ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS

L. Leindler (Szeged, Hungary)

Dedicated to Professor Imre Kátai on his 70th birthday

Abstract. Four of our theorems are generalized such that no monotonicity restriction on the entries of the summability matrix. The origin of these theorems goes back to P. Chandra.

1. Introduction

In the paper [3] we generalized some theorems of P. Chandra. Roughly speaking we replaced by "almost monotone conditions" the classical monotonicity ones claimed at the entries of summability matrix appearing in his theorems. Now we make one step further. We reduce the restrictions further, and do not claim monotonicity conditions at all. Our new results, naturally, include the previous ones as special cases.

Before presenting our theorems we recall some definitions and notations.

Let $f(x)$ be a 2π-periodic continuous function. Let $s_n(f,x)$ denote the n-th partial sum of its Fourier series at x and let $\omega(\delta) = \omega(\delta, f)$ denote the modulus of continuity of f.

We shall use the notation $L \ll R$ at inequalities if there exists a positive constant K such that $L \leq KR$ holds.

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Let $\mathbf{A} := (a_{nk})$ $(k, n = 0, 1, \ldots)$ be a lower triangular infinite matrix of non-negative numbers and let the $\mathbf{A}$-transform of $\{s_n(f, x)\}$ be given by

$$T_n(f) := T_n(f, x) := \sum_{k=0}^{n} a_{nk} s_k(f, x) \quad n = 0, 1 \ldots.$$

2. Theorems

Theorem 1. *Let (a_{nk}) satisfy the following conditions:*

$$a_{nk} \geq 0, \quad \sum_{k=0}^{n} a_{nk} = 1, \quad \textit{and} \quad a_{nk} = 0 \quad \textit{if} \quad k > n. \tag{2.1}$$

Suppose $\omega(t)$ is such that

$$\int_u^\pi t^{-2}\omega(t)dt \ll H(u) \quad (u \to 0+), \tag{2.2}$$

where $H(u) \geq 0$ and

$$\int_0^u H(t)dt \ll uH(u) \quad (u \to 0+). \tag{2.3}$$

Then

$$\|T_n(f) - f\| \ll \alpha_{nn} H(\alpha_{nn}), \tag{2.4}$$

where

$$\alpha_{nk} = \sum_{\nu=0}^{k} |\Delta a_{n\nu}|, \quad \Delta a_{n\nu} := a_{n\nu} - a_{n\nu+1},$$

and $\|\cdot\|$ denotes the supnorm.

Theorem 2. *Let (2.1) and (2.2) hold. Then*

$$\|T_n(f) - f\| \ll \omega(\pi/n) + \alpha_{nn} H(\pi/n). \tag{2.5}$$

If, in addition, $\omega(t)$ satisfies (2.3) then

$$\|T_n(f) - f\| \ll \alpha_{nn} H(\pi/n). \tag{2.6}$$

In the special case

$$a_{nk} \le a_{n,k+1}, \quad k < n, \tag{2.7}$$

Theorems 1 and 2 were proved by P. Chandra [1], furthermore under the additional condition $\alpha_{nn} \ll a_{nn}$, but omitting (2.7), by us [3].

Theorem 3. *Demote*

$$A_{nm} := \sum_{\nu=0}^{m} a_{n\nu} \quad \textit{and} \quad \gamma_{nm} := \sum_{k=m}^{n} |\Delta a_{nk}| \quad (m \le n). \tag{2.8}$$

Then

$$\|T_n(f) - f\| \ll \omega(\pi/n) + \sum_{k=1}^{n} k^{-1} \omega(\pi/k)(A_{n,k+1} + k\gamma_{nk}). \tag{2.9}$$

Theorem 4. *Let (2.2), (2.3) and (2.8) hold. Then*

$$\|T_n(f) - f\| \ll \gamma_{n0} H(\gamma_{n0}). \tag{2.10}$$

We also underline that in the special case

$$a_{nk} \ge a_{n,k+1}, \quad k < n, \tag{2.11}$$

Theorems 3 and 4 were proved by P. Chandra [2], and with the condition $\gamma_{n0} \ll a_{nk}$ instead of (2.11) in [3].

We call the reader's attention to the fact that $\gamma_{n0} = \alpha_{nn}$, consequently Theorems 1 and 4 have the same assertion, therefore Theorem 4 in this general form is superfluous, but its two previous shapes were diverse, and their proofs were dissimilar, too. Now, evidently, it suffices to prove Theorem 1. We have presented Theorem 4 in order to show this special fusion of two theorems by means of generalization.

3. Lemmas

The following two lemmas were proved in [1] and [2] implicitly.

Lemma 1. ([1]) *If (2.2) and (2.3) hold then*

$$\int_0^{\pi/n} \omega(t)dt \ll n^{-2} H(\pi/n).$$

Lemma 2. ([2]) *If (2.2) and (2.3) hold then*

$$\int_0^{u} t^{-1}\omega(t)dt \ll uH(u).$$

Lemma 3. *If τ denotes the integer part of π/t, then*

$$\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t \le A_{n\tau} + \tau\gamma_{n\tau} \tag{3.1}$$

holds uniformly in $0 < t \le \pi$.

Furthermore

$$\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t \ll t^{-1}\alpha_{nn}. \tag{3.2}$$

Remark. Naturally the constant in (3.2) depends on the sequence $\{a_{nk}\}$, but not on t.

Proof. An elementary calculation shows that for $n \ge m \ge 0$

$$\left|\sum_{k=m}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t \sin\frac{t}{2}\right| \le \frac{1}{2}\left[a_{nm} + \sum_{k=m}^{n-1} |\Delta a_{nk}| + a_{nn}\right] \le \gamma_{nm}. \tag{3.3}$$

Hence

$$\left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| \le A_{n\tau} + \left|\sum_{k=\tau}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| \le A_{n\tau} + \tau\gamma_{n\tau}$$

follows, and this verifies (3.1).

It is easy to check that if $m=0$ in (3.3) then the sum γ_{n0} can be replaced by α_{nn}. This modified inequality plainly verifies (3.2). The proof of Lemma 3 is complete.

4. Proofs

Proof of Theorem 1. We have with $\Phi_x(t) := \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$ the following equality

$$T_n(f,x)-f(x) = \frac{2}{\pi}\int\limits_0^\pi \left\{\Phi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^n a_{nk}\sin\left(k+\frac{1}{2}\right)t\right\}dt.$$

Hence

$$\begin{aligned}\|T_n(f)-f\| &\le \frac{2}{\pi}\int\limits_0^\pi \frac{\omega(t)}{2\sin\frac{t}{2}}\left|\sum_{k=0}^n a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt = \\ &= \frac{2}{\pi}\left(\int\limits_0^{\alpha_{nn}} + \int\limits_{\alpha_{nn}}^{\pi}\right) =: I_1+I_2, \text{ say.}\end{aligned} \tag{4.1}$$

By (2.1) the sum in the integral does not exceed 1 and thus using Lemma 2 we have

$$I_1 \ll \int\limits_0^{\alpha_{nn}} t^{-1}\omega(t) \ll \alpha_{nn}H(\alpha_{nn}). \tag{4.2}$$

By (3.2) and (2.2) we also have

$$I_2 \ll \alpha_{nn}\int\limits_{\alpha_{nn}}^{\pi} t^{-2}\omega(t)dt \ll \alpha_{nn}H(\alpha_{nn}). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we obtain (2.4) as asserted.

Proof of Theorem 2. Using again (4.1) in the following form

$$\|T_n(f)-f\| \ll \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) =: J_1 + J_2, \quad \text{say.} \tag{4.4}$$

To the estimation of J_1 we utilize the inequality $|\sin t| \le t$ and (2.1), thus

$$\left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| \le 2nt \sum_{k=0}^{n} a_{nk} \le 2nt, \tag{4.5}$$

whence

$$J_1 \ll n \int_0^{\pi/n} \omega(t)dt \ll \omega(\pi/n)$$

follows.

In the estimation of J_2 we use (3.2) and (2.2), thus

$$J_2 \ll \alpha_{nn} \int_{\pi/n}^{\pi} t^{-2}\omega(t)dt \ll \alpha_{nn} H(\pi/n). \tag{4.6}$$

Henceforth (4.4) and the last two estimations verify (2.5).

The assumption (2.3) insures the application of Lemma 1, thus by (4.5) we get that

$$J_1 \ll n \int_0^{\pi/n} \omega(t)dt \ll n^{-1} H(\pi/n). \tag{4.7}$$

An elementary consideration shows that

$$\alpha_{n,n-1} \ge \max_{\nu} a_{n\nu} - \min_{\nu} a_{n\nu},$$

furthermore

$$\alpha_{nn} = \alpha_{n,n-1} + a_{nn},$$

thus

$$\alpha_{nn} \geq \max_{\nu} a_{n\nu}.$$

Since $A_{nn} = 1$, therefore $\max_{\nu} a_{n\nu} \geq (n+1)^{-1}$. Putting this into (4.7) we get

$$J_1 \ll \alpha_{nn} H(\pi/n).$$

This and (4.6) imply (2.6). The proof is complete.

Proof of Theorem 3. Proceeding as in the proof of (2.5), we obtain that

$$J_1 \ll \omega(\pi/n), \tag{4.8}$$

and in the estimation of J_2 we apply Lemma 3 with (3.1). Then we get that

$$\begin{aligned} J_2 &\ll \int_{\pi/n}^{\pi} t^{-1}\omega(t)(A_{n\tau} + \tau\gamma_{n\tau})dt = \sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} \ll \\ &\ll \sum_{k=1}^{n-1} k^{-1}\omega(\pi/k)(A_{n,k+1} + \gamma_{nk}). \end{aligned} \tag{4.9}$$

Thus (4.8) and (4.9) imply (2.9), as stated.

References

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L. Leindler
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1.
H-6720 Szeged, Hungary
leindler@math.u-szeged.hu