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NOTE ON t–QUASIAFFINE FUNCTIONS

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Dedicated to the 70th birthday of Professor Imre Kátai

Abstract. Given a convex subset D of a vector space and a constant $0 < t < 1$, a function $f : D \to \mathbb{R}$ is called *t-quasiaffine* if, for all $x, y \in D$,

$$\min\{f(x), f(y)\} \leq f\big(tx + (1-t)y\big) \leq \max\{f(x), f(y)\}.$$

If, furthermore, both of these inequalities are strict for $f(x) \neq f(y)$, f is called *strictly t-quasiaffine*. The main results of the paper show that t-quasiaffinity implies $\mathbb{Q}$-quasiaffinity (i.e. t-quasiaffinity for every rational number t in $[0, 1]$). An analogous result is established for strict t-quasiaffinity.

1. Introduction

Let $\mathbb{F}$ be fixed subfield of the set of real numbers $\mathbb{R}$ and let X be a vector space over $\mathbb{F}$ throughout this paper. The two most important particular settings of our investigations are when either $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$.

In what follows, we briefly recall the terminology related to t-convexity of sets and to t-quasiconvexity and t-quasiaffinity of functions.

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Given a constant $t \in [0,1]$, a subset $D \subseteq X$ is called *t-convex* if $tx + (1-t)y \in D$ holds whenever $x, y \in D$. For a given collection $T \subseteq [0,1]$ of numbers, the set D is called *T-convex* if it is t-convex for all $t \in T$ (cf. [2], [3], [8]). Observe, that T-convexity is simply equivalent to convexity in the standard sense if $T = [0,1]$. We say that D is $\mathbb{F}$-convex if it is $\mathbb{F} \cap [0,1]$-convex. In particular, D is $\mathbb{Q}$-convex (rationally convex) if it is $\mathbb{Q} \cap [0,1]$-convex.

The set of all numbers $t \in [0,1]$ such that D is t-convex will be denoted by $T(D)$ in the sequel. Obviously, $0, 1 \in T(D)$ for any set $D \subseteq X$.

Given a t-convex set $D \subseteq X$, a real-valued function $f : D \to \mathbb{R}$ is called *t-quasiconvex* (cf. [2], [3], [8]) if

$$f\big(tx + (1-t)y\big) \le \max\{f(x), f(y)\}, \qquad x, y \in D. \tag{1}$$

If $(-f)$ is t-quasiconvex then f is said to be *t-quasiconcave.* When f is both t-quasiconvex and t-quasiconcave, i.e. when

$$\min\{f(x), f(y)\} \le f\big(tx + (1-t)y\big) \le \max\{f(x), f(y)\}, \qquad x, y \in D, \tag{2}$$

holds, then f is termed a *t-quasiaffine* function.

We say that f is *strictly t-quasiconvex* if it satisfies (1), furthermore,

$$f\big(tx + (1-t)y\big) < \max\{f(x), f(y)\} \qquad \text{if} \quad f(x) \ne f(y). \tag{3}$$

If f and $(-f)$ are strictly t-quasiconvex, i.e. if (2) and

$$\min\{f(x), f(y)\} < f\big(tx+(1-t)y\big) < \max\{f(x), f(y)\} \qquad \text{if} \;\; f(x) \ne f(y) \tag{4}$$

are satisfied then f is called *strictly t-quasiaffine.*

If D is T-convex and (1), (2), (3) and (4) hold for all $t \in T$, then f is said to be *T-quasiconvex*, *T-quasiaffine*, *strictly T-quasiconvex* and *strictly T-quasiaffine*, respectively.

The collection of all numbers $t \in [0,1]$ such that f is t-quasiconvex is denoted by $T(f)$.

The *$\frac{1}{2}$-quasiaffine* functions defined on an interval $I \subset \mathbb{R}$, that is functions satisfying

$$\min\{f(x), f(y)\} \le f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\}, \qquad x, y \in I, \tag{5}$$

were introduced in 1949 by Á. Császár [4], [5]. (They were also called *midpoint-qusiaffine* or *internal.*) Observe that monotone functions $f : I \to \mathbb{R}$ as well as Jensen functions, i.e. solutions of the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \qquad x, y \in I,$$

are always midpoint-quasiaffine. It is known that Jensen functions may be very irregular (see [1], [10]). However, measurable Jensen functions are of the form $f(x) = ax + b$, $x \in I$, hence they are also monotone. Midpoint-quasiaffine functions enjoy a similar property. Namely, Császár [5] proved that if a function $f : I \to \mathbb{R}$ satisfies (5) then it is either monotone or nonmeasurable. Other results of this type were also obtained by Deák [7] and Marcus [13].

Midpoint-quasiaffine functions in a more general setting were investigated by the authors in [14]. It was proved, among others, that under some regularity assumptions every strictly midpoint-quasiaffine function $f : X \to \mathbb{R}$ is of the form $f = g \circ \alpha$ where $\alpha : X \to \mathbb{R}$ is an additive function and $g : \mathbb{R} \to \mathbb{R}$ is monotone. This characterization gives some insight into the result of Császár.

Recently Lewicki [12] extended this characterization to strictly t-quasiaffine functions. The basic role in his paper is played by a theorem stating that if $f : X \to \mathbb{R}$ is strictly t-quasiaffine and $\mathbb{Q}$-radially upper semicontinuous, i.e.

$$\limsup_{r \in \mathbb{Q},\ r \to 0^+} f\big(rx + (1-r)y\big) \le f(y), \qquad x, y \in X,$$

then it is strictly midpoint-quasiaffine. This result is analogous to the result of Kuhn [11] which states that every t-convex function is midpoint-convex. In the proof of Lewicki's theorem the assumptions that the domain of f is the whole space X and f is $\mathbb{Q}$-radially upper semicontinuous are essential. However, using a more sophisticated method we can prove much stronger version of this statement. Namely, we show that every t-quasiaffine function $f : D \to \mathbb{R}$, where D is an $\mathbb{F}$-convex set and $t \in \mathbb{F} \cap]0,1[$, is $\mathbb{Q}$-quasiaffine. An analogous result is also established for strictly t-quasiaffine functions.

2. t-convexity and complementarity of sets

Lemma 1. *Let $D \subseteq X$ be a nonempty set. Then, for the set $T(D)$, we have the following properties:*

(i) if $t \in T(D)$, then $1 - t \in T(D)$;

(ii) if $r, s, t \in T(D)$, then $tr + (1-t)s \in T(D)$.

Proof. Implication (i) is obvious. To prove (ii), assume that $r,s,t\in T(D)$ and fix $x,y\in D$. Then $rx+(1-r)y, sx+(1-s)y\in D$ and, consequently,

$$\big(tr+(1-t)s\big)x+\big(1-tr-(1-t)s\big)y=t\big(rx+(1-r)y\big)+(1-t)\big(sx+(1-s)y\big)\in D,$$

which shows that $tr+(1-t)s\in T(D)$.

As a consequence of the next lemma, we obtain that $T(D)$ is also dense in $[0,1]$ provided that $T(D)\cap]0,1[\neq\emptyset$.

Lemma 2. *Let $S\subseteq[0,1]$ be a nonempty set with the following properties:*
(i) $0,1\in S$ *and* $S\cap]0,1[\neq\emptyset$*;*
(ii) if $r,s,t\in S$*, then* $tr+(1-t)s\in S$*.*
Then S is dense in $[0,1]$.

Proof. Let $t\in S\cap]0,1[$ be fixed and suppose, contrary to our claim, that S is not dense in $[0,1]$. Then there exists an open interval $]a,b[\subseteq[0,1]\setminus S$. Let

$$r=\sup\big(S\cap[0,a]\big)\qquad\text{and}\qquad s=\inf\big(S\cap[b,1]\big).$$

One can see that $S\cap]r,s[=\emptyset$. Take sequences (r_n), (s_n) such that $r_n,s_n\in S$ and $r_n\nearrow r$, $s_n\searrow s$. Since

$$tr_n+(1-t)s_n\longrightarrow tr+(1-t)s\in]r,s[,$$

for a sufficiently large n_0, we have $tr_{n_0}+(1-t)s_{n_0}\in]r,s[$. On the other hand, by property (ii), $tr_{n_0}+(1-t)s_{n_0}\in S$, which contradicts the fact that $S\cap]r,s[=\emptyset$.

In general, given a rational number $r\in]0,1[$, the r-convexity of a set D does not imply its midpoint-convexity, and conversely, the midpoint-convexity of a set does not imply its r-convexity for an arbitrary rational number $r\in$ $\in[0,1]$. For instance, the set of diadic rational numbers $\left\{\frac{k}{2^n}:k\in\mathbb{Z},n\in\mathbb{N}\right\}$ is midpoint-convex but not $\frac{1}{3}$-convex. Similarly, the set of triadic rational numbers $\left\{\frac{k}{3^n}:k\in\mathbb{Z},n\in\mathbb{N}\right\}$ is $\frac{1}{3}$-convex but not midpoint-convex. It is therefore surprising that the midpoint-convexity of a set and its complementary (with respect to a $\mathbb{Q}$-convex set) is equivalent to $\mathbb{Q}$-convexity.

The key for such kind of implications is contained in our next result.

Theorem 3. *Let D be an $\mathbb{F}$-convex set, let $s,t\in\mathbb{F}\cap[0,1]$ with $t(s+1)\leq1$ and assume that A and B are disjoint $\{s,t\}$-convex sets such that $D=A\cup B$. Then A and B are also $\frac{1-t}{1-ts}$-convex.*

Proof. To verify the $\frac{1-t}{1-ts}$-convexity of A, let $x, y \in A$ be arbitrary points and define

$$u := \frac{1-t}{1-ts}x + \frac{t-ts}{1-ts}y, \qquad v := \frac{1-t-ts}{1-ts}x + \frac{t}{1-ts}y.$$

The point u is trivially a convex combination of x and y with coefficients belonging to $\mathbb{F}$. By the assumption $t(s+1) \le 1$, it follows that v is also a convex combination of x and y with coefficients in $\mathbb{F}$. Therefore $u, v \in D = A \cup B$.

To complete the proof, we have to show that $u \in A$. We distinguish two cases.

In the case $v \in A$, the easy-to-check identity $u = sx + (1-s)v$ and the s-convexity of A yields that $u \in A$.

In the case $v \in B$, we use the identity

$$tu + (1-t)v = (1-t)x + ty \tag{6}$$

which is also easy to see. By the t-convexity of A, the right hand side of (6) is an element of A. Hence $tu + (1-t)v \in A$. If the point u were in B, then, by the t-convexity of B, $tu + (1-t)v$ would be an element of B contradicting the disjointness of A and B. Thus, u cannot be in B, i.e. u must belong to A in this case, too.

The proof of the $\frac{1-t}{1-ts}$-convexity of B is analogous.

Remark. If D is an $\mathbb{F}$-affine set (i.e. $tx + (1-t)y \in D$ for all $x, y \in D$ and $t \in \mathbb{F}$), then the condition $t(s+1) \le 1$ of Theorem 3 can be removed. Indeed, due to the $\mathbb{F}$-affinity, the point v constructed in the proof is contained in D (even if it is not in the convex hull of x and y). It seems to be an important question if this remains valid also in the case when D is only an $\mathbb{F}$-convex sets.

Theorem 4. *Let D be an $\mathbb{F}$-convex set, $t \in \mathbb{F} \cap]0,1[$ and assume that A and B are disjoint t-convex sets such that $D = A \cup B$. Then A and B are also $\mathbb{Q}$-convex.*

Proof. Let $\tau \in T(A) \cap T(B)$ be fixed. With the notation $t := s := 1 - \tau$, we can see that the inequality $t(s+1) \le 1$ holds if $\tau \ge \frac{2}{5}$. Therefore, applying Theorem 3, it follows that

$$\frac{1}{2-\tau} = \frac{1}{1+t} = \frac{1-t}{1-ts} \in T(A) \cap T(B), \qquad \tau \in T(A) \cap T(B) \cap \left[\frac{2}{5}, 1\right].$$

By Lemma 1, the set $S := T(A) \cap T(B)$ satisfies property (ii) of Lemma 2. Hence S is dense in $[0,1]$. Thus, we can choose the element $\tau \in T(A) \cap T(B)$ such that $\frac{2}{5} \le \tau \le \frac{1}{2}$ also holds.

Now define $t := \frac{1}{2-\tau}$ and $s := \tau$. The inequality $t(s+1) \le 1$ is equivalent to $\tau \le \frac{1}{2}$ which is valid by the choice of τ. Thus, using Theorem 3 again, we get that

$$\frac{1}{2} = \frac{(2-\tau)-1}{(2-\tau)-\tau} = \frac{1-\frac{1}{2-\tau}}{1-\frac{\tau}{2-\tau}} = \frac{1-t}{1-ts} \in T(A) \cap T(B),$$

i.e. A and B are $\frac{1}{2}$-convex sets.

Finally, we prove the $\mathbb{Q}$-convexity of A and B. For a fixed number $n \ge 1$, denote by (S_n) the following statement: *For all $k \in \{1, \dots, n\}$, the inclusion $\frac{k}{n+1} \in T(A) \cap T(B)$ holds.* Clearly, (S_1) is equivalent to $\frac{1}{2} \in T(A) \cap T(B)$ (what we have already proved).

Assume that (S_n) has been verified. Then we have $\frac{n}{n+1} \in T(A) \cap T(B) \cap$ $\cap \left[\frac{2}{5}, 1\right]$. Therefore, by the first assertion of this proof,

$$\frac{n+1}{n+2} = \frac{1}{2-\frac{n}{n+1}} \in T(A) \cap T(B).$$

Using the second statement of Lemma 1, it follows that

$$\frac{k}{n+2} = \left(1 - \frac{k}{n+1}\right) \cdot 0 + \frac{k}{n+1} \cdot \frac{n+1}{n+2} \in T(A) \cap T(B)$$

for all $k \in \{1, \dots, n+1\}$, which proves the validity of (S_{n+1}) and completes the proof.

As an obvious consequence of this theorem, we get the following result which was established in the particular case $D = X$ in [15].

Corollary 5. *Let D be a $\mathbb{Q}$-convex set and assume that A and B are disjoint sets such that $D = A \cup B$. Then A and B are midpoint convex if and only if they are also $\mathbb{Q}$-convex.*

3. t-quasiaffine functions

For a given function $f : D \subset X \to \mathbb{R}$, we define the upper and lower level sets of f by

$$A(f,c) = \{x \in D \mid f(x) < c\}, \qquad \overline{A}(f,c) = \{x \in D \mid f(x) \le c\}$$

and

$$B(f,c)=\{x\in D \mid f(x)>c\}, \qquad \overline{B}(f,c)=\{x\in D \mid f(x)\geq c\}.$$

The t-convexity property of these sets is related to t-quasiconvexity and t-quasiconcavity of the function f by the following lemma.

Lemma 6. *Let $t\in]0,1[$, D be a t-convex set and $f:D\to\mathbb{R}$. Then the following three properties are equivalent:*

(i) f is a t-quasiconvex function;

(ii) for all $c\in\mathbb{R}$, the level set $A(f,c)$ is t-convex;

(iii) for all $c\in\mathbb{R}$, the level set $\overline{A}(f,c)$ is t-convex.

The proof of this lemma is elementary, therefore, it is omitted. The following lemma describes the basic properties of the set $T(f)$ of a given function f. Its statement is analogous to Lemma 1.

Lemma 7. *Let $D\subseteq X$ be a nonempty $\mathbb{F}$-convex set and $f:D\to\mathbb{R}$. Then, for the set $T(f)$, we have the following properties:*

(i) if $t\in T(f)$, then $1-t\in T(f)$;

(ii) if $r,s,t\in T(f)$, then $tr+(1-t)s\in T(f)$.

Proof. By Lemma 6, we have that

$$T(f)=\bigcap_{c\in\mathbb{R}}T\big(A(f,c)\big).$$

Thus, the statement directly follows from Lemma 1.

As a consequence of Lemma 2 and Lemma 7, we can see that $T(f)$ is dense in $[0,1]$ provided that $T(f)\cap]0,1[\neq\emptyset$. However, the density of $T(f)$ in $[0,1]$ does not imply that $\frac{1}{2}\in T(f)$. In other words, t-quasiconvex functions, in general, need not be midpoint-quasiconvex (in contrast to the theorem of Kuhn [11] stating that t-convex functions are always midpoint-convex). For instance, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0 & \text{if } x=\dfrac{k}{3^n},\ k\in\mathbb{Z},n\in\mathbb{N},\\[2ex] 1 & \text{otherwise.}\end{cases}$$

Then $\frac{1}{3}\in T(f)$, but $\frac{1}{2}\notin T(f)$. Note also that for some $t\in]0,1[$ the t-quasiconvexity implies midpoint-quasiconvexity. For instance, if f is $(2^{-\frac{1}{n}})$-quasiconvex for some $n\in\mathbb{N}$, then it is midpoint-quasiconvex. Indeed, if $t\in$

$\in T(f)$, then, by Lemma 7, $t^2 \in T(f)$, and, by induction, $t^n \in T(f)$ for every $n \in \mathbb{N}$. Therefore, if $2^{-\frac{1}{n}} \in T(f)$, then $\frac{1}{2} = \left(2^{-\frac{1}{n}}\right)^n \in T(f)$.

The following result describes the algebraic structure of the set $T(f) \cap \cap T(-f)$. It is an easy consequence of Theorem 3 and Lemma 6.

Theorem 8. *Let $D \subseteq X$ be a nonempty $\mathbb{F}$-convex set and $f : D \to \mathbb{R}$. Let $s, t \in \mathbb{F} \cap [0,1]$ with $t(s+1) \leq 1$ such that f is $\{s,t\}$-quasiaffine. Then f is also $\frac{1-t}{1-ts}$-quasiaffine.*

Proof. By Lemma 6 and the t-quasiaffinity of f, we have, for all $c \in \mathbb{R}$ that $A(f,c)$ and $\overline{B}(f,c) = \overline{A}(-f,-c)$ are $\{s,t\}$-convex subsets of D. Clearly, $A(f,c)$ and $\overline{B}(f,c)$ are disjoint and $D = A(f,c) \cup \overline{B}(f,c)$. Thus, applying Theorem 3, it follows that the sets $A(f,c)$ and $\overline{B}(f,c)$ are $\frac{1-t}{1-ts}$-convex for all $c \in \mathbb{R}$. Now, by Lemma 6 again, we obtain that f and $-f$ are $\frac{1-t}{1-ts}$-quasiconvex, i.e. f is $\frac{1-t}{1-ts}$-quasiaffine.

Applying Lemma 6 and Theorem 4, we can state and prove the main result of this paper, which generalizes Theorem 1 in [14].

Theorem 9. *Let D be an $\mathbb{F}$-convex set and $t \in \mathbb{F} \cap]0,1[$. Assume that $f : D \to \mathbb{R}$ is a t-quasiaffine function. Then f is also $\mathbb{Q}$-quasiaffine.*

Proof. Arguing in the same way as in the proof of Theorem 8, the t-quasiaffinity of f and Theorem 4 yield that the sets $A(f,c)$ and $\overline{B}(f,c)$ are $\mathbb{Q}$-convex for all $c \in \mathbb{R}$. Now, by Lemma 6 again, it follows that f is $\mathbb{Q}$-quasiaffine.

In view of Theorem 9, we immediately obtain the following

Corollary 10. *Let D be a $\mathbb{Q}$-convex set and $f : D \to \mathbb{R}$. Then $f : D \to \mathbb{R}$ is midpoint-quasiaffine if and only if it is $\mathbb{Q}$-quasiaffine.*

4. Strictly t-quasiaffine functions

The result of this section shows that strict t-quasiaffinity implies strict $\mathbb{Q}$-quasiaffinity.

Theorem 11. *Let D be an $\mathbb{F}$-convex set and $t \in \mathbb{F} \cap]0,1[$. Assume that $f : D \to \mathbb{R}$ is a strictly t-quasiaffine function. Then f is also strictly $\mathbb{Q}$-quasiaffine.*

Proof. In view of Theorem 9, the function f is $\mathbb{Q}$-quasiaffine.

We show first that f is strictly midpoint-quasiaffine. Let $x, y \in D$ such that $f(x) \neq f(y)$. Without loss of generality, we may assume that $f(x) < f(y)$. By the $\mathbb{Q}$-quasiaffinity, we have that (5) holds. We have to prove that both inequalities in (5) are strict. We will only show that $f\left(\frac{x+y}{2}\right) < f(y)$ $\big($the proof of $f(x) < f\left(\frac{x+y}{2}\right)$ is analogous$\big)$. Suppose, on the contrary, that $f\left(\frac{x+y}{2}\right) = f(y)$.

Define

$$u := tx + (1-t)\frac{x+y}{2} \qquad \text{and} \qquad v := t\frac{x+y}{2} + (1-t)y. \tag{7}$$

Then, we have the following identity (cf. [6]):

$$\frac{x+y}{2} = (1-t)u + tv. \tag{8}$$

By (7) and the t-quasiaffinity of f,

$$f(v) = f\left(t\frac{x+y}{2} + (1-t)y\right) = \max\left\{f\left(\frac{x+y}{2}\right), f(y)\right\} = f\left(\frac{x+y}{2}\right).$$

Since $f(x) < f(y) = f\left(\frac{x+y}{2}\right)$, by (7) and the strict t-quasiaffinity of f, we get

$$f(u) = f\left(tx + (1-t)\frac{x+y}{2}\right) < \max\left\{f(x), f\left(\frac{x+y}{2}\right)\right\} = f\left(\frac{x+y}{2}\right).$$

Consequently, using the strict quasiaffinity once more, we obtain

$$f\left(\frac{x+y}{2}\right) = f\left(tv + (1-t)u\right) < \max\{f(u), f(v)\} = f(v) = f\left(\frac{x+y}{2}\right),$$

which is an obvious contradiction showing that f is strictly midpoint-quasiaffine.

Now we will prove (similarly as in the case of Theorem 1 in [14]) that f is strictly $\mathbb{Q}$-quasiaffine. By induction, we can get that

$$\min\{f(x), f(y)\} < f\big(dx + (1-d)y\big) < \max\{f(x), f(y)\} \tag{9}$$

if $f(x) \neq f(y)$ and $d \in]0,1[$ is a dyadic rational number, that is $d = k/2^n$, where $k, n \in \mathbb{N}$, $0 < k < 2^n$. Let $r \in]0,1[\cap \mathbb{Q}$ be arbitrary and $f(x) \neq f(y)$. There exist dyadic rational numbers d', d'' such that $0 < d' < r < d'' < 1$. Then $rx + (1-r)y$ is a $\mathbb{Q}$-convex combination of $d'x + (1-d')y$ and $d''x + (1-d'')y$. Since, by Theorem 9, f is $\mathbb{Q}$-quasiaffine, we have

$$\min\left\{f(d'x + (1-d')y), f(d''x + (1-d'')y)\right\} \leq f\big(rx + (1-r)y\big) \leq$$

$$\le \max\big\{f(d'x+(1-d')y), f(d''x+(1-d'')y)\big\}.$$

On the other hand, we have (9) with $d = d'$ and $d = d''$. These inequalities together with the previous one yield

$$\min\{f(x), f(y)\} < f\big(rx+(1-r)y\big) < \max\{f(x), f(y)\}.$$

Hence f is strictly $\mathbb{Q}$-quasiaffine, which completes the proof.

In view of Theorem 11, we immediately obtain the following

Corollary 12. *Let D be a $\mathbb{Q}$-convex set and $f : D \to \mathbb{R}$. Then $f : D \to \mathbb{R}$ is strictly midpoint-quasiaffine if and only if it is strictly $\mathbb{Q}$-quasiaffine.*

5. M-quasiaffinity

We can generalize the notion of t-quasiaffine functions by replacing the weighted arithmetic mean used in its definition by a more general mean. Given two points $x, y \in X$, we define

$$]x,y[:= \{tx+(1-t)y :\ t \in]0,1[\}, \qquad [x,y] := \{tx+(1-t)y :\ t \in [0,1]\}.$$

Given a convex set $D \subseteq X$, a function $M : D \times D \to D$ is called a *strict mean* on D if

$$M(x,y) \in\ ]x,y[, \qquad x, y \in D,\ x \neq y$$

and

$$M(x,x) = x, \qquad x \in D.$$

Let D be a convex subset of X. A function $f : D \to \mathbb{R}$ is said to be M-quasiaffine if

$$\min\{f(x), f(y)\} \le f\big(M(x,y)\big) \le \max\{f(x), f(y)\}, \qquad x, y \in D.$$

Of course if $t \in]0,1[$, then $M(x,y) = tx+(1-t)y$ is a strict mean and M-quasiaffinity of f coincides with its t-quasiaffinity. In this section, we consider the question if M-quasiaffinity implies midpoint-quasiaffinity. The following example shows that without any additional assumptions such implication does not hold.

Example 2. Let $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a strict mean defined by

$$M(x,y)=\begin{cases}\dfrac{3}{4}x+\dfrac{1}{4}y, & \text{if } x,y\notin\mathbb{Q} \quad\text{and}\quad \dfrac{x+y}{2}\in\mathbb{Q},\\[2ex] \dfrac{1}{2}x+\dfrac{1}{2}y, & \text{otherwise.}\end{cases}$$

The Dirichlet function

$$D(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q},\\[2ex] 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$

is M-quasiaffine (note that $M(x,y)\in\mathbb{Q}$ if $x,y\in\mathbb{Q}$, and $M(x,y)\notin\mathbb{Q}$ if $x,y\notin\mathbb{Q}$). However, by the theorem of Császár, it is not midpoint-quasiaffine (because it is measurable and non-monotone).

This example shows that M-quasiaffine functions need not be midpoint-quasiaffine even if they are measurable. However, if an M-quasiaffine function is continuous, then it is quasiaffine, that is it satisfies (2) for every $t\in[0,1]$.

Theorem 13. *Let X be a Hausdorff topological-vector space, D be a convex subset of X and M be a strict mean on X. If a function $f:D\to\mathbb{R}$ is M-quasiaffine and continuous, then it is quasiaffine.*

Proof. Let $x,y\in D$, $x\neq y$. Define

$$C=\bigl\{z\in D:\min\{f(x),f(y)\}\le f(z)\le\max\{f(x),f(y)\}\bigr\}.$$

Clearly, $x,y\in C$ and C is closed (because f is continuous). To prove that f is quasiaffine, it is enough to show that $[x,y]\subset C$. Suppose, contrary to this claim, that there exists an $z_0\in\,]x,y[\,\setminus C$. Then there exist points $x',y'\in[x,y]\cap$ $\cap C$ such that $]x',y'[\,\cap C=\emptyset$. Since $x',y'\in C$, we have $f(x')\le\max\{f(x),f(y)\}$ and $f(y')\le\max\{f(x),f(y)\}$. Hence, by M-quasiaffinity,

$$f\bigl(M(x',y')\bigr)\le\max\{f(x'),f(y')\}\le\max\{f(x),f(y)\}.$$

Analogously

$$f\bigl(M(x',y')\bigr)\ge\min\{f(x),f(y)\}.$$

Consequently $M(x',y')\in C$. On the other hand we have $M(x',y')\in\,]x',y'[$, because M is a strict mean. This contradicts the fact that $]x',y'[\,\cap C=\emptyset$ and completes the proof.

In the case when $X=\mathbb{R}$ and M is a strict mean which is continuous in both variables, the above result is a consequence of the characterization of quasiconvexity presented in [9].

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