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POISSON DISTRIBUTION FOR A SUM OF ADDITIVE FUNCTIONS ON ARITHMETIC PROGRESSIONS

J. Šiaulys and G. Stepanauskas
(Vilnius, Lithuania)

Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. We consider the limit distribution of values of a sum of additive arithmetic functions the arguments of which run through different arithmetic progressions. The case of the Poisson limit law is studied. The functions considered take at most two values on the set of primes, 0 and 1, and satisfy some additional conditions. Some examples are given.

1. Introduction

A function f defined on positive integers is called additive, if

$$f(n) = \sum_{p^k || n} f(p^k),$$

and strongly additive, if

$$f(n) = \sum_{p|n} f(p).$$

Throughout the paper $p, p_1, \dots$ denote primes, m, n, l, k are positive integers. The additive function depending on real parameter x is denoted by f_x.

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In the article [4] the authors considered the weak convergence of the distributions

$$\nu_x\left(n : n \le x, f_x(n) + g_x(n+1) < u\right) := \frac{1}{[x]} \sum_{\substack{n \le x \\ f_x(n)+g_x(n+1)<u}} 1,$$

as $x \to \infty$, to the Poisson law. In this paper we continue this topic and give a generalization of the result from [4]. We consider more common case, when arguments of the additive functions f_x and g_x run through two different arithmetic progressions. The results given are also related to a paper of I. Kátai [1].

For more detailed introduction and more full list of literature see [4].

2. Main result and examples

Put

$$\mathcal{P}_x^f := (p : p \le ax + b,\ (a,p) = 1,\ f_x(p) = 1),$$
$$\mathcal{P}_x^g := (p : p \le cx + d,\ (c,p) = 1,\ g_x(p) = 1).$$

We consider the additive functions f_x and g_x on two arithmetic progressions $an + b$ and $cn + d$, $n \in \mathbb{N}$, the moduli a, c of which are positive integers and the values b, d are such that $an+b$ and $cn+d$ take only positive integer values for all $n \in \mathbb{N}$. The values a, b, c, d are fixed, do not depend on x, and the requirement

(1) $$(a, b) = (c, d) = 1$$

is fulfilled. Let finally f_x and g_x be a pair of functions which on primes dividing $ad - bc$ satisfies the condition

(2) $$\min_{p \mid ad-bc} (f_x(p), g_x(p)) = 0.$$

Theorem. *Let f_x and g_x, $x \ge 2$, be two sets of strongly additive functions such that $f_x(p), g_x(p) \in \{0, 1\}$ for all primes p, and the conditions (1), (2) be satisfied. Let, in addition,*

(3) $$\lim_{x\to\infty} \frac{1}{\log x} \sum_{p \in \mathcal{P}_x^f \cup \mathcal{P}_x^g} \frac{\log p}{p} = 0.$$

The distribution functions

$$\nu_x\left(n : n \le x, f_x(an+b) + g_x(cn+d) < u\right) \tag{4}$$

converge weakly to the Poisson distribution with a parameter λ *if and only if*

$$\lim_{x\to\infty} \max_{p \in \mathcal{P}_x^f \cup \mathcal{P}_x^g} \frac{1}{p} = 0 \tag{5}$$

and

$$\lim_{x\to\infty} \left(\sum_{p \in \mathcal{P}_x^f} \frac{1}{p} + \sum_{p \in \mathcal{P}_x^g} \frac{1}{p} \right) = \lambda. \tag{6}$$

Example 1. Let $\psi_f(x)$ and $\psi_g(x)$ be two unboundedly increasing functions such that $\log \psi_f(x)/\log x \to 0$ and $\log \psi_g(x)/\log x \to 0$, as $x \to \infty$. Let f_x, g_x be strongly additive and

$$f_x(p) = \begin{cases} 1 & \text{if } \psi_f(x) < p \le \psi_f^{\alpha}(x), \\ 0 & \text{otherwise,} \end{cases}$$

$$g_x(p) = \begin{cases} 1 & \text{if } \psi_g(x) < p \le \psi_g^{\beta}(x), \\ 0 & \text{otherwise,} \end{cases}$$

with some $\alpha, \beta > 1$. Let, finally, a, b, c, d satisfy the conditions

$$(a,b) = (c,d) = 1, \quad ad - bc \ne 0.$$

It follows from Theorem that

$$\lim_{x\to\infty} \nu_x(n : n \le x,\ f_x(an+b) + g_x(cn+d) = k) = \frac{(\log \alpha\beta)^k}{\alpha\beta\, k!}$$

for every fixed $k = 0, 1, 2, \ldots$ If $k = 0$ we have

$$\#\left\{ n : n \le x, \quad \begin{array}{c} an+b \text{ has no prime factors from} \\ \text{the interval } (\psi_f(x), \psi_f^{\alpha}(x)] \text{ and} \\ cn+d \text{ has no prime factors from} \\ \text{the interval } (\psi_g(x), \psi_g^{\beta}(x)] \end{array} \right\} \sim \frac{x}{\alpha\beta \log x},$$

as $x \to \infty$.

Example 2. Let f_x, g_x be strongly additive and

$$f_x(p) = g_x(p) = \begin{cases} 1 & \text{if } \log x < p \le \log^2 x, \\ 0 & \text{otherwise.} \end{cases}$$

Let a, b, c, d satisfy the conditions

$$(a, b) = (c, d) = 1, \quad ad - bc \ne 0.$$

Theorem implies that

$$\lim_{x\to\infty} \nu_x \left(n : n \le x,\ f_x(an+b) + g_x(cn+d) = k\right) = \frac{\log^k 4}{4\, k!}$$

for every fixed $k = 0, 1, 2, \dots$ In case $k = 0$, we get

$$\#\left\{ n : n \le x, \ \begin{array}{c} an+b \text{ and } cn+d \text{ have no prime factors} \\ \text{from the interval } (\log x, \log^2 x] \end{array} \right\} \sim \frac{x}{4 \log x},$$

as $x \to \infty$. If $k = 1$ we have

$$\#\left\{ n : n \le x, \ \begin{array}{c} \text{exactly one prime from the interval} \\ (\log x, \log^2 x] \text{ divides } an+b \text{ or } cn+d \end{array} \right\} \sim \frac{x \log 4}{4 \log x},$$

as $x \to \infty$.

3. Proof of Theorem. Necessity

Suppose that the conditions of Theorem are satisfied. In this part we prove step by step that the weak convergence of the distribution functions (4) implies the relations (5) and (6).

I. First we prove that

$$\max\left(\sum_{p \le ax+b}^{f} \frac{1}{p}, \ \sum_{p \le cx+d}^{g} \frac{1}{p} \right) \ll_\lambda 1. \tag{7}$$

Here and further the superscript f or g by the sign of a sum means that the summation is extended over primes for which $f_x(p) = 1$ or $g_x(p) = 1$, respectively. It is evident that the last estimate (7) yields

$$\sum_{p \in \mathcal{P}_x^f \cup \mathcal{P}_x^g} \frac{1}{p} \ll_\lambda 1 \tag{8}$$

and

$$\max\left({\sum_{p \le w(x)}}^{f} \frac{1}{p},\ {\sum_{p \le w(x)}}^{g} \frac{1}{p} \right) \ll_\lambda 1, \tag{9}$$

where $w(x) = \max(ax+b, cx+d)$.

The weak convergence of the distributions (4) implies that

$$\begin{aligned}
&\liminf_{x\to\infty} \nu_x \left(n : n \le x,\ f_x(an+b) = 0\right) \ge \\
\ge &\liminf_{x\to\infty} \nu_x \left(n : n \le x,\ f_x(an+b) + g_x(cn+d) = 0\right) = \\
= &e^{-\lambda}.
\end{aligned}$$

According to the Halász inequality (see [2])

$$\sum_{\substack{n \le x \\ f_x(an+b)=0}} 1 \le \sum_{\substack{n \le ax+b \\ f_x(n)=0}} 1 \ll (ax+b) \left(\sum_{\substack{p \le ax+b \\ f_x(p) \ne 0}} \frac{1}{p} \right)^{-\frac{1}{2}}.$$

Hence

$$\begin{aligned}
&\limsup_{x\to\infty} {\sum_{p \le ax+b}}^{f} \frac{1}{p} \ll \\
\ll &\limsup_{x\to\infty} \left(\frac{ax+b}{x} \right)^2 \frac{1}{\nu_x^2 \left(n : n \le x, f_x(an+b) = 0\right)} \ll \\
\ll &a^2 e^{2\lambda}
\end{aligned}$$

and

$${\sum_{p \le ax+b}}^{f} \frac{1}{p} \ll_\lambda 1$$

for every $x \ge 2$. Similarly

$${\sum_{p \le cx+d}}^{g} \frac{1}{p} \ll_\lambda 1.$$

Therefore the estimate (7) holds.

II. For the positive integer l define

$$\varphi_{l,x} = \frac{1}{x}\sum_{n\le x}(f_x(an+b)+g_x(cn+d))(f_x(an+b)+g_x(cn+d)-1)\dots \times (f_x(an+b)+g_x(cn+d)-l+1).$$

It follows from the known combinatorial equalities that

$$\varphi_{l,x} = \sum_{k=0}^{l}\binom{l}{k}\frac{1}{x}\sum_{n\le x} f_x(an+b)(f_x(an+b)-1)\dots(f_x(an+b)-k+1)\times \times g_x(cn+d)(g_x(cn+d)-1)\dots(g_x(cn+d)-(l-k)+1).$$

The strong additivity of f_x and g_x implies that

$$\varphi_{l,x} = \sum_{k=0}^{l}\binom{l}{k}\sum_{p_1\in\mathcal{P}_x^f}\dots\sum_{\substack{p_k\in\mathcal{P}_x^f\\ p_k\ne p_1,\dots,p_{k-1}}}\sum_{p_{k+1}\in\mathcal{P}_x^g}\dots\sum_{\substack{p_l\in\mathcal{P}_x^g\\ p_l\ne p_{k+1},\dots,p_{l-1}}} \times\frac{1}{x}\sum_{\substack{n\le x\\ p_1\cdots p_k\mid an+b\\ p_{k+1}\cdots p_l\mid cn+d}} 1. \tag{10}$$

In this part we prove that

$$\sup_{x\ge 2}\varphi_{l,x} \ll_{l,\lambda} 1 \tag{11}$$

for every positive integer l.

If $l=1$, then from (10) we have

$$\varphi_{1,x} = \sum_{p\in\mathcal{P}_x^f}\frac{1}{x}\sum_{\substack{n\le x\\ p\mid an+b}} 1 + \sum_{p\in\mathcal{P}_x^g}\frac{1}{x}\sum_{\substack{n\le x\\ p\mid cn+d}} 1.$$

Since

$$\sum_{\substack{n\le x\\ m\mid an+b}} 1 = \begin{cases} \frac{x}{m}+O(1) & \text{if } (a,m)=1,\\ 0 & \text{if } (a,m)\ne 1,\end{cases} \tag{12}$$

then

$$\varphi_{1,x} = \sum_{p \in \mathcal{P}_x^f} \frac{1}{p} + \sum_{p \in \mathcal{P}_x^g} \frac{1}{p} + O\left(\frac{1}{x} \sum_{p \leq ax+b} 1 + \frac{1}{x} \sum_{p \leq cx+d} 1 \right). \tag{13}$$

Therefore according to the estimate (8), we have

$$\varphi_{1,x} \ll_\lambda 1.$$

If $l \geq 2$, then

$$\begin{aligned}
\varphi_{l,x} \leq & \frac{1}{x} \sum_{n \leq x} \left(f_x(an+b) + g_x(cn+d) \right)^l \ll_l \\
\ll_l & \frac{1}{x} \sum_{n \leq w(x)} \left(f_x^l(n) + g_x^l(n) \right) \ll_l \\
\ll_l & \frac{1}{x} \sum_{n \leq w(x)} \left| f_x(n) - \frac{1}{x} \sum_{n \leq w(x)} f_x(n) \right|^l + \\
& + \frac{1}{x} \sum_{n \leq w(x)} \left| g_x(n) - \frac{1}{x} \sum_{n \leq w(x)} g_x(n) \right|^l + \\
& + \frac{1}{x} \sum_{n \leq w(x)} \left(\frac{1}{x} \sum_{n \leq w(x)} f_x(n) \right)^l + \\
& + \frac{1}{x} \sum_{n \leq w(x)} \left(\frac{1}{x} \sum_{n \leq w(x)} g_x(n) \right)^l .
\end{aligned}$$

Using the Ruzsa moments inequality (see [3]) and the estimate (9), we obtain

$$\begin{aligned}
& \frac{1}{x} \sum_{n \leq w(x)} \left| f_x(n) - \frac{1}{x} \sum_{n \leq w(x)} f_x(n) \right|^l \ll \\
\ll & \frac{1}{w(x)} \sum_{n \leq w(x)} \left| f_x(n) - \frac{1}{x} \sum_{n \leq w(x)} f_x(n) \right|^l \ll_l \\
\ll_l & \left(\sum_{p^k \leq w(x)} \frac{f_x^2(p^k)}{p^k} \right)^{l/2} + \sum_{p^k \leq w(x)} \frac{f_x^l(p^k)}{p^k} \ll_l
\end{aligned}$$

$$\ll_l \left(\sum_{p \le w(x)}^{f} \frac{1}{p} \right)^{l/2} + \sum_{p \le w(x)}^{f} \frac{1}{p} \ll_{l,\lambda} 1. \tag{14}$$

It is not difficult to understand that

$$\frac{1}{x} \sum_{n \le w(x)} \left(\frac{1}{x} \sum_{n \le w(x)} f_x(n) \right)^l \le \frac{1}{x} \sum_{n \le w(x)} \left(\frac{w(x)}{x} \sum_{p \le w(x)}^{f} \frac{1}{p} \right)^l \ll_{l,\lambda} 1. \tag{15}$$

The last two estimates (14) and (15) are true for the function g in place of f, too. So, the estimate (11) follows now immediately from the obtained inequalities.

III. In this part we prove that the conditions of Theorem imply that

$$\lim_{x \to \infty} \varphi_{l,x} = \lambda^l \tag{16}$$

for every fixed positive integer l.

We have from the weak convergence of the distributions (4) that

$$\lim_{x \to \infty} \nu_x \left(n : n \le x, f_x(an+b) + g_x(cn+d) = k \right) = \frac{\lambda^k}{k!} e^{-\lambda}$$

for every fixed positive integer k. Hence, using the estimate (11), for $K > l+2$ we have

$$\begin{aligned}
\varphi_{l,x} =& \sum_{k=l}^{K} k(k-1)\dots(k-l+1)\ \frac{1}{x} \sum_{\substack{n \le x \\ f_x(an+b)+g_x(cn+d)=k}} 1 + \\
&+ \frac{1}{x} \sum_{\substack{n \le x \\ f_x(an+b)+g_x(cn+d)>K}} (f_x(an+b)+g_x(cn+d)) \dots \times \\
&\times (f_x(an+b)+g_x(cn+d)-l+1) \frac{f_x(an+b)+g_x(cn+d)-l}{f_x(an+b)+g_x(cn+d)-l} = \\
=& \sum_{k=l}^{K} k(k-1)\dots(k-l+1) \frac{\lambda^k}{k!} e^{-\lambda} + o_{K,l}(1) + O\left(\frac{\varphi_{l+1,x}}{K-l} \right) = \\
=& \lambda^l + o_{K,l}(1) + O\left(\frac{\lambda^{K+1}}{(K-l+1)!} \right) + O_l\left(\frac{1}{K-l} \right).
\end{aligned}$$

Taking the upper limit in the last equality, as x tends to infinity and then as K tends to infinity, we obtain that the relation (16) holds.

IV. In this part we derive the relations (5) and (6). Because the second of these relations follows from (13) and (16) immediately, it remains to prove (5).

From the condition (3) we have that, for every fixed positive δ

$$\lim_{x\to\infty} \sum_{\substack{p>x^{\delta} \\ p\in\mathcal{P}_x^f\cup\mathcal{P}_x^g}} \frac{1}{p} = 0.$$

Therefore there exist a vanishing function $\delta(x)$ such that

$$\lim_{x\to\infty} x^{\delta(x)} = \infty \tag{17}$$

and

$$\lim_{x\to\infty} \sum_{\substack{p>x^{\delta(x)} \\ p\in\mathcal{P}_x^f\cup\mathcal{P}_x^g}} \frac{1}{p} = 0. \tag{18}$$

Let f_x^* and g_x^* be two new sets of strongly additive functions defined by the equalities:

$$f_x^*(p) = \begin{cases} f_x(p) & \text{if } \ p \le x^{\delta(x)}, \\ \\ 0 & \text{if } \ p > x^{\delta(x)}, \end{cases}$$

$$g_x^*(p) = \begin{cases} g_x(p) & \text{if } \ p \le x^{\delta(x)}, \\ \\ 0 & \text{if } \ p > x^{\delta(x)}. \end{cases}$$

According to the equality (12), for every positive ε we have that

(19)

$$\begin{aligned}
&\nu_x\left(n : n \le x,\ |f_x(an+b) + g_x(cn+d) - f_x^*(an+b) - g_x^*(cn+d)| > \varepsilon\right) \le \\
\le&\nu_x\left(n : n \le x,\ |f_x(an+b) - f_x^*(an+b)| > \frac{\varepsilon}{2}\right) + \\
&\qquad + \nu_x\left(n : n \le x,\ |g_x(cn+d) - f_x^*(cn+d)| > \frac{\varepsilon}{2}\right) \le \\
\le&\nu_x\left(n : n \le x,\ \exists p \mid an+b : f_x(p) \ne f_x^*(p)\right) + \\
&\qquad + \nu_x\left(n : n \le x,\ \exists p \mid cn+d : g_x(p) \ne g_x^*(p)\right) \le \\
\le& \sum_{\substack{p\in\mathcal{P}_x^f \\ p>x^{\delta(x)}}} \frac{1}{[x]} \sum_{\substack{n\le x \\ p\mid an+b}} 1 + \sum_{\substack{p\in\mathcal{P}_x^g \\ p>x^{\delta(x)}}} \frac{1}{[x]} \sum_{\substack{n\le x \\ p\mid cn+d}} 1 \le \\
\le& \sum_{\substack{p\in\mathcal{P}_x^f \\ p>x^{\delta(x)}}} \frac{1}{p} + \sum_{\substack{p\in P_x^g \\ p>x^{\delta(x)}}} \frac{1}{p} + \frac{1}{[x]}\, O\left(\sum_{p\le ax+b} 1 + \sum_{p\le cx+d} 1\right).
\end{aligned}$$

Thus, the weak convergence of distributions (4) and the relation (18) imply that the distribution functions

$$\nu_x\left(n \le x,\ f_x^*(an+b) + g_x^*(cn+d) < u\right) \tag{20}$$

converge also weakly to the Poisson distribution with the same parameter λ. Hence the new functions f_x^* and g_x^* satisfy the conditions of Theorem. The obtained equality (16) implies that

$$\lim_{x\to\infty} \varphi_{l,x}^* = \lambda^l, \tag{21}$$

where

(22)

$$\begin{aligned}\varphi_{l,x}^* =& \frac{1}{x}\sum_{n\le x}(f_x^*(an+b)+g_x^*(cn+d))(f_x^*(an+b)+g_x^*(cn+d)-1)\ldots\times\\ &\times(f_x^*(an+b)+g_x^*(cn+d)-l+1).\end{aligned}$$

If $l=2$ it follows from (21) that the quantity

(23)

$$\begin{aligned}\varphi_{2,x}^* =& \frac{1}{x}\sum_{n\le x} f_x^*(an+b)(f_x^*(an+b)-1)+\\ &+\frac{2}{x}\sum_{n\le x} f_x^*(an+b)g_x^*(cn+d)+\frac{1}{x}\sum_{n\le x} g_x^*(cn+d)(g_x^*(cn+d)-1)\end{aligned}$$

tends to λ^2, as x tends to infinity. The first term of the last sum is equal to

$$\sum_{\substack{p_1\in\mathcal{P}_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^f\\ p_2\le x^{\delta(x)}\\ p_2\ne p_1}} \frac{1}{x}\sum_{\substack{n\le x\\ p_1p_2|an+b}} 1 = \sum_{\substack{p_1\in\mathcal{P}_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^f\\ p_2\le x^{\delta(x)}\\ p_2\ne p_1}} \frac{1}{p_1p_2}+o(1)$$

according to the relation (12). Analogous equality is true for the third term of (23) where the function f is replaced by g. It follows from the well known Chinese residue theorem that

(24)

$$\sum_{\substack{n\le x\\ m_1|an+b\\ m_2|cn+d}} 1 = \begin{cases} \frac{x}{[m_1,m_2]}+O(1) & \text{if } \ (a,m_1)=(c,m_2)=1, (m_1,m_2)|ad-cb,\\ 0 & \text{otherwise.}\end{cases}$$

Hence the second therm of (23) is equal to

$$
\begin{aligned}
&2\sum_{\substack{p_1\in\mathcal{P}_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^g\\ p_2\le x^{\delta(x)}\\ p_2\ne p_1}}\frac{1}{x}\sum_{\substack{n\le x\\ p_1|an+b\\ p_2|cn+d}}1+2\sum_{\substack{p_1\in\mathcal{P}_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^g\\ p_2\le x^{\delta(x)}\\ p_2=p_1}}\frac{1}{x}\sum_{\substack{n\le x\\ p_1|an+b\\ p_2|cn+d}}1=\\
=&2\sum_{\substack{p_1\in\mathcal{P}_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^g\\ p_2\le x^{\delta(x)}\\ p_2\ne p_1}}\frac{1}{p_1p_2}+2\sum_{\substack{p\in\mathcal{P}_x^f\cap\mathcal{P}_x^g\\ p|ad-cb\\ p\le x^{\delta(x)}}}\frac{1}{p}+o(1)=\\
=&2\sum_{\substack{p_1\in P_x^f\\ p_1\le x^{\delta(x)}}}\ \sum_{\substack{p_2\in\mathcal{P}_x^g\\ p_2\le x^{\delta(x)}\\ p_2\ne p_1}}\frac{1}{p_1p_2}+o(1).
\end{aligned}
$$

Putting the obtained expressions into the relation (23) we have that

$$
\begin{aligned}
\varphi^*_{2,x}=&\left(\sum_{\substack{p\in\mathcal{P}_x^f\\ p\le x^{\delta(x)}}}\frac{1}{p}+\sum_{\substack{p\in\mathcal{P}_x^g\\ p\le x^{\delta(x)}}}\frac{1}{p}\right)^2-\\
&-\sum_{\substack{p\in\mathcal{P}_x^f\\ p\le x^{\delta(x)}}}\frac{1}{p^2}-\sum_{\substack{p\in\mathcal{P}_x^g\\ p\le x^{\delta(x)}}}\frac{1}{p^2}-2\sum_{\substack{p\in\mathcal{P}_x^f\cap\mathcal{P}_x^g\\ p\le x^{\delta(x)}}}\frac{1}{p^2}+o(1).
\end{aligned}\tag{25}
$$

From (18) and the proved equality (6), it follows that

$$
\lim_{x\to\infty}\left(\sum_{\substack{p\in\mathcal{P}_x^f\\ p\le x^{\delta(x)}}}\frac{1}{p}+\sum_{\substack{p\in\mathcal{P}_x^g\\ p\le x^{\delta(x)}}}\frac{1}{p}\right)=\lambda.
$$

Because the quantity $\varphi_{2,x}$ tends to λ^2, then the last relation and the expression (25) imply the desired equality (5). The necessity of Theorem is proved.

4. Proof of Theorem. Sufficiency

Suppose that the conditions (5), (6) and the additional condition (3) are satisfied. In this section we derive the weak convergence of distributions (4) to the Poisson law. It was shown that the condition (3) implies the existence of the function $\delta(x)$ which satisfies (17) and (18). Let f_x^* and g_x^* be the sets of strongly additive functions defined in the last part of the previous section. According to the estimate (19), it is sufficient to prove that distribution functions (20) converge weakly to the Poisson law. Let $\varphi_{l,x}^*$ be the factorial moments defined in (22). Using (10), we have that

$$\varphi_{l,x}^* = \sum_{k=0}^{l} \binom{l}{k} \sum_{\substack{p_1 \le x^{\delta(x)} \\ p_1 \in \mathcal{P}_x^f}} \cdots \sum_{\substack{p_k \le x^{\delta(x)} \\ p_k \in \mathcal{P}_x^f \\ p_k \ne p_1, \dots, p_{k-1}}} \sum_{\substack{p_{k+1} \le x^{\delta(x)} \\ p_{k+1} \in \mathcal{P}_x^g}} \cdots \sum_{\substack{p_l \le x^{\delta(x)} \\ p_l \in \mathcal{P}_x^g \\ p_l \ne p_{k+1}, \dots, p_{l-1}}} \times \frac{1}{x} \sum_{\substack{n \le x \\ p_1 \cdots p_k | an+b \\ p_{k+1} \cdots p_l | cn+d}} 1.$$

By (12) and (24)

$$\varphi_{l,x}^* = \sum_{k=0}^{l} \binom{l}{k} \sum_{\substack{p_1 \le x^{\delta(x)} \\ p_1 \in \mathcal{P}_x^f}} \cdots \sum_{\substack{p_k \le x^{\delta(x)} \\ p_k \in \mathcal{P}_x^f \\ p_k \ne p_1, \dots, p_{k-1}}} \sum_{\substack{p_{k+1} \le x^{\delta(x)} \\ p_{k+1} \in \mathcal{P}_x^g \\ p_{k+1} \ne p_1, \dots, p_k}} \cdots \sum_{\substack{p_l \le x^{\delta(x)} \\ p_l \in \mathcal{P}_x^g \\ p_l \ne p_1, \dots, p_{l-1}}} \times \left(\frac{1}{p_1 \dots p_l} + O\left(\frac{1}{x}\right)\right).$$

And after some combinatorial calculations we obtain

$$\varphi_{l,x}^* = \sum_{k=0}^{l} \binom{l}{k} \left(\sum_{\substack{p \le x^{\delta(x)} \\ p \in \mathcal{P}_x^f}} \frac{1}{p} \right)^k \left(\sum_{\substack{p \le x^{\delta(x)} \\ p \in \mathcal{P}_x^g}} \frac{1}{p} \right)^{l-k} + \\ + O_l \left(\max_{\substack{p \le x^{\delta(x)} \\ p \in \mathcal{P}_x^f \cup \mathcal{P}_x^g}} \frac{1}{p} \left(\max \left(1, \sum_{\substack{p \le x^{\delta(x)} \\ p \in \mathcal{P}_x^f \cup \mathcal{P}_x^g}} \frac{1}{p} \right) \right)^{l-1} \right) + \\ + O_l \left(\frac{1}{x} \sum_{p_1 \le x^{\delta(x)}} \sum_{p_2 \le x^{\delta(x)}} \cdots \sum_{p_l \le x^{\delta(x)}} 1 \right).$$

The last equality and the conditions (5), (6), (17), (18) yield that

$$\lim_{x\to\infty} \varphi^*_{l,x} = \lambda^l \tag{26}$$

for every fixed positive integer l.

Let $\psi^*_x(t)$ be the almost characteristic function of the distribution (20), i.e.

$$\psi^*_x(t) = \frac{1}{x}\sum_{n\le x} e^{it(f^*_x(an+b)+g^*_x(cn+d))}$$

for $t \in \mathbb{R}$.

If r and q are positive integers, then

$$\left| e^{itr} - 1 - \sum_{j=1}^{q-1} \binom{r}{j} (e^{it}-1)^j \right| \le \binom{r}{q} \left| e^{it} - 1 \right|^q .$$

Thus, for positive integer L

$$\left| \psi^*_x(t) - 1 - \sum_{l=1}^{L} \frac{(e^{it}-1)^l}{l!} \varphi^*_{l,x} \right| \le \frac{|e^{it}-1|^{L+1}\varphi^*_{L+1,x}}{(L+1)!} .$$

According to (26), we obtain

$$\left| \psi^*_x(t) - 1 - \sum_{l=1}^{L} \frac{(e^{it}-1)^l}{l!} \lambda^l \right| \le o_L(1) + O\left(\frac{(2\lambda)^{L+1}}{(L+1)!} \right)$$

for sufficiently large x. From the last inequality, it follows that

$$\lim_{x\to\infty} \frac{1}{[x]} \sum_{n\le x} e^{it(f^*_x(an+b)+g^*_x(cn+d))} = e^{\lambda(e^{it}-1)}$$

for each real number t. Hence the distribution functions (20) converge weakly to the Poisson law with parameter λ. Theorem is proved.

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J. Šiaulys
Department of Probability Theory
and Number Theory
Vilnius University
Naugarduko 24
03225 Vilnius, Lithuania
jonas.siaulys@mif.vu.lt

G. Stepanauskas
Department of Mathematical
Computer Science
Vilnius University
Naugarduko 24
03225 Vilnius, Lithuania
gediminas.stepanauskas@mif.vu.lt