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ON TWO GAMES IN THE REAL LINE

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Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. Consider the next game. The ammunitions of player $\mathcal{A}$ are: $\{\lambda_1,\ \lambda_2,\ \ldots\}$ $(\lambda_1 > \lambda_2 > \ldots \geq 0)$. The ammunitions of player $\mathcal{B}$ are: $\{\omega_1,\ \omega_2,\ \ldots\}$ $(\omega_1 > \omega_2 > \ldots \geq 0)$. The game starts from 0. In the nth step $\mathcal{A}$ chooses $\varepsilon_n \in \{0, \pm 1\}$ and calculates y_n:

$$x_{n-1} = \varepsilon_n \lambda_n + y_n.$$

Then $\mathcal{B}$ chooses $\delta_n \in \{0, \pm 1\}$ and calculates x_n:

$$y_n = \delta_n \omega_n + x_n.$$

The goal of player $\mathcal{A}$ is to reach x_0 as $\sum_{n=1}^{\infty} (\varepsilon_n \lambda_n + \delta_n \omega_n)$, but player $\mathcal{B}$ tries to prevent $\mathcal{A}$ from it. In the case $\lambda_n = \lambda^n$ and $\omega_n = \omega^n$ I managed to determine the winning set of $\mathcal{A}$, while in the case $\lambda_n = \lambda^n$ and $\omega_n = a\lambda^n$ for some values of a I managed to set up only a conjecture, but for the rest values the winning set of $\mathcal{A}$ is also known.

1. Introduction

Consider the next game. The ammunitions of player $\mathcal{A}$ are: $\{\lambda_1,\ \lambda_2,\ \ldots\}$, $(\lambda_1 > \lambda_2 > \ldots \geq 0)$. The ammunitions of player $\mathcal{B}$ are: $\{\omega_1,\ \omega_2,\ \ldots\}$, $(\omega_1 > > \omega_2 > \ldots \geq 0)$. The game starts from 0. In the 1st step $\mathcal{A}$ chooses $\varepsilon_1 \in \in \{0, \pm 1\}$, and calculates y_1:

$$x_0 = \varepsilon_1 \lambda_1 + y_1.$$

Then $\mathcal{B}$ chooses $\delta_1 \in \{0, \pm 1\}$, and calculates x_1:

$$y_1 = \delta_1 \omega_1 + x_1.$$

After that $\mathcal{A}$ follows again. In the nth step $\mathcal{A}$ chooses $\varepsilon_n \in \{0, \pm 1\}$ and calculates y_n:

$$x_{n-1} = \varepsilon_n \lambda_n + y_n.$$

Then $\mathcal{B}$ chooses $\delta_n \in \{0, \pm 1\}$ and calculates x_n:

$$y_n = \delta_n \omega_n + x_n.$$

It is clear that

$$x_0 = \varepsilon_1 \lambda_1 + \delta_1 \omega_1 + \ldots + \varepsilon_n \lambda_n + \delta_n \omega_n + x_n.$$

The goal of player $\mathcal{A}$ is to choose ε_n so that $x_0 = \sum_{n=1}^{\infty} (\varepsilon_n \lambda_n + \delta_n \omega_n)$, i.e. that $x_n \to 0$ $(n \to \infty)$. The purpose of $\mathcal{B}$ is to frustrate it. We would like to determine the set E of those $x_0 (\in \mathbb{R})$ for which $\mathcal{A}$ wins. We say that $\mathcal{B}$ wins if $\mathcal{A}$ does not win. Thus $\mathcal{B}$ wins if $x \notin E$.

We introduce some notations:

$L_n := \sum_{k=n+1}^{\infty} \lambda_k,$

$W_n := \sum_{k=n+1}^{\infty} \omega_k,$

$M_n := L_n - W_n - \omega_n,$

$N_n := -(L_n - W_n) + \lambda_n + \omega_n,$

$H_n := [-(L_n - W_n), -N_{n+1}] \cup [-M_{n+1}, M_{n+1}] \cup [N_{n+1}, (L_n - W_n)].$

1.1. Statement. *$x_0 \in E$ if and only if $x_n \in [-(L_n - W_n), L_n - W_n]$ for every n.*

Proof. *Sufficiency.* $L_n - W_n$ tends to zero (because L_n and W_n tend to zero), so if the condition holds, then x_n tends to zero, consequently $\mathcal{A}$ wins.

Necessity. Suppose that $x_0 > L_0 - W_0$. Then for the choice $\delta_j = -1$ $(j = 1, 2, \ldots)$ of $\mathcal{B}$, $\mathcal{A}$ cannot reach x_0, the maximal number which can be represented in the form $\varepsilon_1 \lambda_1 + \delta_1 \omega_1 + \ldots$ is $L_0 - W_0$. Similarly, if for some integer n $x_n > L_n - W_n$, x_n cannot be written as $\varepsilon_{n+1} \lambda_{n+1} + \delta_{n+1} \omega_{n+1} + \ldots$, if $\mathcal{B}$ chooses $\delta_{n+1} = \delta_{n+2} = \ldots = -1$. Thus, if $x_0 \in E$, then $x_n \leq L_n - W_n$ $(n \in \mathbb{N})$. It is clear that $E = -E$. Thus $-x_n \leq L_n - W_n$ $(n \in \mathbb{N})$, and so

$$x_n \in [-(L_n - W_n), L_n - W_n] \ (n \in \mathbb{N}).$$

For arbitrary $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\omega_i\}_{i=1}^{\infty}$ it would be too hard to say anything, so we examine the next two cases:

1. $\lambda_n = \lambda^n$ and $\omega_n = \omega^n$ $\left(a = \frac{\omega}{\lambda} < 1,\ 1/3 < \lambda < 1 \right)$,
2. $\lambda_n = \lambda^n$ and $\omega_n = a\lambda^n$, where $0 < a < 1$ and $1/3 < \lambda < 1$.

2. The first case

2.1. Statement. *If* $a > \frac{\lambda}{\lambda^2-\lambda+1}$, *then* E *is the empty set.*

Proof. Let us solve the inequality $\omega_1 > L_1 - W_1$:

$$\begin{aligned} \omega_1 &> L_1 - W_1, \\ a\lambda &> \frac{\lambda^2}{1-\lambda} - \frac{a^2\lambda^2}{1-a\lambda}, \\ a\lambda(1-\lambda)(1-a\lambda) &> \lambda^2(1-a\lambda) - a^2\lambda^2(1-\lambda), \\ a - a^2\lambda - a\lambda + a^2\lambda^2 &> \lambda - a\lambda^2 - a^2\lambda + a^2\lambda^2, \\ a(\lambda^2 - \lambda + 1) &> \lambda, \\ a &> \frac{\lambda}{\lambda^2 - \lambda + 1}. \end{aligned}$$

Whilst $|y_1| \geq 0$ for all x_0, thus with suitable choice of δ_1 $\mathcal{B}$ can achieve that $|x_1| > \omega_1$, which means that $\mathcal{A}$ cannot win.

2.2. Statement. *If* $a \leq \frac{3\lambda-1}{3\lambda^2-3\lambda+2}$, *then* $E = [-(L_0 - W_0), L_0 - W_0]$.

Proof. If $|x_0| > L_0 - W_0$, then $\mathcal{A}$ cannot win because of Statement 1.1. Therefore, and from the symmetry $E = -E$ we may assume that $0 \leq x_{n-1} \leq$ $\leq L_{n-1} - W_{n-1}$. If $\lambda^n < 2x_{n-1}$, then $\mathcal{A}$ chooses $\varepsilon_n = 1$, and $\varepsilon_n = 0$ in the case $0 \leq x_{n-1} \leq \lambda^n/2$. Then

$$x_{n-1} = \varepsilon_n \lambda^n + \delta_n \omega^n + x_n.$$

We have to prove that $|x_n| \leq L_n - W_n$ for every choice $\delta_n \in \{-1,\ 0,\ 1\}$. We distinguish three cases.

Case I. $\lambda^n \leq x_{n-1}$.

Then $\varepsilon_n = 1$, $x_n = x_{n-1} - \lambda^n - \delta_n \omega^n$, therefore $x_n \le x_{n-1} - \lambda^n + \omega^n$, and $x_n \ge -\omega^n$, consequently

$$|x_n| \le x_{n-1} - \lambda^n + \omega^n \le L_{n-1} - W_{n-1} - \lambda^n + \omega^n = L_n + \lambda^n - W_n - \omega^n - \lambda^n + \omega^n =$$

$$L_n - W_n.$$

Case II. $x_{n-1} < \lambda^n < 2x_{n-1}$.

In this case

$$|x_n| \le \lambda^n - x_{n-1} + \omega^n < \lambda^n/2 + \omega^n.$$

Case III. $2x_{n-1} \le \lambda^n$.

In this case

$$|x_n| \le x_{n-1} + \omega^n < \lambda^n/2 + \omega^n.$$

Solve the inequality $\lambda^n/2 + \omega^n \le L_n - W_n$:

$$\lambda^n/2 + a^n\lambda^n \le \frac{\lambda^{n+1}}{1-\lambda} - \frac{a^{n+1}\lambda^{n+1}}{1-a\lambda},$$

$$1 - \lambda - a\lambda + a\lambda^2 + 2a^n - 2a^n\lambda - 2a^{n+1}\lambda + 2a^{n+1}\lambda^2 \le$$

$$\le 2\lambda - 2a\lambda^2 - 2a^{n+1}\lambda + 2a^{n+1}\lambda^2,$$

$$a^n(2 - 2\lambda) + a(3\lambda^2 - \lambda) \le 3\lambda - 1.$$

Since $a < 1$

$$a^n(2 - 2\lambda) + a(3\lambda^2 - \lambda) \le a(2 - 2\lambda) + a(3\lambda^2 - \lambda) = a(3\lambda^2 - 3\lambda + 2).$$

Thus if $a \le \frac{3\lambda - 1}{3\lambda^2 - 3\lambda + 2}$, then the inequality holds.

2.3. Statement. *If* $\frac{\lambda}{\lambda^2 - \lambda + 1} \ge a > \frac{3\lambda - 1}{3\lambda^2 - 3\lambda + 2}$ *and* $\lambda > 1/2$, *then* $E = H_0$.

Proof. The proof is divided into three parts:

A) If $x_0 \notin H_0$, then $\mathcal{A}$ cannot win.

B) If n is large enough, then $H_n = [-(L_n - W_n), L_n - W_n]$.

C) If $x_n \in H_n$, then $x_{n+1} \in H_{n+1}$.

A) Because of Statement 1.1 $\mathcal{A}$ cannot win if $|x_0| > L_0 - W_0$. Assume that $x_0 > L_1 - W_1 - \omega$. If $\mathcal{A}$ chooses $\varepsilon_1 = 0$ or $\varepsilon_1 = -1$, then for the choice $\delta_1 = -1$ $x_1 > Ł_1 - W_1$ follows, so $\mathcal{A}$ must choose $\varepsilon_1 = 1$. If $x_0 < \lambda + \omega - (L_1 - W_1)$ holds, too, then for the choice $\delta_1 = 1 \quad x_1 < -(L_1 - W_1)$ follows.

B) We have

$$M_{n+1} - N_{n+1} = \{L_{n+1} - W_{n+1} - \omega^{n+1}\} - \{\lambda^{n+1} + \omega^{n+1} - (L_{n+1} - W_{n+1})\} =$$

$$= 2(L_{n+1} - W_{n+1}) - \lambda^{n+1} - 2\omega^{n+1} =$$

$$= 2\left\{\frac{\lambda^{n+2}}{1-\lambda} - \frac{a^{n+2}\lambda^{n+2}}{1-a\lambda}\right\} - \lambda^{n+1} - 2a^{n+1}\lambda^{n+1},$$

and so

$$\frac{M_{n+1} - N_{n+1}}{\lambda^{n+1}} = 2\lambda\left\{\frac{1}{1-\lambda} - \frac{a^{n+2}}{1-a\lambda}\right\} - 1 - 2a^{n+1}.$$

The right hand side tends to $\frac{2\lambda}{1-\lambda} - 1$ as $n \to \infty$, therefore B) holds if $\lambda > 1/3$.

C) We have

2.4. Statement. $x_n \in H_n \Longrightarrow y_{n+1} \in [-M_{n+1}, M_{n+1}]$.

The choice of $\mathcal{A}$ for ε_{n+1} is given:

if $x_n \in [-M_{n+1}, M_{n+1}]$, then $\mathcal{A}$ must choose $\varepsilon_{n+1} = 0$,

if $|x_n| \in [N_{n+1}, (L_n - W_n)]$, then $\mathcal{A}$ must choose $\varepsilon_{n+1} = \operatorname{sgn}(x_n)$, otherwise $\mathcal{B}$ can choose δ_{n+1} to satisfy $|x_{n+1}| > L_{n+1} - W_{n+1}$.

$$\begin{aligned}
&[N_{n+1}, L_n - W_n] - \lambda^{n+1} = \\
=&[\lambda^{n+1} + \omega^{n+1} - (L_{n+1} - W_{n+1}), L_n - W_n] - \lambda^{n+1} = \\
=&[-(L_{n+1} - W_{n+1} - \omega^{n+1}), L_{n+1} - W_{n+1} - \omega^{n+1}] = [-M_{n+1}, M_{n+1}].
\end{aligned}$$

After that, if $\delta_{n+1} = 0$, then:

I.: $L_k - W_k - \omega^k < L_{k+1} - W_{k+1} - \omega^{k+1}$,

if $\delta_{n+1} = \pm 1$, then:

II.: $\omega^k - (L_k - W_k) + \omega^k \geq \lambda^{k+1} + \omega^{k+1} - (L_{k+1} - W_{k+1})$ should hold.

I.: $L_k - W_k - \omega^k = \lambda^{k+1} - \omega^{k+1} + L_{k+1} - W_{k+1} - \omega^k \leq L_{k+1} - W_{k+1} - \omega^{k+1}$,
$\lambda^{k+1} \leq a^k \cdot \lambda^k$,
$\lambda \leq a^k$.

II.: $\omega^k - (L_{k+1} - W_{k+1}) - \lambda^{k+1} + \omega^{k+1} + \omega^k \geq \lambda^{k+1} + \omega^{k+1} - (L_{k+1} - W_{k+1})$,
$2\omega^k \geq 2\lambda^{k+1}$,
$a^k \geq \lambda$.

We need: if $\frac{\lambda^n}{2} + \omega^n > L_n - W_n \Leftrightarrow a^n(2 - 2\lambda) + a(3\lambda^2 - \lambda) > 3\lambda - 1$ then $a^{n-1} \geq \lambda$.

Suppose indirectly that $a^{n-1} < \lambda \Leftrightarrow a^n < a\lambda$. Then

$$3\lambda - 1 < a^n(2 - 2\lambda) + a(3\lambda^2 - \lambda) < a\lambda(2 - 2\lambda) + a(3\lambda^2 - \lambda) \Leftrightarrow a > \frac{3\lambda - 1}{\lambda^2 + \lambda}.$$

We show the statement $\frac{3\lambda-1}{\lambda^2+\lambda} > \frac{\lambda}{\lambda^2-\lambda+1}$:

$3\lambda^3 - 4\lambda^2 + 4\lambda - 1 > \lambda^3 + \lambda^2,$

$2\lambda^3 - 5\lambda^2 + 4\lambda - 1 = 2(\lambda-1)^2(\lambda - \frac{1}{2}) > 0 \Leftrightarrow \lambda > \frac{1}{2}.$

2.5. Statement. *If* $\frac{\lambda}{\lambda^2-\lambda+1} \geq a > \frac{3\lambda-1}{3\lambda^2-3\lambda+2}$ *and* $\lambda \leq 1/2$, *then* E *is the emptyset or the union of finitely many intervals.*

We determine E. First we find the smallest n such that $H_n = [-(L_n - -W_n), L_n - W_n]$. If $x_{n-1} \in H_{n-1}$, then $\mathcal{A}$ wins, so $\mathcal{B}$ wins if $x_{n-1} \notin H_{n-1}$. It can happen if $y_{n-1} \notin [-M_{n-1}, M_{n-1}]$ or $y_{n-1} \in [-M_{n-1}, M_{n-1}]$ and $|x_{n-1}| \in \in (M_n, N_n)$. The latter case stands if $y_{n-1} \in [-M_{n-1}, M_{n-1}]$ and

$$y_{n-1} \in \pm(M_n, N_n) \cup \pm[(M_n, N_n) + \omega_{n-1}] \cup \pm[(M_n, N_n) - \omega_{n-1}],$$

i.e.

$$y_{n-1} \in$$

$$\in \Big(\pm(M_n, N_n) \cup \pm[(M_n, N_n) + \omega_{n-1}] \cup \pm[(M_n, N_n) - \omega_{n-1}]\Big) \cap [-M_{n-1}, M_{n-1}].$$

It is realized if and only if $x_{n-2} \in B_{n-2}$, where

$$B_{n-2} =$$

$$= \Big(\pm(M_n, N_n) \cup \pm[(M_n, N_n) + \omega_{n-1}] \cup \pm[(M_n, N_n) - \omega_{n-1}]\Big) \cap [-M_{n-1}, M_{n-1}] \cup$$

$$\cup \bigg(\Big(\pm(M_n, N_n) \cup \pm[(M_n, N_n) + \omega_{n-1}] \cup$$

$$\cup \pm[(M_n, N_n) - \omega_{n-1}]\Big) \cap [-M_{n-1}, M_{n-1}]\bigg) + \lambda_{n-1} \cup$$

$$\cup \bigg(\Big(\pm(M_n, N_n) \cup \pm[(M_n, N_n) + \omega_{n-1}] \cup \pm[(M_n, N_n) - \omega_{n-1}]\Big) \cap$$

$$\cap [-M_{n-1}, M_{n-1}]\bigg) - \lambda_{n-1}.$$

Hence $\mathcal{B}$ wins if $|x_{n-2}| > L_{n-2} - W_{n-2}$ or $x_{n-2} \in B_{n-2}$. This holds if $y_{n-2} \in \Big(B_{n-2} \cup (B_{n-2} + \omega_{n-2}) \cup (B_{n-2} - \omega_{n-2})\Big) \cap [-M_{n-2}, M_{n-2}]$ what is true if $x_{n-3} \in B_{n-3}$, where

$$B_{n-3} = \Big(B_{n-2} \cup (B_{n-2} + \omega_{n-2}) \cup (B_{n-2} - \omega_{n-2})\Big) \cap [-M_{n-2}, M_{n-2}] \cup$$

$$\cup\Big(\big(B_{n-2}\cup(B_{n-2}+\omega_{n-2})\cup(B_{n-2}-\omega_{n-2})\big)\cap[-M_{n-2},M_{n-2}]\Big)+\lambda_{n-2}\cup$$

$$\cup\Big(\big(B_{n-2}\cup(B_{n-2}+\omega_{n-2})\cup(B_{n-2}-\omega_{n-2})\big)\cap[-M_{n-2},M_{n-2}]\Big)-\lambda_{n-2}.$$

So $\mathcal{B}$ wins if $|x_{n-3}|>L_{n-3}-W_{n-3}$ or $x_{n-3}\in B_{n-3}$. Continuing this procedure we can determine the sets $B_{n-4},B_{n-5},\dots,B_0$ and we can say that $\mathcal{B}$ wins if $|x_0|>L_0-W_0$ or $x_0\in B_0$, hence $\mathcal{A}$ wins if $x_0\in[-(L_0-W_0),L_0-W_0]\setminus B_0$.

Examples.

If $\lambda=0.35$ and $\omega=0.155$, then E is the empty set.

If $\lambda=0.35$ and $\omega=0.156$, then

$$E=[-0.00004941524,0.00004941524]\cup\pm[0.3499505848,0.3500494152].$$

If $\lambda=0.35$ and $\omega=0.154$, then

$$E=\pm[0.00494655847,0.0064284416]\cup\pm[0.3435715584,0.3450534415]\cup$$
$$\cup\pm[0,3549465585,0.3564284416].$$

If $\lambda=0.35$ and $\omega=0.153$, then

$$E=\pm[0.00455100581,0.0048020712]\cup\pm[0.0049416788,0.0064829942]\cup$$
$$\cup\pm[0.3435170058,0.3450583212]\cup\pm[0.3451979288,0.3454489942]\cup$$
$$\cup\pm[0.3545510058,0.3548020712]\cup\pm[0.3549416788,0.3564829942].$$

3. The second case

3.1. Statement. *If $a>\lambda$, then E is the empty set.*

Proof. Let us solve the inequality $\omega_1>L_1-W_1$:

$$\begin{aligned}\omega_1&>L_1-W_1,\\ a\lambda&>\frac{\lambda^2}{1-\lambda}-\frac{a\lambda^2}{1-\lambda},\\ a\lambda(1-\lambda)&>\lambda^2-a\lambda^2,\\ a-a\lambda&>\lambda-a\lambda,\\ a&>\lambda.\end{aligned}$$

Whilst $|y_1| \geq 0$ for all x_0 thus with suitable choice of δ_1 $\mathcal{B}$ can obtain that $|x_1| > \omega_1$, which means that $\mathcal{A}$ cannot win.

3.2. Statement. *If* $a \leq \frac{3}{2}\lambda - \frac{1}{2}$, *then* $E = [-(L_0 - W_0), L_0 - W_0]$.

Proof. If $|x_0| > L_0 - W_0$, then $\mathcal{A}$ cannot win because of Statement 1.1. Let $S = [-(L_0 - W_0), L_0 - W_0] = [-(1-a)L_0, (1-a)L_0]$. It is enough to prove that for each $x_0 \in S$ there exists a choice of $\varepsilon_1 \in \{-1,\ 0,\ 1\}$ such that for every choice of $\delta_1 \in \{-1,\ 0,\ 1\}$ $x_1 \in \lambda S$, where $x_1 = x_0 - \varepsilon_1\lambda - \delta_1 a\lambda$. The choice $\varepsilon_1 = 0$ is suitable if $x_0 \in \lambda S \cap (\lambda S - a\lambda) \cap (\lambda S + a\lambda) = T$. The right hand side is an interval,

$$T = [-(1-a)\lambda L_0 + a\lambda, (1-a)\lambda L_0 - a\lambda],$$

since $(1-a)\lambda L_0 - a\lambda \geq -(1-a)\lambda L_0 + a\lambda$. Indeed, this inequality holds true, since $2(1-a)\lambda L_0 \geq 2a\lambda$, which holds if and only if $(1-a)\frac{\lambda}{1-\lambda} \geq a$, i.e. if $\lambda \geq a$. Similarly, $\varepsilon_1 = 1$ is suitable if $x_0 \in T + \lambda$, and $\varepsilon_1 = -1$ is suitable if $x_0 \in T - \lambda$. Finally we observe that

$$(T - \lambda) \cup T \cup (T + \lambda) = S,$$

if $(1-a)\lambda L_0 - a\lambda - \lambda \geq -(1-a)\lambda L_0 + a\lambda$ which holds if and only if $a \leq \frac{3}{2}\lambda - \frac{1}{2}$.

3.3. Statement. *If* $a = \lambda$, *then* $E = \{0, \pm\lambda\}$.

Proof. If $x_0 \notin \{0, \pm\lambda\}$, then with suitable choice of δ_1 $\mathcal{B}$ can obtain that $|x_1| > \omega_1$, which means that $\mathcal{A}$ cannot win. If $x_0 \in \{0, \pm\lambda\}$, then with the choices $\varepsilon_1 = \operatorname{sgn}(x_0)$ and $\varepsilon_i = -\delta_{i-1}$ $i \geq 2$ $\mathcal{A}$ wins trivially.

3.1. Remarks on the case $\lambda > a > \frac{3}{2}\lambda - \frac{1}{2}$

1. Conjecture. *If* $\lambda > a > \frac{3}{2}\lambda - \frac{1}{2}$, *then* E *is the empty set.*

3.1.1. Lemma. *If* $x_n \in \pm(M_{n+1}, N_{n+1})$, *then* $\mathcal{A}$ *cannot win.*

3.1.2. Remark. $M_{n+1} < N_{n+1}$ *if* $a > \frac{3\lambda}{2} - \frac{1}{2}$.

Proof of the remark.

$$\begin{aligned} M_{n+1} &< N_{n+1}, \\ L_{n+1} - W_{n+1} - \omega_{n+1} &< -(L_{n+1} - W_{n+1}) + \omega_{n+1} + \lambda_{n+1}, \\ 2(L_{n+1} - W_{n+1}) &< \lambda_{n+1} + 2\omega_{n+1}, \\ (L_{n+1} - W_{n+1}) &< \lambda_{n+1}/2 + \omega_{n+1}. \end{aligned}$$

We have seen previously that it holds if and only if $a > \frac{3\lambda}{2} - \frac{1}{2}$.

Proof of the lemma. We can assume that $|x_n| > 0$. If $\mathcal{A}$ chooses $\varepsilon_{n+1} = 0$ or $\varepsilon_{n+1} = -1$, then for the choice $\delta_{n+1} = -1$ we get

$$x_{n+1} > L_{n+1} - W_{n+1} - \omega_{n+1} + \omega_{n+1} = L_{n+1} - W_{n+1}.$$

If $\mathcal{A}$ chooses $\varepsilon_{n+1} = 1$, then $\mathcal{B}$ chooses $\delta_{n+1} = 1$, thus

$$x_{n+1} < -(L_{n+1} - W_{n+1}) + \omega_{n+1} + \lambda_{n+1} - \lambda_{n+1} - \omega_{n+1} = -(L_{n+1} - W_{n+1}).$$

So for any choice of $\mathcal{A}$ $\mathcal{B}$ can obtain $|x_{n+1}| > (L_{n+1} - W_{n+1})$, so $\mathcal{A}$ cannot win.

3.1.3. Lemma. *If $|x_{n-1}| \le M_n$, then $\mathcal{A}$ must choose $\varepsilon_n = 0$, in the opposite case $\mathcal{B}$ wins. If $|x_{n-1}| \ge N_n$, then $\mathcal{A}$ must choose $\varepsilon_n = sgn(x_{n-1})$, if not then $\mathcal{B}$ wins.*

Proof. If $|x_{n-1}| \le L_n - W_n - \omega_n$ and $\varepsilon_n = 1$, then $\mathcal{B}$ chooses $\delta_n = 1$, thus

$$x_n \le L_n - W_n - \omega_n - \lambda_n - \omega_n \le L_n - W_n - 2(L_n - W_n) = -(L_n - W_n).$$

If $\varepsilon_n = -1$, then $\mathcal{B}$ reaches his goal with $\delta_n = -1$. In the case $|x_{n-1}| \ge N_n$ we get the proof similarly to the proof of the previous lemma.

3.1.4. Lemma. *If $a > \frac{2\lambda^2}{1+\lambda}$, then*

$$M_n \le N_{n+1}.$$

Proof.

$$\begin{aligned}
M_n &\le N_{n+1},\\
L_n - W_n - \omega_n &\le -(L_{n+1} - W_{n+1}) + \omega_{n+1} + \lambda_{n+1},\\
L_{n+1} + \lambda_{n+1} - W_{n+1} - \omega_{n+1} - \omega_n &\le -L_{n+1} + W_{n+1} + \omega_{n+1} + \lambda_{n+1},\\
2L_{n+1} &\le 2W_{n+1} + 2\omega_{n+1} + \omega_n,\\
\frac{2\lambda^{n+2}}{1-\lambda} &\le a\left(\frac{2\lambda^{n+2}}{1-\lambda} + 2\lambda^{n+1} + \lambda^n\right),\\
2\lambda^2 &\le a(2\lambda^2 + 2\lambda - 2\lambda^2 + 1 - \lambda) = a(1+\lambda),\\
\frac{2\lambda^2}{1+\lambda} &\le a.
\end{aligned}$$

3.1.5. Statement. *If $a > \frac{2\lambda^2}{1+\lambda}$, then E is the empty set.*

3.1.6. Remark.

$$\frac{2\lambda^2}{1+\lambda} > 3\lambda/2 - 1/2.$$

Proof of the remark. Let us solve the inequality

$$\begin{aligned} 4\lambda^2 &> 3\lambda^2 + 2\lambda - 1, \\ \lambda^2 - 2\lambda + 1 &> 0, \\ (\lambda - 1)^2 &> 0. \end{aligned}$$

Proof. Let the strategy of $\mathcal{B}$ be the following: choose δ_n so that $|x_n| > > L_n - W_n$ holds if he can, otherwise choose $\delta_n = 0$. Whilst $\mathcal{A}$ tries to win it is given what to choose because of Lemma 3.2. After the first step we get $|y_1| \leq M_1$. Whilst $M_n \leq N_{n+1}$ $\mathcal{A}$ must choose $\varepsilon_n = 0$ always. M_n tends to zero decreasingly so there will be such an n for which $M_{n-1} \geq |y_1| = \ldots = = |x_{n-1}| = |y_n| > L_n - W_n - \omega_n$ holds. For this n $|x_n| > L_n - W_n$ also holds.

Lemma 3.1.7. If $a > \frac{\lambda^2(1+\lambda)}{1+\lambda^2}$, then

$$N_{n+1} + \omega_n - \lambda_{n+1} > M_{n+2}.$$

Proof.

$$\begin{aligned} N_{n+1} + \omega_n - \lambda_{n+1} &> M_{n+2}, \\ -L_{n+2} - \lambda_{n+2} + W_{n+2} + \omega_{n+2} + \omega_{n+1} + \omega_n &> L_{n+2} - W_{n+2} - \omega_{n+2}, \\ 2W_{n+2} + 2\omega_{n+2} + \omega_{n+1} + \omega_n &> 2L_{n+2} + \lambda_{n+2}, \\ a\left(\frac{2\lambda^{n+3}}{1-\lambda} + 2\lambda^{n+2} + \lambda^{n+1} + \lambda^n\right) &> \frac{2\lambda^{n+3}}{1-\lambda} + \lambda^{n+2}, \\ a(2\lambda^3 + 2\lambda^2 - 2\lambda^3 + \lambda - \lambda^2 + 1 - \lambda) &= 2\lambda^3 + \lambda^2 - \lambda^3, \\ a(\lambda^2 + 1) &> \lambda^3 + \lambda^2, \\ a &> \frac{\lambda^2(1+\lambda)}{1+\lambda^2}. \end{aligned}$$

3.1.8. *If $a > \frac{\lambda^2(1+\lambda)}{1+\lambda^2}$, then E is the empty set.*

Proof. Let $\mathcal{B}$ choose $\delta_n = 0$ while he cannot obtain $|x_n| > L_n - W_n$ or $|y_n| > N_n$ holds. In the latter case let $\mathcal{B}$ start with choice $\delta_n = -1$. Whilst the distance $L_n - W_n - |x_n|$ will not change and $L_n - W_n$ tends to zero there will be such an l that

$$M_k < |x_l| < N_k$$

holds, thus $\mathcal{A}$ cannot win.

Lemma 3.1.9. *If the winning set of $\mathcal{A}$ is E, then the winning set of $\mathcal{A}$ after the first step is $\lambda \cdot E$.*

Proof. The winning strategy of $\mathcal{A}$ for some $x_0 \in E$ is the same as the winning strategy for λx_0 in the game played with the ammunitions $\{\lambda_2,\ \lambda_3, \dots\}$ and $\{\omega_2,\ \omega_3, \dots\}$, therefore the lemma is true.

3.1.10. Statement. *If $\frac{3}{2}\lambda - \frac{1}{2} < a < \lambda$, then there is no interval on which $\mathcal{A}$ wins.*

Proof. Suppose indirectly that $\mathcal{A}$ wins on the interval $[a, b]$. We can assume that $b < M_1$ and $b - a$ is maximal. In the first step $\mathcal{A}$ must choose $\varepsilon_1 = 0$. If $\mathcal{B}$ chooses $\delta_1 = 0$, then $\lambda \cdot H$ must contain $[a, b]$ but it is a contradiction because the length of the longest interval contained by $\lambda \cdot E$ is $\lambda \cdot (b - a)$.

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