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# ON THE AVERAGE OF SOME $q$–ADDITIVE FUNCTIONS

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*Dedicated to Professor Imre Kátai for his 70th birthday*

## 1. Introduction

Let $q$ be an arbitrary fixed natural number $\geq 2$. Then every natural number $n$ can be written uniquely as

$$n = \sum_{k=0}^{\infty} a_k(n) q^k, \quad 0 \leq a_k(n) \leq q-1$$

($q$-adic expansion of $n$).

A complex-valued arithmetical function $g(n)$ is called *$q$-additive* according to [5], if $g(n)$ satisfies the relations

$$g(0) = 0 \quad \text{and} \quad g(n) = \sum_{k=0}^{\infty} g(a_k(n) q^k), \quad n \in \mathbf{N},$$

whenever $n$ has the above $q$-adic expansion.

It is clear that when we give the values of the function $g$ on the set $\{rq^k; 1 \leq \leq r \leq q-1, \ k = 0, 1, \ldots\}$, then the $q$-additive function $g(n)$ is determined uniquely by the above relations, and vice versa.

The most famous example of $q$-additive functions is *the function sum of digits* $S_q(n)$, which is defined by $S_q(n) = \sum_{k=0}^{\infty} a_k(n)$. When $q = 10$, its values are, for example, $S_{10}(9) = 9$, $S_{10}(10) = 1$, ..., $S_{10}(99) = 18$, $S_{10}(100) = 1$, .... The values of $S_{10}(n)$ fluctuate largely because of *the raising-up* to the upper decimal point, but the raising-up occurs regularly, so we can expect both regularity and irregularity in the behaviour of $S_q(n)$.

Some mathematicians studied about the mean value of $S_q(n)$, see [3], [2], [7], [9], and in 1975 H. Delange proved the following very interesting result:

**Theorem A.** (Delange [4]) *We have, for any* $N \in \mathbf{N}$, *that*

$$\frac{1}{N}\sum_{n=0}^{N-1} S_q(n) = \frac{q-1}{2\log q}\log N + F\left(\frac{\log N}{\log q}\right),$$

*where the function* $F(x)$ *is a periodic function with period 1, defined by either of the following two ways* (I) *and* (II)*:*

$$(I)\qquad \begin{aligned} F(x) = & \frac{q-1}{2}(1+[x]-x)+ \\ & + q^{1+[x]-x}\sum_{r=0}^{\infty} q^{-r} \int_0^{q^r(q^{-1-[x]+x})} \left([qt]-q[t]-\frac{q-1}{2}\right)dt, \end{aligned}$$

*where* $[x]$ *denotes the largest integer not exceeding* $x$.

$$(II)\qquad F(x) = \sum_{k\in\mathbf{Z}} C_k e^{2\pi i k x},$$

*whose Fourier coefficients are given by*

$$C_0 = \frac{q-1}{2\log q}(\log(2\pi)-1) - \frac{q+1}{4},$$

$$C_k = i\frac{q-1}{2\pi k}\frac{\zeta\left(\frac{2\pi i k}{\log q}\right)}{1+\frac{2\pi i k}{\log q}}, \quad k \neq 0,$$

*where* $\zeta(s)$ *denotes the Riemann zeta-fuction.*

This theorem realizes both of the regularity and irregularity of the function $S_q(n)$ beautifully, and it must be remarked that Delange proved this result only by elementary calculations.

This result was generalized for much more general $q$-additive functions by Mauclaire and Murata [6], whereas their proof based on complex function theory. Their results are a little complicated but in the essence their proof goes as follows (we change their notations into our language):

For a $q$-additive function $g(n)$, they introduced generic functions by power series,

$$G_r(z) = \sum_{k=0}^{\infty} g(rq^k)z^k, \quad 1 \le r \le q-1,$$

and they assumed

(H1) every $G_r(z)$ converges at $z = 0$, and represents a rational function in $z$.

They further assumed

(H2) every pole of $G_r(z)$ is contained in the circle $|z| < \sqrt{q}$.

Then their theorem states

**Theorem B.** (Mauclaire-Murata [6]) *Under the above notations and assumptions, we have, for a natural number $N$, an explicit formula*

$$\frac{1}{N}\sum_{n=0}^{N-1} g(n) = \text{a finite sum of terms of } C\cdot(\log N)^{\alpha} +$$

$$+ \text{ a finite sum of terms of } D\cdot N^{\beta}(\log N)^{\gamma}\cdot F_{\delta}\left(\frac{\log N}{\log q}\right),$$

*where every function $F_\delta(x)$ is a periodic function with period 1. Moreover the leading coefficients $C$'s and $D$'s, the exponents $\alpha$'s $\beta$'s and $\gamma$'s, and the Fourier coefficients of the periodic functions $F_\delta(x)$ are all determined explicitly from the function-theoretical properties of the generic functions $G_r(z)$.*

When the maximum amplitude of the oscillating terms is smaller than the leading $C\cdot(\log N)^{\alpha}$ term, then the average of $g(n)$ is given as "main term + + very exact error terms", and when the oscillating amplitude is greater than any $C\cdot(\log N)^{\alpha}$ terms, this means that the average of $g(n)$ oscillates and does not have average order.

This paper is, in a certain sense, a sequel of the paper [6], and here we consider about the above condition (H2). Let $h$ be a 2-additive function defined by

$$h(0) = 0 \quad \text{and} \quad h(2^k) = 2^{-k/2}, \quad k = 0, 1, \ldots. \tag{1}$$

Then the only one generic function $G(z) = \sum_{k=0}^{\infty} h(2^k)z^k = -\sqrt{2}/(z-\sqrt{2})$ does not satisfy the assumption (H2), and we cannot apply Theorem B to this function $h(n)$.

In this paper, first we limit our attention to $q$-additive functions $g(n)$ which satisfy

$$g(rq^k) = r \cdot g(q^k), \quad \text{for all} \quad 1 \le r \le q-1 \quad \text{and} \quad k \in \mathbf{N} \cup \{0\},$$

and we will prove the following two facts,

(i) when we think of a more general average value $A_{g,m}(x)$ of this $q$-additive function $g(n)$ (the definition appears in the next section), then this average has the same expression - a finite sum of terms of $C \cdot (\log N)^{\alpha} +$ + a finite sum of oscillating terms - as in Theorem B,

(ii) and as an application of this first result, we can weaken the assumption (H2) into

(H3) very pole of $G_r(z)$ is contained in the circle $|z| < q$,

(cf. Theorem 2 in Section 4), and for the function $h(n)$ defined by (1), we can conclude that

$$\frac{1}{N}\sum_{n=0}^{N-1} h(n) = \frac{(q-1)\sqrt{q}}{2(\sqrt{q}-1)} + \frac{1-q^{3/2}}{\sqrt{N}\log q}\left(\frac{3}{2}\Phi\left(\frac{\log N}{\log q}\right) + \frac{1}{\log q}\frac{d\Phi}{dx}\left(\frac{\log N}{\log q}\right)\right),$$

where

$$\Phi(x) = \sum_{b \in \mathbf{Z}} e^{2\pi i b x} \frac{\zeta\left(-\frac{1}{2} + \frac{2\pi i b}{\log q}\right)}{\left(-\frac{1}{2} + \frac{2\pi i b}{\log q}\right)\left(\frac{1}{2} + \frac{2\pi i b}{\log q}\right)\left(\frac{3}{2} + \frac{2\pi i b}{\log q}\right)},$$

i.e. the average of $h(n)$ is the sum of main term and two oscillating terms.

We think that the phenomenon described in Theorem B must be held generally for all $q$-additive functions providing the assumption (H1). In fact, for the average value of $q$-additive function which does not satisfy (H3), we can separate the oscillating terms and can derive the Fourier type expressions for these oscillating terms. But without assumptions such as (H2) and (H3), we can not prove the convergence of the series.

Throughout of this paper $[x]$ denotes the largest integer not exceeding $x$ and $\{x\} = x - [x]$.

## 2. A $q$-additive function $g_f(n)$ and an $m$-tuple average of $g_f(n)$

Let us introduce a weight function $f : \mathbf{N} \cup \{0\} \to \mathbf{C}$. We define the values of an arithmetic function $g_f$ by $g_f(rq^k) = r \cdot f(k)$ and for general $n = \sum_{k=0}^{\infty} a_k(n) q^k$, we define

$$g_f(0) = 0 \quad \text{and} \quad g_f(n) = \sum_{k=0}^{\infty} g(a_k(n)q^k) = \sum_{k=0}^{\infty} a_k(n) f(k), \quad n \in \mathbf{N}. \tag{2}$$

Then $g_f(n)$ is a $q$-additive function defined by the weight function $f(k)$, and its generic functions are

$$G_r(z) = \sum_{k=0}^{\infty} g(rq^k) z^k = r \sum_{k=0}^{\infty} f(k) z^k, \quad 1 \le r \le q-1.$$

So we need only

$$G_f(z) = \sum_{k=0}^{\infty} f(k) z^k.$$

First we remark that this $q$-additive function $g_f(n)$ has a relation with the fractional part of $n/q^k$.

**Lemma 1.** *Let $f(k)$ satisfy the following assumption:*

$$\lim_{k\to\infty} \left| \frac{f(k)}{f(k+1)} \right| > \frac{1}{q}. \tag{A1}$$

*Then*

$$g_f(n) = \sum_{k=1}^{\infty} (qf(k-1) - f(k)) \left\{ \frac{n}{q^k} \right\}.$$

**Proof.** Under the assumption (A1) the power series $\sum_{j=0}^{\infty} f(j) z^j$ has the radius of convergence $r$ with $r > q^{-1}$, and so, the series $\sum_{j=0}^{\infty} f(j) q^{-j}$ converges. Since

$$\sum_{j=k}^{\infty} (qf(j) - f(j+1)) q^{k-1-j} = f(k), \tag{3}$$

substituting (3) into (2), we have

$$g_f(n) = \sum_{k=0}^{\infty} a_k(n) \sum_{j=k}^{\infty} (qf(j) - f(j+1))q^{k-1-j} =$$
$$= \sum_{j=0}^{\infty} (qf(j) - f(j+1))q^{-j} \sum_{k=0}^{j} a_k(n)q^{k-1}.$$

Since

$$0 \le q^{-j} \sum_{k=0}^{j} a_k(n)q^{k-1} < 1 \quad \text{and} \quad q^{-j} \sum_{k=j+1}^{\infty} a_k(n)q^{k-1} \in \mathbf{N} \cup \{0\},$$

we see that

$$q^{-j} \sum_{k=0}^{j} a_k(n)q^{k-1} = \left\{ q^{-j} \sum_{k=0}^{\infty} a_k(n)q^{k-1} \right\} = \left\{ \frac{n}{q^{j+1}} \right\}.$$

This completes the proof.

As for the average of $g_f(n)$, we have the following expression.

**Lemma 2.** *Let $N \in \mathbf{N}$. Under the assumption* (A1), *we have*

$$\frac{1}{N} \sum_{n=0}^{N-1} g_f(n) = \sum_{k=1}^{\infty} (qf(k-1) - f(k)) \frac{1}{N} \int_0^N \left( \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right) dv.$$

**Proof.** From Lemma 1 it follows that

$$\frac{1}{N} \sum_{n=0}^{N-1} g_f(n) = \sum_{k=1}^{\infty} (qf(k-1) - f(k)) \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \frac{n}{q^k} \right\}.$$

Let us consider the inner sum on the right-hand side. We have

$$\sum_{n=0}^{N-1} \left\{ \frac{n}{q^k} \right\} = \int_0^N \left\{ \frac{[v]}{q^k} \right\} dv =$$
$$= \int_0^N \left\{ \left[ \frac{v}{q^k} \right] + \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right\} dv. \tag{4}$$

Making use of the inequality

$$0 \le \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} < 1,$$

(4) becomes to

$$\sum_{n=0}^{N-1} \left\{ \frac{n}{q^k} \right\} = \int_0^N \left( \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right) dv$$

and this completes the proof.

Now we generalize the usual average and put the definition.

**Definition.** Let $m \in \mathbf{N}$, and $x$ be a positive real number. The $m$-tuple average $A_{g_f,m}(x)$ of $g_f$ is defined by

$$A_{g_f,m}(x) = \sum_{k=1}^{\infty} (qf(k-1) - f(k)) \times \\ \times \frac{1}{x^m} \int_0^x \int_0^{v_{m-1}} \cdots \int_0^{v_1} \left( \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right) dv dv_1 \cdots dv_{m-1}.$$

**Remark.** When we take $m = 1$ and $x = N \in \mathbf{N}$, then from Lemma 2, we have

$$A_{g_f,1}(N) = \frac{1}{N} \sum_{n=0}^{N-1} g_f(n).$$

**Lemma 3.** *Let $0 < \alpha < 1$. Let $\mathcal{B}_m(x)$ be the function defined by*

$$\mathcal{B}_m(x) = \int_0^x \int_0^{v_{m-1}} \cdots \int_0^{v_1} \{v\} dv dv_1 \cdots dv_{m-1}.$$

*Then we have*

$$\mathcal{B}_m(x) = -\frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\zeta(s) x^{s+m}}{s(s+1) \cdots (s+m)} ds, \tag{5}$$

*where $\zeta(s)$ is the Riemann zeta-function.*

**Proof.** Changing the order of integrations from right, we have

$$\mathcal{B}_m(x) = \int_0^x \{v\} \frac{(x-v)^{m-1}}{(m-1)!} dv. \tag{5}$$

Here we recall of a famous formula

$$-\frac{\zeta(s)}{s} = \int_0^\infty \frac{\{x\}}{x^{s+1}} dx, \quad 0 < \sigma < 1 \tag{6}$$

(see [8] p.14, (2.1.5)). We want to repeat integration by parts to the right-hand side of (7).

To do this we firstly derive some poperties for $\mathcal{B}_m(x)$. From (6) it follows that

$$|\mathcal{B}_m(x)| \le \frac{x^m}{m!}, \quad x \to \infty, \quad \text{and} \quad \mathcal{B}_m(x) = \frac{x^{m+1}}{(m+1)!}, \quad x \to 0+.$$

Moreover, from the definition of $\mathcal{B}_m(x)$ it follows that $\mathcal{B}_1(x)$ is continuous and $\frac{d}{dx}\mathcal{B}_1(x) = \{x\}$ for $x \in (0,\infty) - \mathbf{N}$, and $\mathcal{B}_m(x)$, $m \ge 2$, is a $C^{m-1}$ function and $\frac{d}{dx}\mathcal{B}_m(x) = \mathcal{B}_{m-1}(x)$ for $x > 0$.

By these properties of $\mathcal{B}_m(x)$ we can repeat integration by parts to the integral on the right-hand side of (7) and get

$$-\frac{\zeta(s)}{s(s+1)\cdots(s+m)} = \int_0^\infty \frac{\mathcal{B}_m(x)}{x^{s+m+1}} dx, \quad 0 < \sigma < 1. \tag{8}$$

Applying the Mellin inversion formula to (8), we obtain the expression of this lemma.

**Lemma 4.** *We assume the assumption (A1), and let $r$ be the radius of convergence of the generic function $G_f(z) = \sum_{k=0}^\infty f(k)z^k$. When we choose a positive number $\alpha$ with $q^{-1} < q^{-\alpha} < \min(r,1)$, then*

$$A_{g_f,m}(x) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)} (1-q^{1-s}) G_f(q^{-s}) ds.$$

**Proof.** Let us consider the $m$-tuple integrals in Definition 1. Changing the order of integrations from right, we have, by (6),

$$\int_0^x \int_0^{v_{m-1}} \cdots \int_0^{v_1} \left( \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right) dv dv_1 \cdots dv_{m-1} =$$

$$= \int_0^x \left( \left\{ \frac{v}{q^k} \right\} - \frac{\{v\}}{q^k} \right) \frac{(x-v)^{m-1}}{(m-1)!} dv =$$

$$= q^{km} \mathcal{B}_m \left( \frac{x}{q^k} \right) - \frac{1}{q^k} \mathcal{B}_m(x).$$

Hence we have, by (5),

$$A_{g_f,m}(x) = \sum_{k=1}^{\infty} (qf(k-1) - f(k)) \times$$

$$\times \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{\zeta(s) x^s}{s(s+1)\cdots(s+m)} (q^{-k} - (q^{-s})^k) ds.$$

Under our choice of $\alpha$ we can change the order of summation and integration, then by direct computation, we have

$$\sum_{k=1}^{\infty} (qf(k-1) - f(k))(q^{-k} - (q^{-s})^k) = (1 - q^{1-s}) \sum_{k=0}^{\infty} f(k)(q^{-s})^k,$$

and this completes the proof.

This lemma shows that we can derive an exact value of $A_{g_f,m}(x)$ from function-theoretical properties of $G_f(z)$.

At the end of this section we remark that among the averages $A_{g_f,m}(x)$, $m \in \mathbf{N}$, we have the following relation.

**Lemma 5.** *Under the assumption (A1), $A_{g_f,1}(x)$ is continuous, and*

$$\frac{d}{dx}(x^m A_{g_f,m}(x)) = x^{m-1} A_{g_f,m-1}(x), \quad m \geq 2, \quad x \in (0, \infty).$$

**Proof.** By the expression of $A_{g_f,1}(x)$ in Lemma 4 we see that $A_{g_f,1}(x)$ is continuous.

Multiply the equation of Lemma 4 by $x^m$, then differentiate it. By the estimate $|\zeta(\alpha+it)| \ll (1+|t|)^{(1/2)(1-\alpha)+\varepsilon}$, where $\varepsilon > 0$ can be chosen arbitrarily small, we can change the order of differentiation and integration, and obtain this lemma.

## 3. Analytic properties of $A_{g_f,m}$

In order to study the $m$-tuple average $A_{g_f,m}(x)$ we will shift the contour of integration in Lemma 4 to left. For this purpose, here we put a new assumption (cf. Assumption (H1)):

(A2) $G_f(z) = \sum_{k=0}^{\infty} f(k)z^k$ is continued to a rational function for which the degree of the polynomial on the numerator is less than that on the denominator. Moreover, the poles $P$ of $G_f(z)$ satisfy $|P| > q^{-1}$.

Obviously, the assumption (A2) is stronger than (A1).

The main result of this section is Theorem 1 (at the end of this section).

Let us consider the partial-fraction decomposition for $G_f(z)$. Let $\Pi$ be the set of all poles $P$ of $G_f(z)$, and $d_P$ the order of the pole $P$ of $G_f(z)$. Then $G_f(z)$ is expressed as

$$G_f(z) = \sum_{P\in\Pi} \sum_{l=1}^{d_P} \frac{C_{P,l}}{(z-P)^l}, \quad C_{P,l} \in \mathbf{C},$$

and

$$\frac{1}{(z-P)^l} = \frac{1}{z^l} \sum_{k=0}^{\infty} \binom{k+l-1}{k} \left(\frac{P}{z}\right)^k, \qquad |z| > |P|. \tag{9}$$

From Lemma 4 and the assumption (A2), for a suitably chosen positive number $\alpha$ with $q^{-1} < q^{-\alpha} < \min\left(\min_{P\in\Pi} |P|, 1\right)$, we have

$$A_{g_f,m}(x) = \sum_{P\in\Pi} \sum_{l=1}^{d_P} C_{P,l} \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)} \frac{1-q^{1-s}}{(q^{-s}-P)^l} ds. \tag{10}$$

Now let us choose $m \in \mathbf{N}$ as $\max_{P\in\Pi}|P| < q^{m-1/2}$, and choose a negative number $\beta$ such that

$$\max\left(q^{m-1}, \max_{P\in\Pi}|P|\right) < q^{-\beta} < q^{m-1/2}. \tag{11}$$

This $\beta$ satisfies $-m+1/2 < \beta < -m+1$. For $s = \beta + it$, by the estimate $|\zeta(\beta+it)| \ll (1+|t|)^{1/2-\beta}$, we have

$$\left|\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}\right| \ll \frac{|\zeta(\beta+it)|}{(1+|t|)^{m+1}} \ll \frac{1}{(1+|t|)^{(m-1/2+\beta)+1}},$$

and we can shift the contour of integration in (10) to the vertical line $\Re\, s = \beta$. Then

$$A_{g_f,m}(x) =$$

$$\begin{aligned}
&= \sum_{P\in\Pi}\sum_{l=1}^{d_P} C_{P,l}\Bigg(\sum_{\substack{w=0,-1,\ldots,-m+1\\ \text{and } P\neq q^{-w}}} \operatorname*{Res}_{s=w}\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}+\\
&\quad + \sum_{\substack{w=0,-1,\ldots,-m+1\\ \text{and } P=q^{-w}}} \operatorname*{Res}_{s=w}\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}+\\
&\quad + \sum_{\substack{w:P=q^{-w}\\ \text{and } w\neq 0,-1,\ldots,-m+1}} \operatorname*{Res}_{s=w}\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}+\\
&\quad + \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}ds\Bigg) =\\
&= \sum_{P\in\Pi}\sum_{l=1}^{d_P} C_{P,l}(R_1+R_2+R_3+I), \quad \text{say.}
\end{aligned} \tag{12}$$

First we consider $I$. By (11) we see that $|q^{-s}| = q^{-\beta} > |P|$. Hence, by (9),

$$I = \sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}(1-q^{1-s})(q^{k+l})^s ds. \tag{13}$$

As for this integral we can calculate it *exactly*.

**Lemma 6.** *Let* $m \in \mathbf{N}$ *and choose* $\beta$ *as* $-m+1/2 < \beta < -m+1$. *Let* $B_t(x)$ *be the* $t$*-th Bernoulli polynomial and* $B_t$ *the* $t$*-th Bernoulli number definded by*

$$\frac{ze^{xz}}{e^z-1} = \sum_{t=0}^{\infty} \frac{B_t(x)}{t!} z^t \quad \text{and} \quad B_t = B_t(0),$$

*respectively (see [1] p.264). Then*

$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}ds = \frac{B_{m+1}(\{x\})-B_{m+1}}{(m+1)!}. \tag{14}$$

*Especially, we have, for* $N \in \mathbf{N}$,

$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\zeta(s)N^{s+m}}{s(s+1)\cdots(s+m)}ds = 0.$$

**Proof.** We prove (14) by induction. The assertion in the case $m = 1$ is a special case of [6] Lemma 5, but we shall give here a little different proof for the remaining process. It follows from Lemma 3 that

$$\int_0^x \{v\}dv = -\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\zeta(s)x^{s+1}}{s(s+1)}ds.$$

We shift the contour of integration to the vertical line $\Re\, s = \beta$ with $-1/2 < < \beta < 0$, then

$$\int_0^x \{v\}dv = -\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\zeta(s)x^{s+1}}{s(s+1)}ds - \zeta(0)x.$$

By the facts $\zeta(0) = -1/2$, $\int_n^{n+1} B_1(\{v\})dv = 0$ if $n \in \mathbf{N}$, and $\frac{d}{dv}B_2(v) = 2B_1(v)$, we have

$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\zeta(s)x^{s+1}}{s(s+1)}ds = \int_0^x \left(\{v\}-\frac{1}{2}\right)dv =$$

$$= \int_0^{\{x\}} B_1(v)dv = \frac{B_2(\{x\})-B_2}{2},$$

which verifies our assertion in the case $m = 1$.

Now we start from

$$-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\zeta(s)x^{s+m-1}}{s(s+1)\cdots(s+m-1)}ds = \frac{B_m(\{x\})-B_m}{m!},$$

where $-m+3/2<\gamma<-m+2$ (assumption of induction). Let $-m+1/2<$ $<\beta<-m+1$. Then

(15)

$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}ds = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}ds + \operatorname*{Res}_{s=-m+1}\frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}.$$

The second term on the right-hand of (15) is equal to $B_m x/m!$, because of the fact $\zeta(-m+1) = (-1)^{m+1}B_m/m$ (see [1] p.266). Let us differentiate the first term on the right-hand side of (15). Then, the change of the order of differentiation and integration is valid, and we have

$$\frac{d}{dx}\left(-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}ds\right) = \frac{B_m(\{x\})-B_m}{m!}.$$

This asserts that the first term on the right-hand side of (15) is equal to $\int_0^x((B_m(\{v\})-B_m)/m!)dv$. Hence, by $\int_n^{n+1}B_m(\{v\})dv = 0$ if $n\in\mathbf{N}$, and $\frac{d}{dv}B_{m+1}(v) = (m+1)B_m(v)$, we have

$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{\zeta(s)x^{s+m}}{s(s+1)\cdots(s+m)}ds = \int_0^x\frac{B_m(\{v\})-B_m}{m!}dv + \frac{B_m x}{m!} =$$

$$= \int_0^{\{x\}}\frac{B_m(v)}{m!}dv = \frac{B_{m+1}(\{x\})-B_{m+1}}{(m+1)!}.$$

This completes the proof.

Combining Lemma 6 with (13), we have

$$I = \sum_{k=0}^{\infty} \binom{k+l-1}{k} \times \tag{16}$$

$$\times P^k \left( \frac{-B_{m+1}(\{xq^{k+l}\}) + B_{m+1}}{(xq^{k+l})^m (m+1)!} + q \frac{B_{m+1}(\{xq^{k+l-1}\}) - B_{m+1}}{(xq^{k+l-1})^m (m+1)!} \right).$$

**Computation of $R_1$.** To compute $R_1$ we use the equation

$$\frac{1}{s(s+1)\cdots(s+m)} = \frac{1}{m!} \sum_{a=0}^{m} \binom{m}{a} \frac{(-1)^a}{s+a}. \tag{17}$$

Since $\zeta(-j) = (-1)^j B_{j+1}/(j+1)$, we have

$$R_1 = \frac{1}{m!} \sum_{\substack{j=0,1,\dots,m-1 \\ \text{and } P \neq q^j}} \binom{m}{j} \frac{B_{j+1}}{j+1} x^{-j} \frac{1-q^{j+1}}{(q^j - P)^l}.$$

Hence

$$\sum_{P \in \Pi} \sum_{l=1}^{d_P} C_{P,l} R_1 = \frac{1}{m!} \sum_{\substack{j=0,1,\dots,m-1 \\ \text{and } P \neq q^j}} \binom{m}{j} \frac{B_{j+1}}{j+1} \frac{1-q^{j+1}}{x^j} G_f(q^j). \tag{18}$$

**Computation of $R_2$.** We have

$$\begin{aligned} R_2 = & \sum_{\substack{j=0,1,\dots,m-1 \\ \text{and } P = q^j}} \operatorname*{Res}_{s=0} \frac{\zeta(s-j) x^{s-j}}{(s-j)\cdots(s-j+m)} \frac{1-q^{1-s+j}}{q^{jl}(q^{-s}-1)^l} = \\ = & \sum_{\substack{j=0,1,\dots,m-1 \\ \text{and } P = q^j}} \left( \frac{1}{(xq^l)^j} \operatorname*{Res}_{s=0} \frac{\zeta(s-j) x^s}{(s-j)\cdots(s-j+m)} \frac{1}{(q^{-s}-1)^l} - \right. \\ & \left. - \frac{q^{j+1}}{(xq^l)^j} \operatorname*{Res}_{s=0} \frac{\zeta(s-j)(x/q)^s}{(s-j)\cdots(s-j+m)} \frac{1}{(q^{-s}-1)^l} \right). \end{aligned}$$

We calculate these residues. We have, near $s = 0$,

$$\begin{aligned} \frac{1}{(q^{-s}-1)^l} = & \frac{1}{(-s\log q)^l} \sum_{n=0}^{\infty} \sum_{\substack{n_1+\dots+n_l=n \\ n_i=0,1,2,\dots,n}} \frac{B_{n_1}\cdots B_{n_l}}{n_1!\cdots n_l!} (-s\log q)^n = \\ = & \frac{1}{(-s\log q)^l} \sum_{n=0}^{\infty} C_{n,l} (-s\log q)^n, \quad \text{say.} \end{aligned} \tag{19}$$

We put the Taylor expansion of $\zeta(s)$ at $s=-j$ as

$$\zeta(s-j)=\sum_{n=0}^{\infty}\frac{\zeta^{(n)}(-j)}{n!}s^n=\sum_{n=0}^{\infty}D_{n,j}s^n,\quad \text{say.} \tag{20}$$

By (17) we have, near $s=0$,

$$\frac{1}{(s-j)\cdots(s-j+m)}= \tag{21}$$

$$\sum_{n=0}^{\infty}\frac{1}{m!}\sum_{\substack{a=0,\ldots,m\\ a\neq j}}\binom{m}{a}\frac{(-1)^{a+1}}{(j-a)^{n+1}}s^n+\frac{1}{m!}\binom{m}{j}\frac{(-1)^j}{s}=$$

$$=\sum_{n=-1}^{\infty}E_{n,j,m}s^n,\quad \text{say.}$$

Combining (19)-(21), we have

$$\operatorname*{Res}_{s=0}\frac{\zeta(s-j)y^s}{(s-j)\cdots(s-j+m)}\frac{1}{(q^{-s}-1)^l}=$$

$$=\frac{1}{(-\log q)^l}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3=0,1,2,\ldots,l\\ n_4=-1,0,1,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,j}E_{n_4,j,m}}{n_1!}(-\log q)^{n_2}(\log y)^{n_1},$$

and hence,

$$R_2=\frac{1}{(-\log q)^l}\times \tag{22}$$

$$\times\sum_{\substack{j=0,1,\ldots,m-1\\ \text{and } P=q^j}}\left(\frac{1}{(xq^l)^j}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3=0,1,2,\ldots,l\\ n_4=-1,0,1,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,j}E_{n_4,j,m}}{n_1!}(-\log q)^{n_2}(\log x)^{n_1}-\right.$$

$$\left.-\frac{q^{j+1}}{(xq^l)^j}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3=0,1,2,\ldots,l\\ n_4=-1,0,1,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,j}E_{n_4,j,m}}{n_1!}(-\log q)^{n_2}(\log(x/q))^{n_1}\right).$$

**Computation of $R_3$.** We have

$$R_3 = \sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} \operatorname*{Res}_{s=-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}} \frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l}+$$

$$+\,\delta(P)\operatorname*{Res}_{s=-\frac{\mathrm{Log}P}{\log q}} \frac{\zeta(s)x^s}{s(s+1)\cdots(s+m)}\frac{1-q^{1-s}}{(q^{-s}-P)^l} =$$

$$= \sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} R_{3,1} + \delta(P)R_{3,2}, \quad \text{say,}$$

where $\mathrm{Log}P = \log|P| + i\mathrm{Arg}P$, $0 \le \mathrm{Arg}P < 2\pi$, and

$$\delta(P) = \begin{cases} 1 & \text{if } -\frac{\mathrm{Log}P}{\log q} \neq 0, -1, \ldots -m+1, \\ 0 & \text{otherwise.} \end{cases}$$

The calculation of $\delta(P)R_{3,2}$ is similar as $R_{3,1}$, and we consider only $R_{3,1}$.

$$R_{3,1} = \operatorname*{Res}_{s=0} \frac{\zeta\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)x^{s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}}}{\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)\cdots\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}+m\right)}\frac{1-Pq^{1-s}}{(Pq^{-s}-P)^l} =$$

$$= \frac{e^{2\pi ib\frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q}+l}}\operatorname*{Res}_{s=0} \frac{\zeta\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)x^s}{\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)\cdots\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}+m\right)}\frac{1}{(q^{-s}-1)^l}-$$

$$-\frac{qe^{2\pi ib\frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q}+l-1}}\operatorname*{Res}_{s=0} \frac{\zeta\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)(x/q)^s}{\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)\cdots\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}+m\right)}\frac{1}{(q^{-s}-1)^l}.$$

We use again the Taylor expansion of $\zeta(s)$,
(23)

$$\zeta\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right) = \sum_{n=0}^{\infty}\frac{\zeta^{(n)}\left(-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)}{n!}s^n = \sum_{n=0}^{\infty} D_{n,P,b}s^n, \quad \text{say,}$$

and from (17) we have, near $s=0$,

$$\frac{1}{\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}\right)\cdots\left(s-\frac{\mathrm{Log}P}{\log q}+\frac{2\pi ib}{\log q}+m\right)} =$$

$$
(24)\qquad \begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{m!} \sum_{a=0}^{m} \binom{m}{a} \frac{(-1)^{a+1}}{\left(\frac{\operatorname{Log} P}{\log q} - \frac{2\pi i b}{\log q} - a\right)^{n+1}} s^n = \\
&= \sum_{n=0}^{\infty} E_{n,P,b,m} s^n, \quad \text{say.}
\end{aligned}
$$

Combining (19), (23) and (24), we have

$$
\operatorname*{Res}_{s=0} \frac{\zeta\left(s - \frac{\operatorname{Log} P}{\log q} + \frac{2\pi i b}{\log q}\right) y^s}{\left(s - \frac{\operatorname{Log} P}{\log q} + \frac{2\pi i b}{\log q}\right) \cdots \left(s - \frac{\operatorname{Log} P}{\log q} + \frac{2\pi i b}{\log q} + m\right)} \frac{1}{(q^{-s}-1)^l} =
$$

$$
= \frac{1}{(-\log q)^l} \sum_{\substack{n_1+n_2+n_3+n_4=l-1 \\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}} \frac{C_{n_2,l} D_{n_3,P,b} E_{n_4,P,b,m}}{n_1!} (-\log q)^{n_2} (\log y)^{n_1},
$$

and hence,

$$
\begin{aligned}
R_3 = \frac{1}{(-\log q)^l} \sum_{\substack{b \in \mathbf{Z} \\ b \neq 0}} \Bigg( & \frac{e^{2\pi i b \frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q} + l}} \sum_{\substack{n_1+n_2+n_3+n_4=l-1 \\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}} \frac{C_{n_2,l} D_{n_3,P,b} E_{n_4,P,b,m}}{n_1!} \times \\
& \times (-\log q)^{n_2} (\log x)^{n_1} - \\
& - \frac{q e^{2\pi i b \frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q} + l - 1}} \sum_{\substack{n_1+n_2+n_3+n_4=l-1 \\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}} \frac{C_{n_2,l} D_{n_3,P,b} E_{n_4,P,b,m}}{n_1!} \times \\
& \times (-\log q)^{n_2} (\log(x/q))^{n_1} \Bigg) + \\
+ \frac{\delta(P)}{(-\log q)^l} \Bigg( & \frac{1}{P^{\frac{\log x}{\log q} + l}} \sum_{\substack{n_1+n_2+n_3+n_4=l-1 \\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}} \frac{C_{n_2,l} D_{n_3,P,0} E_{n_4,P,0,m}}{n_1!} \times \\
& \times (-\log q)^{n_2} (\log x)^{n_1} - \\
& - \frac{q}{P^{\frac{\log x}{\log q} + l - 1}} \sum_{\substack{n_1+n_2+n_3+n_4=l-1 \\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}} \frac{C_{n_2,l} D_{n_3,P,0} E_{n_4,P,0,m}}{n_1!} \times \\
& \times (-\log q)^{n_2} (\log(x/q))^{n_1} \Bigg).
\end{aligned}
$$

Substituting (16), (18), (22), (25) into (12), we obtain the following

**Theorem 1.** *We assume the assumption (A2) and let* $\Pi$ *be the set of all poles of the generic function* $G_f(z)$. *Choose* $m \in \mathbf{N}$ *as* $\max_{P\in\Pi}|P| < q^{m-1/2}$. *Then we have the generalized average* $A_{g_f,m}(x)$ *has an expression*

$$A_{g_f,m}(x) = \text{a finite sum of terms of } C\cdot x^{\alpha}(\log x)^{\beta} + \\ + \text{ a finite sum of oscillating terms,}$$

*and we can calculate all of those coefficients explicitly. More precisely we have*

$$A_{g_f,m}(x) = M_1 + M_2 + M_3 +$$

$$+ \frac{1}{(m+1)!x^m} \sum_{P\in\Pi}\sum_{l=1}^{d_P} C_{P,l} \sum_{k=0}^{\infty} \binom{k+l-1}{k} P^k \times$$

$$\times \frac{(-B_{m+1}(\{xq^{k+l}\}) + B_{m+1}) + q^{m+1}(B_{m+1}(\{xq^{k+l-1}\}) - B_{m+1})}{q^{(k+l)m}}$$

*with*

$$M_1 = \sum_{P\in\Pi}\sum_{l=1}^{d_P} \frac{C_{P,l}}{(-\log q)^l} \times$$

$$\times \sum_{\substack{j=0,1,\dots,m-1\\ \text{and } P=q^j}} \left( \frac{1}{(xq^l)^j} \sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3=0,1,2,\dots,l\\ n_4=-1,0,1,\dots,l-1}} \frac{C_{n_2,l}D_{n_3,j}E_{n_4,j,m}}{n_1!}(-\log q)^{n_2}(\log x)^{n_1} - \right.$$

$$\left. - \frac{q^{j+1}}{(xq^l)^j} \sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3=0,1,2,\dots,l\\ n_4=-1,0,1,\dots,l-1}} \frac{C_{n_2,l}D_{n_3,j}E_{n_4,j,m}}{n_1!}(-\log q)^{n_2}(\log(x/q))^{n_1} \right),$$

*where* $C_{n_2,l}$ $D_{n_3,j}$ $E_{n_4,j,m}$ *are defined by (19), (20), (21), respectively,*

$$M_2 = \sum_{P\in\Pi}\sum_{l=1}^{d_P} \frac{C_{P,l}}{(-\log q)^l} \times$$

$$\times \left( \sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} \left( \frac{e^{2\pi i b \frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q}+l}} \sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3,n_4=0,1,2,\dots,l-1}} \frac{C_{n_2,l}D_{n_3,P,b}E_{n_4,P,b,m}}{n_1!} \times \right.\right.$$

$$\times(-\log q)^{n_2}(\log x)^{n_1}-$$

$$-\frac{qe^{2\pi ib\frac{\log x}{\log q}}}{P^{\frac{\log x}{\log q}+l-1}}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,P,b}E_{n_4,P,b,m}}{n_1!}\times$$

$$\left.\times(-\log q)^{n_2}(\log(x/q))^{n_1}\right)+$$

$$+\delta(P)\left(\frac{1}{P^{\frac{\log x}{\log q}+l}}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,P,0}E_{n_4,P,0,m}}{n_1!}\times\right.$$

$$\times(-\log q)^{n_2}(\log x)^{n_1}-$$

$$-\frac{q}{P^{\frac{\log x}{\log q}+l-1}}\sum_{\substack{n_1+n_2+n_3+n_4=l-1\\ n_1,n_2,n_3,n_4=0,1,2,\ldots,l-1}}\frac{C_{n_2,l}D_{n_3,P,0}E_{n_4,P,0,m}}{n_1!}\times$$

$$\left.\left.\times(-\log q)^{n_2}(\log(x/q))^{n_1}\right)\right),$$

*where* $C_{n_2,l}$ $D_{n_3,P,b}$ $E_{n_4,P,b,m}$ *are defined by (19), (23), (24), respectively, and*

$$M_3=\frac{1}{m!}\sum_{\substack{j=0,1,\ldots,m-1\\ \text{and }\ P\neq q^j}}\binom{m}{j}\frac{B_{j+1}}{j+1}\frac{1-q^{j+1}}{x^j}G_f(q^j).$$

## 4. Examples and an application

In this last section, we discuss about some examples and remaining problems.

**Example 1.** Let $f(k)=1$ for $k\in\mathbf{N}$. Then $g_f(n)$ is *the function sum of digits.* We have

$$F(z)=\sum_{k=0}^{\infty}f(k)z^k=\frac{-1}{z-1},$$

$\Pi = \{1\}$, $P = 1$, $l = 1$, $C_{P,l} = -1$. Since $\max_{P\in\Pi} |P| = 1 < q^{m-1/2}$ for any $m \in \mathbf{N}$, we can choose any $m \in \mathbf{N}$ in Theorem 1. By direct computation,

$$A_{g_f,m}(x) =$$

$$\begin{aligned}
=&\frac{q-1}{\log q}\frac{\log x}{2m!} + \frac{q-1}{2m!\log q}\left(\sum_{a=1}^{m}\binom{m}{a}\frac{(-1)^a}{a} + \log(2\pi)\right) - \frac{q+1}{4m!} - \\
&- \frac{q-1}{\log q}\sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} e^{2\pi i b\frac{\log x}{\log q}} \frac{\zeta\left(\frac{2\pi i b}{\log q}\right)}{\left(\frac{2\pi i b}{\log q}\right)\cdots\left(\frac{2\pi i b}{\log q}+m\right)} + \\
&+ \frac{1}{m!}\sum_{j=1}^{m-1}\binom{m}{j}\frac{B_{j+1}}{j+1}\frac{1-q^{j+1}}{x^j}\frac{1}{1-q^j} - \\
-&\frac{1}{(m+1)!x^m}\sum_{k=0}^{\infty}\frac{(-B_{m+1}(\{xq^{k+1}\}) + B_{m+1}) + q^{m+1}(B_{m+1}(\{xq^k\}) - B_{m+1})}{q^{(k+1)m}}.
\end{aligned}$$

Especially, when we choose $m = 1$ and $x = N \in \mathbf{N}$, then

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} g_f(n) =& \frac{q-1}{\log q}\frac{\log N}{2} + \frac{q-1}{2\log q}(-1 + \log(2\pi)) - \frac{q+1}{4} - \\
&- \frac{q-1}{\log q}\sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} e^{2\pi i b\frac{\log N}{\log q}} \frac{\zeta\left(\frac{2\pi i b}{\log q}\right)}{\left(\frac{2\pi i b}{\log q}\right)\left(\frac{2\pi i b}{\log q}+1\right)},
\end{aligned}$$

which is Theorem A of Delange [4].

In the case $\sqrt{q} \le \max_{P\in\Pi} |P| < q$ we can study $A_{g_f,1}(x)$ more precisely.

**Theorem 2.** *We assume the assumption (A2) and let $\Pi$ be the set of all poles of the generic funciton $G_f(z)$. In the case $\sqrt{q} \le \max_{P\in\Pi} |P| < q$ we have*

$$A_{g_f,1}(N) = \frac{1}{N}\sum_{n=0}^{N-1} g_f(n) = \frac{1}{N}\frac{d}{dx}\left(x^2(M_1 + M_2 + M_3)\right)\Big|_{x=N} + \frac{(q^2-1)G_f(q)}{12N},$$

*where $M_1$, $M_2$, $M_3$ are the same quantities as in Theorem 1, and $M_2$ is a $C^1$ function in $x \in (0,\infty)$.*

**Proof.** Since $\sqrt{q} \le \max_{P\in\Pi}|P| < q$, we can choose $m = 2$ in Theorem 1. Then

$$x^2 A_{g_f,2}(x) = x^2(M_1 + M_2 + M_3) + \Psi(x), \tag{26}$$

where $M_1$, $M_2$, $M_3$ are the same quantities as in Theorem 1, and

$$\begin{aligned}\Psi(x) =& \frac{1}{6}\sum_{P\in\Pi}\sum_{l=1}^{d_P} C_{P,l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\times \\ &\times \frac{(-B_3(\{xq^{k+l}\}) + B_3) + q^3(B_3(\{xq^{k+l-1}\}) - B_3)}{q^{(k+l)2}}.\end{aligned} \tag{27}$$

We shall verify that the series of $\Psi(x)$ is termwisely differentiable in $x$. It is known that

$$B_n(\{x\}) = -\frac{n!}{(2\pi i)^n}\sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}}\frac{e^{2\pi i b x}}{b^n}, \quad n \ge 2$$

(see [1] p.267). This asserts that $B_3(\{x\})$ is differentiable in $x \in \mathbf{R}$ and $\frac{d}{dx}B_3(\{x\}) = 3B_2(\{x\})$. By $\max_{x\in\mathbf{R}}|B_2(\{x\})| = 1/6$ and (9), we have

$$\sum_{k=0}^{\infty}\binom{k+l-1}{k}|P|^k\left|\frac{d}{dx}\frac{(-B_3(\{xq^{k+l}\}) + B_3) + q^3(B_3(\{xq^{k+l-1}\}) - B_3)}{q^{(k+l)2}}\right| =$$

$$= 3\sum_{k=0}^{\infty}\binom{k+l-1}{k}|P|^k\left|\frac{-B_2(\{xq^{k+l}\}) + q^2 B_2(\{xq^{k+l-1}\})}{q^{k+l}}\right| \le$$

$$\le \frac{1+q^2}{2}\frac{1}{q^l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}\frac{|P|^k}{q^k} = \frac{1+q^2}{2}\frac{1}{(q-|P|)^l}.$$

Therefore, we can differentiate the series of (27) termwisely. $x^2A_{g_f,2}(x)$ in (26) is a $C^1$ function in $x \in (0,\infty)$ by Lemma 5, and $x^2M_1$ and $x^2M_3$ in (26) are also $C^1$ functions in $x \in (0,\infty)$ by those forms. Therefore, $x^2M_2$, and consequently, $M_2$ is a $C^1$ function in $x \in (0,\infty)$. We have

$$\frac{d}{dx}\Psi(x) =$$

$$= \frac{1}{2}\sum_{P\in\Pi}\sum_{l=1}^{d_P} C_{P,l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\frac{-B_2(\{xq^{k+l}\}) + q^2B_2(\{xq^{k+l-1}\})}{q^{k+l}} =$$

$$=\frac{1}{2}\sum_{P\in\Pi}\sum_{l=1}^{d_P}C_{P,l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\times$$
$$\times\frac{(-B_2(\{xq^{k+l}\})+B_2)+q^2(B_2(\{xq^{k+l-1}\})-B_2)}{q^{k+l}}+$$
$$+\frac{q^2-1}{12}\sum_{P\in\Pi}\sum_{l=1}^{d_P}\frac{C_{P,l}}{(q-P)^l}=$$
$$=\frac{1}{2}\sum_{P\in\Pi}\sum_{l=1}^{d_P}C_{P,l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\times$$
$$\times\frac{(-B_2(\{xq^{k+l}\})+B_2)+q^2(B_2(\{xq^{k+l-1}\})-B_2)}{q^{k+l}}+$$
$$+\frac{(q^2-1)G_f(q)}{12}.$$

Differentiating the both sides of (26), we obtain

$$\begin{aligned}A_{g_f,1}(x)=&\frac{1}{x}\frac{d}{dx}\Big(x^2(M_1+M_2+M_3)\Big)+\\&+\frac{1}{2x}\sum_{P\in\Pi}\sum_{l=1}^{d_P}C_{P,l}\sum_{k=0}^{\infty}\binom{k+l-1}{k}P^k\times\\&\times\frac{(-B_2(\{xq^{k+l}\})+B_2)+q^2(B_2(\{xq^{k+l-1}\})-B_2)}{q^{k+l}}+\\&+\frac{(q^2-1)G_f(q)}{12x}.\end{aligned}\tag{28}$$

When we choose $x=N\in\mathbf{N}$ in (28), the second term on the right-hand side is equal to 0. This completes the proof.

**Example 2.** Let $f(k)=q^{-k/2}$, and consider $g_f(n)$ (cf. $q$-additive function $h(n)$ in Section 1 formula (1)). We have

$$G_f(z)=\sum_{k=0}^{\infty}f(k)z^k=\frac{-\sqrt{q}}{z-\sqrt{q}},$$

$\Pi=\{\sqrt{q}\}$, $P=\sqrt{q}$, $l=1$, $C_{P,l}=-\sqrt{q}$, and now we can apply Theorem 2. Theorem 2 gives

$$\frac{1}{N}\sum_{n=0}^{N-1}g_f(n)=\frac{1}{N}\frac{d}{dx}\Big(x^2(M_1+M_2+M_3)\Big)\Big|_{x=N}+\frac{(q^2-1)\sqrt{q}}{12N(\sqrt{q}-q)}.$$

By direct computation, $M_1 = 0$,

$$\frac{1}{N}\frac{d}{dx}(x^2 M_3)\Big|_{x=N} = \frac{(q-1)\sqrt{q}}{2(\sqrt{q}-1)} + \frac{q^2-1}{12N(\sqrt{q}-1)},$$

and

$$\frac{1}{N}\frac{d}{dx}(x^2 M_2)\Big|_{x=N} = \frac{1-q^{3/2}}{N\log q}\frac{d}{dx}\left(x^{3/2}\Phi\left(\frac{\log x}{\log q}\right)\right)\Big|_{x=N},$$

where

$$\Phi(x) = \sum_{b\in\mathbf{Z}} e^{2\pi ibx}\frac{\zeta\left(-\frac{1}{2}+\frac{2\pi ib}{\log q}\right)}{\left(-\frac{1}{2}+\frac{2\pi ib}{\log q}\right)\left(\frac{1}{2}+\frac{2\pi ib}{\log q}\right)\left(\frac{3}{2}+\frac{2\pi ib}{\log q}\right)}.$$

Since $\Phi\left(\frac{\log x}{\log q}\right)$ is a $C^1$ function in $x \in (0,\infty)$, $\Phi(x)$ is a $C^1$ function in $x \in \mathbf{R}$. Moreover, $\frac{d\Phi}{dx}(x)$ is periodic with period 1 and continuous, because $\Phi(x)$ is periodic with period 1. Hence $\frac{d\Phi}{dx}\left(\frac{\log x}{\log q}\right)$ is bounded, and

$$\frac{d}{dx}\left(x^{3/2}\Phi\left(\frac{\log x}{\log q}\right)\right) = \frac{3}{2}\sqrt{x}\Phi\left(\frac{\log x}{\log q}\right) + \frac{\sqrt{x}}{\log q}\frac{d\Phi}{dx}\left(\frac{\log x}{\log q}\right) = O(\sqrt{x}).$$

Thus, we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1} g_f(n) = \frac{(q-1)\sqrt{q}}{2(\sqrt{q}-1)} + O\left(\frac{1}{\sqrt{N}}\right).$$

**Example 3.** Let us take the weight function $f(k) = q^{-k}$, then

$$G_f(z) = \sum_{k=0}^{\infty} f(k)z^k = \frac{-q}{z-q},$$

$\Pi = \{q\}$, $P = q$, $l = 1$, $C_{P,l} = -q$. We can choose $m \in \mathbf{N}$ as $m \geq 2$, and when $m = 2$, we have

$$A_{g_f,2}(x) =$$

$$
\begin{aligned}
&= \frac{q}{4} + \frac{q^2-1}{\log q}\zeta(-1)\frac{\log x}{x} - \left(\frac{(q^2+1)\zeta(-1)}{2} - \frac{q^2-1}{\log q}\zeta'(-1)\right)\frac{1}{x} - \\
(29) \qquad & - \frac{q^2-1}{\log q}\cdot\frac{1}{x}\sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} e^{2\pi i b\frac{\log x}{\log q}} \frac{\zeta\left(-1+\frac{2\pi i b}{\log q}\right)}{\left(-1+\frac{2\pi i b}{\log q}\right)\left(\frac{2\pi i b}{\log q}\right)\left(\frac{2\pi i b}{\log q}+1\right)} - \\
& - \frac{1}{6x^2}\sum_{k=0}^{\infty}\frac{(-B_3(\{xq^{k+1}\})+B_3)+q^3(B_3(\{xq^k\})-B_3)}{q^{k+1}}.
\end{aligned}
$$

Here appear two periodic functions

$$
\Phi(x) = \sum_{\substack{b\in\mathbf{Z}\\ b\neq 0}} e^{2\pi i b x}\frac{\zeta\left(-1+\frac{2\pi i b}{\log q}\right)}{\left(-1+\frac{2\pi i b}{\log q}\right)\left(\frac{2\pi i b}{\log q}\right)\left(\frac{2\pi i b}{\log q}+1\right)},
$$

$$
\Psi(x) = \sum_{k=0}^{\infty}\frac{(-B_3(\{xq^{k+1}\})+B_3)+q^3(B_3(\{xq^k\})-B_3)}{q^{k+1}}.
$$

We see that $\Phi(x)$ and $\Psi(x)$ are periodic with period 1. Differentiating (29) in $x$, then, by Lemma 5,

$$
A_{g_f,1}(x) =
$$

$$
\begin{aligned}
&= \frac{q}{2} + \frac{q^2-1}{\log q}\zeta(-1)\frac{\log x}{x} + \left(-\frac{(q^2+1)\zeta(-1)}{2} + \frac{q^2-1}{\log q}(\zeta(-1)+\zeta'(-1)\right)\frac{1}{x} + \\
&+ \frac{1}{x}\frac{d}{dx}\left(-\frac{q^2-1}{\log q}x\Phi\left(\frac{\log x}{\log q}\right) - \frac{1}{6}\Psi(x)\right).
\end{aligned}
$$

Because of the estimate $\zeta(-1+it) \ll |t|^{3/2}$ and the fact

$$
\frac{d}{dx}B_3(\{xq^{k+l}\}) = 3q^{k+l}B_2(\{xq^{k+l}\}),
$$

we can not apply termwise differentiation on $\Phi(x)$ and $\Psi(x)$, and we can not derive the similar results as Theorem 2 (or Example 2) for this $g_f$. This means we can prove the existence of the average $A_{g_f,1}$, but not in the expression as in the Theorem B.

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