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A DISCRETE LIMIT THEOREM ON THE COMPLEX PLANE FOR THE HURWITZ ZETA–FUNCTION WITH AN ALGEBRAIC IRRATIONAL PARAMETER

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In honour of Professor Imre Kátai on the occasion of his 70th birthday

1. Introduction

Let $s = \sigma + it$ be a complex variable. The Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter α, $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. The function $\zeta(s,\alpha)$ is a meromorphic function, the point $s = 1$ is its simple pole with residue 1. If $\alpha = 1$, then $\zeta(s,\alpha)$ reduces to the Riemann zeta-function $\zeta(s)$.

The value distribution of the function $\zeta(s,\alpha)$, as of other zeta-functions, can be described by limit theorems in the sense of weak convergence of probability measures in various spaces. In [10] limit theorems of such a kind were proved in the case of rational or transcendental α, while in [9], [11] and [12] the function $\zeta(s,\alpha)$ with an algebraic irrational parameter α was investigated. All above mentioned theorems are of continuous type, since they deal with probability measures defined by translations $\zeta(\sigma + it,\alpha)$ or $\zeta(s + i\tau,\alpha)$, where t or τ varies continuously in the interval $[0, T]$. Also, there exist discrete limit theorems when t or τ takes values from some discrete set, for example, from a certain arithmetical progression.

Denote by $\mathcal{B}(\mathcal{S})$ the class of Borel sets of a metric space S, and let, for $N \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\mu_N(\dots) = \frac{1}{N+1} \sum_{\substack{l=0 \\ \dots}}^{N} 1,$$

where in place of dots a condition satisfied by l is to be written. Discrete limit theorems for the function $\zeta(s,\alpha)$ with rational or transcendental α were obtained in [6]. We will recall a discrete limit theorem with transcendental α.

Let

$$\widehat{\Omega} = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\} \stackrel{def}{=} \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, with the product topology and pointwise multiplication the infinite-dimensional torus $\widehat{\Omega}$ is a compact topological Abelian group. Therefore, on $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}))$ the probability Haar measure $\widehat{m}_H$ can be defined, and this gives a probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \widehat{m}_H)$. Denote by $\widehat{\omega}(m)$ the projection of $\widehat{\omega} \in \widehat{\Omega}$ to the coordinate space γ_m, and, for $\sigma > \frac{1}{2}$, on the probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \widehat{m}_H)$ define the complex-valued random variable $\zeta(\sigma, \alpha, \widehat{\omega})$ by

$$\zeta(\sigma, \alpha, \widehat{\omega}) = \sum_{m=0}^{\infty} \frac{\widehat{\omega}(m)}{(m+\alpha)^{\sigma}}.$$

Theorem 1. *Suppose that α is a transcendental number, $h > 0$ is a fixed number such that $\exp\{\frac{2\pi}{h}\}$ is irrational, and $\sigma > \frac{1}{2}$. Then the probability measure*

$$\mu_N(\zeta(\sigma + ilh, \alpha) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \tag{1}$$

converges weakly to the distribution of the random variable $\zeta(\sigma, \alpha, \widehat{\omega})$ as $N \to \to \infty$.

The proof of Theorem 1 is based on the linear independence over the field of rational numbers $\mathbb{Q}$ of the system

$$L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$$

with transcendental α.

The aim of this paper is to obtain the weak convergence of probability measure (1) in the case of algebraic irrational α. For this, we will adapt the method proposed in [11].

For α algebraic irrational, J.W.S. Cassels proved [2] that at least 51 percent of the elements of the system $L(\alpha)$ are linearly independent over $\mathbb{Q}$. Let $I(\alpha)$ be a maximal linearly independent subset of $L(\alpha)$. We suppose that $I(\alpha) \neq L(\alpha)$, since otherwise we have the same situation as in the case of transcendental α. Denote $D(\alpha) = L(\alpha) \setminus I(\alpha)$. For any element $d_m \in D(\alpha)$, the system $\{d_m\} \cup I(\alpha)$, clearly, is linearly dependent over $\mathbb{Q}$. Therefore, there exists a finite number of elements $i_{m_1}, \dots, i_{m_n} \in I(\alpha)$ such that, for some $k_0(m), \dots, k_n(m) \in \in \mathbb{Z} \setminus \{0\}$,

$$k_0(m)d_m + k_1(m)i_{m_1} + \dots + k_n(m)i_{m_n} = 0.$$

This implies the relation

$$m + \alpha = (m_1 + \alpha)^{-\frac{k_1(m)}{k_0(m)}} \dots (m_n + \alpha)^{-\frac{k_n(m)}{k_0(m)}}. \tag{2}$$

Now let $\mathcal{M}(\alpha) = \{m \in \mathbb{N}_0 : \log(m + \alpha) \in I(\alpha)\}$ and $\mathcal{N}(\alpha) = \{m \in \mathbb{N}_0 : \log(m + \alpha) \in D(\alpha)\}$. Define the torus

$$\Omega = \prod_{m \in \mathcal{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathcal{M}(\alpha)$. Then, similarly as above, Ω is a compact topological Abelian group, and we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H is the Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathcal{M}(\alpha)$.

If $m \in \mathcal{N}(\alpha)$ and relation (2) takes place, then we define $\omega(m)$ by

$$\omega(m) = \omega(m_1)^{-\frac{k_1(m)}{k_0(m)}} \dots \omega(m_n)^{-\frac{k_n(m)}{k_0(m)}},$$

where the principal values of the roots are taken. Thus, the functions $\omega(m)$ are defined for all $m \in \mathbb{N}_0$. Now, for $\sigma > \frac{1}{2}$, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ we define the complex-valued random element $\zeta(s, \alpha, \omega)$ by

$$\zeta(\sigma, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m + \alpha)^{\sigma}}.$$

There exists a Dubickas conjecture, see [3], [4], that there are algebraic irrational numbers α such that the product

$$\prod_{m=0}^{\infty} (m + \alpha)^{k_m},$$

where only a finite number of integers k_m are distinct from zero, for every collection $\underline{k} = (k_0, k_1, \dots)$, is irrational. Denote by $\mathcal{D}$ a class of algebraic irrational numbers with this property.

Theorem 2. *Suppose that α is algebraic irrational and $\alpha \in \mathcal{D}$, $h > 0$ is a fixed number such that $\exp\{\frac{2\pi}{h}\}$ is rational, and $\sigma > \frac{1}{2}$. Then the probability measure*

$$P_{N,\sigma}(A) \stackrel{def}{=} \mu_N(\zeta(\sigma + ilh, \alpha) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the distribution $P_{\zeta,\sigma}$ of the random variable $\zeta(\sigma, \alpha, \omega)$ as $N \to \infty$.

2. A limit theorem on the torus

In this section, we will consider the weak convergence of the probability measure

$$Q_N(A) \underset{def}{=} \mu_N(((m+\alpha)^{-ilh} : \ m \in \mathcal{M}(\alpha)) \in A), \quad A \in \mathcal{B}(\Omega).$$

Theorem 3. *Suppose that α and h are as in Theorem 2. Then the probability measure Q_N converges weakly to the Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ as $N \to \infty$.*

Proof. The dual group of Ω is isomorphic to

$$\bigoplus_{m \in \mathcal{M}(\alpha)} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathcal{M}(\alpha)$. An element $\underline{k} = \{k_m : m \in \mathcal{M}(\alpha)\} \in$ $\in \bigoplus_{m \in \mathcal{M}(\alpha)} \mathbb{Z}_m$, where only a finite number of integers k_m are non-zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_{m \in \mathcal{M}(\alpha)} \omega^{k_m}(m), \quad \omega \in \Omega.$$

Therefore, the Fourier transform $g_N(\underline{k})$ of the measure Q_N is given by

$$g_N(\underline{k}) = \int_{\Omega} \prod_{m \in \mathcal{M}(\alpha)} \omega^{k_m}(m) dQ_N,$$

where only a finite number of integers k_m are non-zero. Thus, we have that

$$
\begin{aligned}
g_N(\underline{k}) &= \frac{1}{N+1}\sum_{l=0}^{N}\prod_{m\in\mathcal{M}(\alpha)}(m+\alpha)^{-ik_m lh} = \\
&= \frac{1}{N+1}\sum_{l=0}^{N}\exp\left\{-ilh\sum_{m\in\mathcal{M}(\alpha)}k_m\log(m+\alpha)\right\}.
\end{aligned}
\tag{3}
$$

The system $I(\alpha)$ is linearly independent over $\mathbb{Q}$. Moreover, since $\alpha\in\mathcal{D}$, the number

$$\prod_{m\in\mathcal{M}(\alpha)}(m+\alpha)^{k_m} = \exp\left\{\sum_{m\in\mathcal{M}(\alpha)}k_m\log(m+\alpha)\right\}$$

is irrational. Since, by the choice of the number h, $\exp\{\frac{2\pi r}{h}\}$ is rational for every integer r, we find from (3) that

$$g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k}=\underline{0}, \\ \dfrac{1-\exp\{-i(N+1)h\sum\limits_{m\in\mathcal{M}(\alpha)}k_m\log(m+\alpha)\}}{(N+1)(1-\exp\{-ih\sum\limits_{m\in\mathcal{M}(\alpha)}k_m\log(m+\alpha)\})} & \text{if} \quad \underline{k}\neq\underline{0}. \end{cases}$$

Consequently,

$$\lim_{N\to\infty}g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k}=\underline{0}, \\ 0 & \text{if} \quad \underline{k}\neq\underline{0}. \end{cases}$$

This and Theorem 1.4.2 of [5] show that the measure Q_N converges weakly to m_H as $N\to\infty$.

3. Discrete limit theorems for absolutely convergent Dirichlet series

Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and let, for $m,n\in\mathbb{N}_0$,

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}.$$

Define

$$\zeta_n(s,\alpha)=\sum_{m=0}^{\infty}\frac{v_n(m,\alpha)}{(m+\alpha)^s}.$$

In [10] it was observed that the latter series converges absolutely for $\sigma>\frac{1}{2}$. Let $\omega_0(m)$ be a fixed element from the set of the functions $\omega(m)$ defined above. Then the series

$$\zeta_n(s,\alpha,\omega_0)=\sum_{m=0}^{\infty}\frac{\omega_0(m)v_n(m,\alpha)}{(m+\alpha)^s}$$

also converges absolutely for $\sigma>\frac{1}{2}$. Define on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ two probability measures $P_{N,n,\sigma}$ and $\widehat{P}_{N,n,\sigma}$ by

$$\mu_N(\zeta_n(\sigma+ilh,\alpha)\in A)$$

and

$$\mu_N(\zeta_n(\sigma+ilh,\alpha,\omega_0)\in A),$$

respectively.

Theorem 4. *Suppose that α and h are as in Theorem 2 and $\sigma>\frac{1}{2}$. Then on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{n,\sigma}$ such that both the measures $P_{N,n,\sigma}$ and $\widehat{P}_{N,n,\sigma}$ converge weakly to $P_{n,\sigma}$ as $N\to\infty$.*

Proof. Define the function $u_{n,\sigma}:\Omega\to\mathbb{C}$ by the formula

$$u_{n,\sigma}(\{\omega(m):m\in\mathcal{M}(\alpha)\})=\sum_{m=0}^{\infty}\frac{\omega(m)v_n(m,\alpha)}{(m+\alpha)^{\sigma}}.$$

Since the latter series converges uniformly in ω, the function $u_{n,\sigma}$ is continuous. Moreover,

$$u_{n,\sigma}(\{(m+\alpha)^{-ilh}:m\in\mathcal{M}(\alpha)\})=\zeta_n(\sigma+it,\alpha),$$

hence $P_{N,n,\sigma}=Q_N u_{n,\sigma}^{-1}$, where Q_N is the probability measure considered in Theorem 3, and

$$Q_N u_{n,\sigma}^{-1}(A)=Q_N(u_{n,\sigma}^{-1}A),\quad A\in\mathcal{B}(\mathbb{C}).$$

Therefore, Theorem 3 together with Theorem 5.1 of [1] show that the measure $P_{N,n,\sigma}$ converges weakly to $m_H u_{n,\sigma}^{-1}$ as $N\to\infty$.

Now define $u:\Omega\to\Omega$ by

$$u(\{\omega(m):m\in\mathcal{M}(\alpha)\})=\{\omega(m)\omega_0(m):m\in\mathcal{M}(\alpha)\}.$$

Then, obviously,

$$u_{n,\sigma}(u(\{(m+\alpha)^{-ilh} : m \in \mathcal{M}(\alpha)\})) = \zeta_n(\sigma + it, \alpha, \omega_0).$$

Therefore, similarly as in the case of the measure $P_{N,n,\sigma}$, we obtain that the measure $\widehat{P}_{N,n,\sigma}$, as $N \to \infty$, converges weakly to the measure $m_H(u_{n,\sigma}u)^{-1} = (m_H u^{-1})u_{n,\sigma}^{-1} = m_H u_{n,\sigma}^{-1}$ in view of the invariance of the Haar measure m_H. The theorem is proved.

4. Approximation in the mean

The functions $\zeta_n(s,\alpha)$ and $\zeta_n(s,\alpha,\omega)$ are auxiliary. To pass from them to $\zeta(s,\alpha)$ and $\zeta(s,\alpha,\omega)$ we need some results on approximation in the mean.

Theorem 5. *Let* $\sigma > \frac{1}{2}$. *Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{l=0}^{N} |\zeta(\sigma + ilh, \alpha) - \zeta_n(\sigma + ilh, \alpha)| = 0.$$

Proof. Let

$$l_n(s,\alpha) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)(n+\alpha)^s,$$

where σ_1 is the same as in Section 3, and $\Gamma(s)$ denotes the Euler gamma-function. Then in [6] it was obtained that, for $\sigma_2 > \frac{1}{2}$ and $\sigma > \sigma_2$,

$$\zeta(\sigma + it, \alpha) - \zeta_n(\sigma + it, \alpha) \ll$$

$$\ll \int_{-\infty}^{\infty} |\zeta(\sigma_2 + it + i\tau, \alpha) l_n(\sigma_2 - \sigma + i\tau, \alpha)| d\tau + \left| \frac{l_n(1 - \sigma - it, \alpha)}{1 - \sigma - it} \right|.$$

Hence we find that

$$\frac{1}{N+1} \sum_{l=0}^{N} |\zeta(\sigma + ilh, \alpha) - \zeta_n(\sigma + ilh, \alpha)| \ll \tag{4}$$

$$\ll \int_{-\infty}^{\infty} (|l_n(\sigma_2 - \sigma + i\tau, \alpha)|) \frac{1}{N} \sum_{l=0}^{N} |\zeta(\sigma_2 + ilh + i\tau, \alpha)| d\tau + o(1)$$

as $N \to \infty$. By Theorem 3.3.1 of [10] the mean square of $\zeta(s,\alpha)$

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it,\alpha)|^2 dt$$

is bounded for $\sigma > \frac{1}{2}$, $\sigma \neq 1$. This implies the estimate

$$\frac{1}{T}\int_0^T |\zeta'(\sigma+it,\alpha)|^2 dt \ll 1.$$

Now an application of the Gallagher lemma (see, for example [13], Lemma 1.4), shows that

$$\frac{1}{N}\sum_{l=0}^{N} |\zeta(\sigma_2+ilh+i\tau)| \ll \left(\frac{1}{N}\sum_{l=0}^{N} |\zeta(\sigma_2+ilh+i\tau)|^2\right)^{1/2} \ll 1+|\tau|.$$

Therefore, the left-hand side of (4) is estimated as

$$O\left(\int_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+i\tau)|(1+|\tau|)d\tau\right)+o(1). \tag{5}$$

Since $\sigma_2-\sigma<0$, the definition of $l_n(s,\alpha)$ yields

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+i\tau)|(1+|\tau|)d\tau = 0,$$

and this together with (5) proves the theorem.

Theorem 6. *Let $\sigma > \frac{1}{2}$ and α is algebraic irrational. Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{l=0}^{N} |\zeta(\sigma+ilh,\alpha,\omega)-\zeta_n(\sigma+ilh,\alpha,\omega)| = 0$$

for almost all $\omega \in \Omega$.

Proof. In [11], Lemma 8, it was proved that, for $\sigma > \frac{1}{2}$ and almost all $\omega \in \Omega$,

$$\frac{1}{T}\int_0^T |\zeta(\sigma+it,\alpha,\omega)|^2 dt \ll 1.$$

Therefore, the further proof runs in the same way as that of Theorem 5.

5. Proof of Theorem 2

On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ define one more probability measure

$$\widehat{P}_{N,\sigma}(A) = \mu_N(\zeta(\sigma + it, \alpha, \omega) \in A).$$

Theorem 7. *Suppose that α and $h > 0$ are as in Theorem 2, and $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that both the measures $P_{N,\sigma}$ and $\widehat{P}_{N,\sigma}$ converge weakly to P_σ as $N \to \infty$.*

Proof. By Theorem 4 both the measures $P_{N,n,\sigma}$ and $\widehat{P}_{N,n,\sigma}$ converge weakly to the same measure $P_{n,\sigma}$ as $N \to \infty$. We will prove that the family of probability measures $\{P_{n,\sigma} : n \in \mathbb{N}_0\}$ is tight, i.e. for every $\varepsilon > 0$ there exists a compact subset K such that $P_{n,\sigma}(K) \geq 1 - \varepsilon$ for all $n \in \mathbb{N}_0$.

Let M be an arbitrary positive number. Then the Chebyshev inequality yields

$$(6) \qquad P_{N,n\sigma}(\{z \in \mathbb{C} : |z| > M\}) = \mu_N(|\zeta_n(\sigma + ilh, \alpha)| > M) \leq$$

$$\leq \frac{1}{M(N+1)} \sum_{l=0}^{N} |\zeta_n(\sigma + ilh, \alpha)|.$$

An application of the Gallagher lemma gives the estimate

$$(7) \qquad \frac{1}{N+1} \sum_{l=0}^{N} |\zeta_n(\sigma + ilh, \alpha)| \ll \left(\frac{1}{N} \int_0^N |\zeta_n(\sigma + it, \alpha)|^2 dt \right)^{1/2}.$$

Moreover, since the series for $\zeta_n(s, \alpha)$ is absolutely convergent for $\sigma > \frac{1}{2}$, we have that

$$\lim_{N \to \infty} \frac{1}{N} \int_0^N |\zeta_n(\sigma + it, \alpha)|^2 dt = \sum_{m=0}^{\infty} \frac{v_n^2(m)}{(m+\alpha)^{2\sigma}} \ll \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} < \infty.$$

This, (6) and (7) show that

$$(8) \qquad \sup_{n \in \mathbb{N}} \limsup_{N \to \infty} P_{N,n,\sigma}(\{z \in \mathbb{C} : |z| > M\}) \leq CR,$$

where

$$R = \left(\sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} \right)^{1/2}.$$

Now let $\varepsilon > 0$ be arbitrary, and $M = CR\varepsilon^{-1}$. Then in virtue of (8)

$$\limsup_{N\to\infty} P_{N,n,\sigma}(\{z \in \mathbb{C} : |z| > M\}) \le \varepsilon. \tag{9}$$

The weak convergence of the measure $P_{N,n,\sigma}$ to $P_{n,\sigma}$ as $N \to \infty$ implies that of the probability measure

$$\mu_N(|\zeta_n(\sigma + ilh, \alpha)| \in A), \quad A \in \mathcal{B}(\mathbb{R}),$$

to the measure $P_{n,\sigma}u^{-1}$, where $u : \mathbb{C} \to \mathbb{R}$ is given by $u(z) = |z|$. Hence Theorem 2.1 of [1] and (9) give

$$P_{n,\sigma}(\{z \in \mathbb{C} : |z| > M\}) \le \liminf_{N\to\infty} P_{N,n,\sigma}(\{z \in \mathbb{C} : |z| > M\}) \le$$

$$\le \limsup_{N\to\infty} P_{N,n,\sigma}(\{z \in \mathbb{C} : |z| > M\}) \le \varepsilon. \tag{10}$$

Now we put $K_\varepsilon = \{z \in \mathbb{C} : |z| > M\}$. Then the set K_ε is compact, and by (10)

$$P_{n,\sigma}(K_\varepsilon) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$, i.e. the family $\{P_{n,\sigma} : n \in \mathbb{N}_0\}$ is tight. By the Prokhorov theorem, Theorem 6.1 of [1], this family is relatively compact. Therefore, there exists $\{P_{n_1,\sigma}\} \subset \{P_{n,\sigma}\}$ such that $P_{n_1,\sigma}$ converges to some measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $n_1 \to \infty$.

Define a discrete random variable θ_N on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ by the distribution law

$$\mathbb{P}(\theta_N = lh) = \frac{1}{N+1}, \quad l = 0, 1, \dots, N.$$

Let $X_{N,n}(\sigma) = \zeta_n(\sigma + i\theta_N, \alpha)$, and denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then the weak convergence of $P_{N,n,\sigma}$ to $P_{n,\sigma}$, as $N \to \infty$, is equivalent to

$$X_{N,n}(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} X_n(\sigma), \tag{11}$$

where $X_n(\sigma)$ is the random variable with distribution $P_{n,\sigma}$. Moreover, the weak convergence of $P_{n_1,\sigma}$ to P_σ, as $n_1 \to \infty$, implies the relation

$$X_{n_1}(\sigma) \xrightarrow[n_1\to\infty]{\mathcal{D}} P_n. \tag{12}$$

By Theorem 5 we have that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}(|X_N(\sigma) - X_{N,n}(\sigma)| \geq \varepsilon) =$$

$$= \lim_{n\to\infty} \limsup_{N\to\infty} \mu_N(|\zeta(\sigma+ilh,\alpha) - \zeta_n(\sigma+ilh,\alpha)| \geq \varepsilon) \leq$$

$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{l=0}^{N} |\zeta(\sigma+ilh,\alpha) - \zeta_n(\sigma+ilh,\alpha)| = 0.$$

Now this, (11), (12) and Theorem 4.2 of [1] give the relation

$$X_N(\sigma) \xrightarrow[N\to\infty]{\mathcal{D}} P_\sigma, \tag{13}$$

which is equivalent to the weak convergence of $P_{N,\sigma}$ to P_σ as $N \to \infty$. Moreover, (13) shows that the measure P_σ is independent of the choice of the sequence $\{P_{n_1,\sigma}\}$. Therefore, the relation

$$X_n(\sigma) \xrightarrow[n\to\infty]{\mathcal{D}} P_\sigma \tag{14}$$

takes place.

Now define

$$\widehat{X}_{N,n}(\sigma) = \zeta_n(\sigma + i\theta_N, \alpha, \omega)$$

and

$$\widehat{X}_N(\sigma) = \zeta(\sigma + i\theta_N, \alpha, \omega).$$

Then the above way together with (14) leads to weak convergence of $\widehat{P}_{N,\sigma}$ to P_σ as $N \to \infty$. The theorem is proved.

From Theorem 7 it follows that for the full proof of Theorem 2 it suffices to show the coincidence of the measures P_σ and $P_{\zeta,\sigma}$. For this, we need some results of ergodicity theory. We set

$$a_{h,\alpha} = \{(m+\alpha)^{-ih} : m \in \mathcal{M}(\alpha)\},$$

and define the measurable measure preserving transformation $\varphi_{h,\alpha}$ on Ω by $\varphi_{h,\alpha}(\omega) = a_{h,\alpha}\omega$, $\omega \in \Omega$. A set $A \in \mathcal{B}(\Omega)$ is called invariant with respect to the transformation $\varphi_{h,\alpha}$ if the sets A and $A_{h,\alpha} = \varphi_{h,\alpha}(A)$ differ one from another by a set of zero m_H-measure. All invariant sets form a sub-σ-field of $\mathcal{B}(\Omega)$. If this σ-field consists only of sets having m_H-measure equal to 0 or 1, then the transformation $\varphi_{h,\alpha}$ is ergodic.

Lemma 8. *The transformation $\varphi_{h,\alpha}$ is ergodic.*

Proof. Let $\chi : \Omega \to \gamma$ be a character of the group Ω. In the proof of Theorem 3 it was observed that

$$\chi(\omega) = \prod_{m\in\mathcal{M}(\alpha)} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero.

Let χ be a non-principal character. Then we have that

$$\chi(a_{h,\alpha}) = \prod_{m\in\mathcal{M}(\alpha)} (m+\alpha)^{-ihk_m}.$$

By hypotheses on α and h, $\chi(a_{h,\alpha}) \neq 1$. Therefore, the further proof runs in the same way as, for example in [7], Lemma 7.

Denote by $\mathbb{E}(X)$ the expectation of the random element X.

Lemma 9. *Let T be a measurable measure preserving ergodic transformation on the space $(\widetilde{\Omega}, \mathcal{B}(\widetilde{\Omega}), m)$. Then, for every $g \in L^1(\widetilde{\Omega}, \mathcal{B}(\widetilde{\Omega}), m)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(T^k(\widetilde{\omega})) = \mathbb{E}(g)$$

for almost all $\widetilde{\omega} \in \widetilde{\Omega}$.

The lemma is the Birkhoff theorem. Its proof can be found, for example, in [8], §1.2.

Proof of Theorem 2. Let A be a continuity set of the measure P_σ in Theorem 7, i.e. $P_\sigma(\partial A) = 0$, where ∂ denotes the boundary operator. Then Theorem 7 and Theorem 2.1 of [1] show that, for $\sigma > \frac{1}{2}$,

$$\lim_{N\to\infty} \mu_N(\zeta(\sigma + ilh, \alpha) \in A) = P_\sigma(A). \tag{15}$$

Now we fix the set A, and on $(\Omega, \mathcal{B}(\Omega), m_H)$ define a random variable θ by the formula

$$\theta(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(\sigma, \alpha, \omega) \in A, \\ 0 & \text{if} \quad \zeta(\sigma, \alpha, \omega) \notin A. \end{cases}$$

Then we have that

$$\mathbb{E}(\theta) = \int_{\Omega} \theta dm_H = m_H(\omega \in \Omega : \zeta(\sigma, \alpha, \omega) \in A) = P_{\zeta,\sigma}(A). \tag{16}$$

In view of Lemmas 8 and 9, for almost all $\omega \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{l=0}^{N} \theta(\varphi_{h,\alpha}^{l}(\omega)) = \mathbb{E}(\theta). \tag{17}$$

However, by the definition of θ and $\varphi_{h,\alpha}$,

$$\frac{1}{N+1} \sum_{l=0}^{N} \theta(\varphi_{h,\alpha}^{l}(\omega)) = \mu_N(\zeta(\sigma + ilh, \alpha, \omega) \in A).$$

Therefore, this, (16) and (17) show that, for almost all $\omega \in \Omega$,

$$\lim_{N\to\infty} \mu_N(\zeta(\sigma + ilh, \alpha, \omega) \in A) = P_{\zeta,\sigma}(A).$$

Thus, by (15), $P_\sigma(A) = P_{\zeta,\sigma}(A)$ for all continuity sets A of the measure P_σ. However, the system of all continuity sets constitute the determining class, therefore, $P_\sigma(A) = P_{\zeta,\sigma}(A)$ for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved.

References

[1] **Billingsley P.,** *Convergence of probability measures*, John Wiley, New York, 1968.

[2] **Cassels J.W.S.,** Footnote to a note of Davenport and Heilbronn, *J. London Math. Soc.,* **36** (1961), 171-184.

[3] **Dubickas A.,** Multiplicative dependence of quadratic polynomials, *Lith. Math. J.,* **38** (3) (1998), 225-231.

[4] **Drungilas P. and Dubickas A.,** A multiplicative dependence of shifted algebraic numbers, *Coloq. Math.*, **96** (1) (2003), 75-81.

[5] **Heyer H.,** *Probability measures on locally compact groups*, Springer Verlag, Berlin, 1977.

[6] **Ignatavičiūtė J.,** *Value-distribution of the Lerch zeta-function. Discrete version,* Doctoral Thesis, Vilnius University, 2003.

[7] **Kačinskaitė R. and Laurinčikas A.,** On the value distribution of the Matsumoto zeta-function, *Acta Math. Hungar.,* **105** (4) (2004), 339-359.

[8] **Krengel U.,** *Ergodic theorems,* Walter de Gruyter, Berlin-New York, 1985.

[9] **Laurinčikas A.,** A limit theorem for the Hurwitz zeta-function with algebraic irrational parameter, *Zapiski Nauchn. Sem. POMI*, **322** (2005), 125–134.

[10] **Laurinčikas A. and Garunkštis R.,** *The Lerch zeta-function,* Kluwer Academic Publishers, 2002.

[11] **Laurinčikas A. and Steuding J.,** A limit theorem for the Hurwitz zeta-function with an algebraic irrational parameter, *Arch. Math.,* **85** (2005), 419-432.

[12] **Laurinčikas A. and Steuding J.,** A limit theorem in the space of analytic functions for the Hurwitz zeta-function with an algebraic irrational parameter (submitted)

[13] **Montgomery H.L.,** *Topics in multiplicative number theory*, Springer Verlag, 1971.

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