty} \left|\Delta_1 \theta\left(\frac{k}{n+1}\right)\right| (k+1)^{p_0}\right.$$

$$\left| \int_{1/(n+1)}^{1/2} (f(x-t)-f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} dt_1 \right|^{p_0} \Bigg)^{1/p_0} dt_2 \leq$$

$$\leq C \int_0^{1/(n+1)} \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right|^{q_0} (k+1)^{p_0 q_0} \right)^{1/p_0 q_0} \tag{14}$$

$$\left(\sum_{k=0}^{\infty} \left| \int_{1/(n+1)}^{1/2} (f(x-t)-f(x)) \frac{\sin(\pi(2k+1)t_1)}{\sin(\pi t_1)} dt_1 \right|^{p_0 q_0'} \right)^{1/p_0 q_0'} dt_2.$$

Since $p_0 = p'/q_0'$ and so $1/p_0 q_0 - 1/p_0 = -1/p'$, the inequality

$$\left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right|^{q_0} (k+1)^{p_0 q_0} \right)^{1/p_0 q_0} \leq C(n+1)^{1-1/p'}$$

follows from (6). Applying Hausdorff-Young inequality we obtain

$$\left(\sum_{k=0}^{\infty} \left| \int_0^{1/2} \frac{f(x-t)-f(x)}{\sin(\pi t_1)} \mathbf{1}_{(1/(n+1),1/2)} \sin(\pi(2k+1)t_1)\, dt_1 \right|^{p_0 q_0'} \right)^{1/p_0 q_0'} \leq$$

$$\leq C_p \left(\int_{1/(n+1)}^{1/2} \left| \frac{f(x-t)-f(x)}{t_1} \right|^p dt_1 \right)^{1/p}. \tag{15}$$

Thus, by (14) and (15),

$$\left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}}^{(2,1)} f(x)|^{p_0} \right)^{1/p_0} \leq$$

$$\leq C_p (n+1)^{1-1/p'} \int_0^{1/(n+1)} \left(\int_{1/(n+1)}^{1/2} \left| \frac{f(x-t)-f(x)}{t_1} \right|^p dt_1 \right)^{1/p} dt_2 \leq \tag{16}$$

$$\leq C_p (n+1)^{1-2/p'} \left(\int_0^{1/(n+1)} \int_{1/(n+1)}^{1/2} \left| \frac{f(x-t)-f(x)}{t_1} \right|^p dt_1 dt_2 \right)^{1/p}.$$

Defining

$$g(v) := \int_0^{v_1}\int_0^{v_2} |f(x-t)-f(x)|^p dt_1 dt_2$$

we conclude that

$$|g(v)| \le 4^p \epsilon^p |v_1||v_2| \qquad (|v_1|,|v_2|<\delta)$$

and

$$|g(v)| \le B(x)|v_1||v_2| \qquad (v\in \mathbb{T}^2),$$

where $B(x) := 2^{p-1}(M_{s,p}f(x)^p + f(x)^p)$. Using this and integrating by parts we conclude

$$\int_{1/(n+1)}^{1/2}\int_0^{1/(n+1)} \left|\frac{f(x-t)-f(x)}{t_1}\right|^p dt_1 dt_2 =$$

$$= \int_{1/(n+1)}^{1/2} \frac{\partial g(t_1,1/(n+1))/\partial t_1}{t_1^p} dt_1 =$$

$$= \Big[g(t_1,1/(n+1))t_1^{-p}\Big]_{1/(n+1)}^{1/2} + p\int_{1/(n+1)}^{1/2} g(t_1,1/(n+1))t_1^{-p-1} dt_1 \le$$

$$\le C_p B(x)(n+1)^{-1} + C_p \int_{1/(n+1)}^{\delta} \epsilon^p t_1^{-p}(n+1)^{-1} dt_1 +$$

$$+ p\int_{\delta}^{1/2} B(x) t_1^{-p}(n+1)^{-1} dt_1 \le$$

$$\le C_p B(x)(n+1)^{-1} + C_p\epsilon^p (n+1)^{p-2} + C_p B(x)\delta^{1-p}(n+1)^{-1}.$$

Note that $M_{s,p}f(x)$ is finite by the definition of the Lebesgue points. Taking into account (16) we obtain

$$\begin{aligned}(17)\qquad &\left(\sum_{k=0}^{\infty}\left|\Delta_1\theta\left(\frac{k}{n+1}\right)\right| |s_{\mathbf{k}}^{(2,1)}f(x)|^{p_0}\right)^{1/p_0} \le \\ &\le C_p(n+1)^{1-2/p'}(B(x)^{1/p}(n+1)^{-1/p} + \epsilon(n+1)^{1-2/p}) \le \\ &\le C_p\epsilon + C_p B(x)^{1/p}(n+1)^{-1/p'},\end{aligned}$$

which is small enough if $n+1$ is large enough. The other integrals corresponding to $s_{\mathbf{k}}^{(2,i)}f(x)$ $(i=2,3,4)$ and $s_{\mathbf{k}}^{(i)}f(x)$ $(i=3,4)$ can be handled similarly.

If f is bounded on $\mathbb{T}^d$ and continuous on G then every $x \in G$ is a strong p-Lebesgue point of f. If $K \subset G$ is compact then there exists $\delta_1 > 0$ such that the set $K + \delta_1 \subset G$ is compact. f is uniformly continuous on $K + \delta_1$ and so $|f(x+u) - f(x)| < \epsilon$ whenever $|u| < \delta$ and $x \in K$ with some $\delta < \delta_1$. This means that (12) and hence (13) hold uniformly on K. Since f is bounded, $M_{s,p}f(x)$ is bounded, too. Thus (17) holds uniformly, which finishes the proof.

Note that $L_p(\log L)^{d-1}(\mathbb{T}^d) \supset L_r(\mathbb{T}^d)$ for all $p < r < \infty$.

Corollary 1. *Suppose that (6) is satisfied and* $1 < p < \infty$. *If* $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ *or* $f \in L_r(\mathbb{T}^d)$ *for some* $p < r < \infty$ *then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}} f(x) - f(x)|^q \right)^{1/q} = 0 \tag{18}$$

for each strong p*-Lebesgue point of* f *and all* $0 < q < \infty$. *If in addition* $f \in L_\infty(\mathbb{T}^d)$ *is continuous on an open set* $G \subset \mathbb{T}^d$ *(resp. on* $\mathbb{T}^d$*) then (18) holds uniformly on every* $K \subset G$ *compact set (resp. on* $\mathbb{T}^d$*).*

Proof. Observe that $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ implies $f \in L_{p_1}(\log L)^{d-1}(\mathbb{T}^d)$ if $p_1 < p$. If x is a strong p-Lebesgue point of f then it is also a strong p_1-Lebesgue point. If in addition $1 < p_1 < q_0 \wedge 2$ then, by Theorem 1, (10) holds for $p_0 = p_1'/q_0'$. Letting $p_1 \to 1$ we have $p_0 \to \infty$. Now Lemma 1 finishes the proof.

Corollary 2. *Suppose that (6) is satisfied and* $1 < p < \infty$. *If* $f \in L_p(\mathbb{T}^d)$ *then for all* $0 < q < \infty$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \Delta_1 \theta \left(\frac{k}{n+1} \right) \right| |s_{\mathbf{k}} f(x) - f(x)|^q \right)^{1/q} = 0 \quad a.e.$$

In the special case $\theta(x) = \max(0, 1-|x|)$ and $d = 1$ Corollary 2 was proved in Marcinkiewicz [7] and Zygmund [21].

6. Some summability methods

In this section we consider some well known summability methods as special cases of the strong θ-summation. Of course, there are a lot of other summability methods which could be considered as special cases. It is easy to see that (7) and so (6) is satisfied all in the next examples. The elementary computations are left to the reader.

Example 1 (Weierstrass summation). $\theta(x) = e^{-|x|^\gamma}$ for some $0 < < \gamma < \infty$. Note that if $\gamma = 1$ then we obtain the Abel means.

Example 2 (Fejér summation).

$$\theta(x) = \begin{cases} 1 - |x|, & \text{if } 0 \le |x| \le 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Example 3 (Riesz summation). For $0 < \alpha < \infty, 0 < \gamma < \infty$ let

$$\theta(x) = \begin{cases} (1 - |x|^\gamma)^\alpha, & \text{if } 0 \le |x| \le 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Example 4 (de La Vallée-Poussin summation). Let

$$\theta(x) = \begin{cases} 1, & \text{if } |x| \le 1/2, \\ -2|x| + 2, & \text{if } 1/2 < |x| \le 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Example 5 (Jackson-de La Vallée-Poussin summation). Let

$$\theta(x) = \begin{cases} 1 - 3x^2/2 + 3|x|^3/4, & \text{if } |x| \le 1, \\ (2 - |x|)^3/4, & \text{if } 1 < |x| \le 2, \\ 0, & \text{if } |x| > 2 \end{cases} \qquad (x \in \mathbb{R}).$$

Example 6. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ is even, $\theta(\alpha_j) = \beta_j$ $(j = = 0, 1, \ldots, m)$, $\theta(x) = 0$ for $x \ge \alpha_m$, θ_9 is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

Example 7 (Rogosinski summation). Let

$$\theta(x) = \begin{cases} \cos \pi x/2, & \text{if } |x| \le 1 + 2j, \\ 0, & \text{if } |x| > 1 + 2j \end{cases} \qquad (j \in \mathbb{N}).$$

7. Strong Marcinkiewicz-Cesàro summability

The following example cannot be defined by a single function θ. Let

$$\theta = \big(\theta(k, n+1), \quad k \in \mathbb{Z}, \quad n \in \mathbb{N}\big)$$

be a double sequence, which is even in the first parameter. Then the strong Marcinkiewicz-θ-summability in (9) can be defined by replacing $\Delta_1\theta\left(\frac{k}{n+1}\right)$ by

$$\Delta_1\theta(k, n+1) := \theta(k, n+1) - \theta(k+1, n+1).$$

If we suppose that

(19)

$$\begin{cases} \theta(0, n+1) = 1 \quad (n \in \mathbb{N}), \qquad \lim_{n\to\infty} \theta(k, n+1) = 0 \quad (k \in \mathbb{Z}), \\ \sum_{k=0}^{\infty} |\Delta_1\theta(k, n+1)|(1+k)^{\beta} \le C(n+1)^{\beta} \ \text{ for all } \ 0 \le \beta < \infty \ \text{ and } \ n \in \mathbb{N}, \\ \text{there exists } \ 1 < q_0 < \infty \ \textit{ such that for all } \ 0 \le \beta < \infty \ \text{ and } \ n \in \mathbb{N} \\ \sum_{k=0}^{\infty} |\Delta_1\theta(k, n+1)|^{q_0}(1+k)^{\beta} \le C(n+1)^{\beta+1-q_0}. \end{cases}$$

then we can prove the analogues to Theorem 1 and Corollaries 1 and 2:

Corollary 3. *Suppose that (19) is satisfied and* $1 < p < \infty$*. If* $f \in L_p(\log L)^{d-1}(\mathbb{T}^d)$ *or* $f \in L_r(\mathbb{T}^d)$ *for some* $p < r < \infty$ *then*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} |\Delta_1\theta(k, n+1)||s_{\mathbf{k}}f(x) - f(x)|^q \right)^{1/q} = 0$$

for each strong p-Lebesgue point of f *and all* $0 < q < \infty$*. If in addition* $f \in L_\infty(\mathbb{T}^d)$ *is continuous on an open set* $G \subset \mathbb{T}^d$ *(resp. on* $\mathbb{T}^d$*) then the convergence holds uniformly on every* $K \subset G$ *compact set (resp. on* $\mathbb{T}^d$*).*

Corollary 4. *Suppose that (19) is satisfied and* $1 < p < \infty$*. If* $f \in L_p(\mathbb{T}^d)$ *then for all* $0 < q < \infty$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty} |\Delta_1\theta(k, n+1)||s_{\mathbf{k}}f(x) - f(x)|^q \right)^{1/q} = 0 \quad a.e.$$

To define the *strong Marcinkiewicz-Cesàro summability* let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \qquad (n \in \mathbb{N}, \alpha > 0).$$

It is known (see Zygmund [22, p.77]) that

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \qquad A_n^\alpha \sim n^\alpha \qquad (n \in \mathbb{N}).$$

Let

$$\theta(k, n+1) = \begin{cases} \frac{A_{n-|k|}^\alpha}{A_n^\alpha}, & \text{if } 0 \le |k| \le n, \\ 0, & \text{if } |k| > n \end{cases}$$

for some $0 < \alpha < \infty$. If $\alpha = 1$, we obtain again the Marcinkiewicz-Fejér summability. The first condition of (19) is obvious and to see the second and third ones observe that $\Delta_1\theta(k, n+1) = \frac{A_{n-k}^{\alpha-1}}{A_n^\alpha}$ and

$$\sum_{k=0}^{\infty} |\Delta_1\theta(k, n+1)|^q (1+k)^\beta \le Cn^{-\alpha q} \sum_{k=0}^{n} (n-k)^{(\alpha-1)q}(1+k)^\beta \le$$
$$\le Cn^{-\alpha q} n^\beta n^{1+(\alpha-1)q} \le C(n+1)^{\beta+1-q},$$

whenever $1 \le q < 1/(1-\alpha)$ if $0 < \alpha < 1$ and $1 \le q < \infty$ if $\alpha \ge 1$. Consequently Corollaries 3 and 4 hold for the Marcinkiewicz-Cesàro summability.

8. Fourier transforms

In this section we describe briefly the analogous results for Fourier transforms. The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(u) e^{-2\pi \imath x \cdot u} \, du \qquad (x \in \mathbb{R}^d).$$

The Fourier transform can be extended to the $L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ spaces in the usual way (see e.g. Butzer and Nessel [1]). For $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ let

$$s_T f(x) := \int_{-T_1}^{T_1} \dots \int_{-T_d}^{T_d} \hat{f}(t) e^{2\pi \imath x \cdot t} \, dt = \int_{\mathbb{R}^d} f(u) \prod_{j=1}^{d} \frac{\sin 2\pi T_j (x_j - u_j)}{\pi (x_j - u_j)} du.$$

Since this last integral is well defined also in case $f \in L_p(\mathbb{R}^d)$ $(2 < p < \infty)$, we define $s_T f$ in this case by this integral.

For $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ the *Marcinkiewicz-Fejér means* are defined by

$$\sigma_T f(x) := \int_0^T \left(1 - \frac{|t_1| \vee \ldots \vee |t_d|}{T}\right) \hat{f}(t) e^{2\pi \imath x\cdot t}\, dt = \frac{1}{T}\int_0^T s_{\mathbf{u}} f(x) du$$

$(T \in \mathbb{R}_+)$. If $(2 < p < \infty)$ then the second integral can be regarded as the definition of $\sigma_T f$. It is known that for all $f \in L_1(\mathbb{R}^d)$

$$\lim_{T\to\infty} \sigma_T f(x) = f(x) \qquad \text{a.e.}$$

(see Weisz [14, 17]). We say that the Fourier transform of a function f is *strongly Marcinkiewicz summable* with the exponent q at $x \in \mathbb{R}^d$ if

$$\lim_{T\to\infty} \left(\frac{1}{T}\int_0^T |s_{\mathbf{u}} f(x) - f(x)|^q \, du\right)^{1/q} = 0.$$

To introduce the *Marcinkiewicz-θ-summability* we will assume that $\theta \in$ $\in C_0(\mathbb{R})$ is even, $\theta(0) = 1$, θ is differentiable on $\mathbb{R}$ except of finitely many points, $\theta'(x)(1+x)^d \in L_1(\mathbb{R}_+)$. Then the *Marcinkiewicz-θ-means* are defined by

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} \theta\left(\frac{|t_1| \vee \ldots \vee |t_d|}{T}\right) \hat{f}(t) e^{2\pi \imath x\cdot t}\, dt =$$

$$= \frac{-1}{T}\int_0^\infty \theta'\left(\frac{u}{T}\right) \int_{-u}^{u} \ldots \int_{-u}^{u} \hat{f}(t) e^{2\pi \imath x\cdot t}\, dt\, du =$$

$$= \frac{-1}{T}\int_0^\infty \theta'\left(\frac{u}{T}\right) s_{\mathbf{u}} f(x)\, du$$

for all $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$. Again, $\sigma_T^\theta f$ can be defined by the last integral for all $f \in L_p(\mathbb{R}^d)$ $(2 < p < \infty)$. Under some conditions on θ we [16] have verified that for all $f \in L_1(\mathbb{R}^d)$,

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \qquad \text{a.e.}$$

The Fourier transform of a function f is called *strongly Marcinkiewicz-θ-summable* with the exponent q at $x \in \mathbb{R}^d$ if

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_0^\infty \left|\theta'\left(\frac{u}{T}\right)\right| |s_{\mathbf{u}}f(x)-f(x)|^q\,du\right)^{1/q}=0.$$

Assuming

$$(20)\qquad \begin{cases} \theta\in C_0(\mathbb{R}) \text{ is even and } \theta(0)=1, \\ \theta \text{ is differentiable on } \mathbb{R} \text{ except of finitely many points,} \\ \text{there exists } 1<q_0<\infty \text{ such that} \\ |\theta'(x)|^{q_0}(1+x)^\alpha \text{ is integrable over } \mathbb{R}_+ \text{ for all } 0\le\alpha<\infty \end{cases}$$

we can prove the analogues of all results above:

Corollary 5. *Suppose that (20) is satisfied and* $1<p<\infty$*. If* $f\in$ $\in L_s(\mathbb{R}^d)\cap L_p(\log L)^{d-1}(\mathbb{R}^d)$ *for some* $1\le s<\infty$ *or* $f\in L_r(\mathbb{R}^d)$ *for some* $p<r<\infty$ *then*

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_0^\infty \left|\theta'\left(\frac{u}{T}\right)\right| |s_{\mathbf{u}}f(x)-f(x)|^q\,du\right)^{1/q}=0$$

for each strong p-Lebesgue point of f *and all* $0<q<\infty$*. If in addition* $f\in L_\infty(\mathbb{R}^d)$ *is continuous on an open set* $G\subset\mathbb{R}^d$ *(resp. uniformly continuous on* $\mathbb{R}^d$*) then the convergence holds uniformly on every* $K\subset G$ *compact set (resp. on* $\mathbb{R}^d$*).*

Since $f\in L_1(\mathbb{R}^d)\cap L_p(\log L)^{d-1}(\mathbb{R}^d)$ implies $f\in L_p(\mathbb{R}^d)$, we obtain

Corollary 6. *Suppose that (20) is satisfied and* $1<p<\infty$*. If* $f\in L_p(\mathbb{R}^d)$ *then for all* $0<q<\infty$

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_0^\infty \left|\theta'\left(\frac{u}{T}\right)\right| |s_{\mathbf{u}}f(x)-f(x)|^q\,du\right)^{1/q}=0\quad a.e.$$

Note that the last condition of (20) implies that $|\theta'(x)|(1+x)^\alpha$ is integrable over $\mathbb{R}_+$ for all $0\le\alpha<\infty$ (see (8)). Examples 1-7 all satisfy (20).

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