Annales Univ. Sci. Budapest., Sect. Comp. 28 (2008) 197-212

ON THE DISTRIBUTION OF THE NUMBER OF DIGITS NEEDED TO WRITE THE FACTORIZATION OF AN INTEGER

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Dedicated to Professor Imre Kátai on the occasion of his 70th birthday

Abstract. Let $F_q(n)$ be the number of digits needed to write the factorization of n in base q. Several authors have studied the cardinality of the set of economical numbers, that is those integers n for which $F_q(n) \leq \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1$. The fact that the set of economical numbers is of zero density in the set of integers reveals nothing about the *normal* behavior of $F_q(n)$. In this note we study the central distribution of the function $F_q(n)$ and show that it is Gaussian.

1. Introduction and notations

Let $F_q(n)$ be the number of digits needed to write the factorization of n in base q. For example, $F_{10}(125) = F_{10}(5^3) = 2$ and $F_{10}(30) = F_{10}(2 \cdot 3 \cdot \cdot 5) = 3$. In 1995, Santos [7] introduced the notation of *economical numbers in base q*, $q \geq 2$, namely those integers n for which $F_q(n) \leq \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1$, meaning that the number of digits needed to write the factorization of n is smaller or equal to the number of digits appearing in its digital expansion in base q. Since then, several authors have studied the counting function of economical numbers, in particular De Koninck and Luca [3], [4], and more recently De Koninck, Doyon and Luca [5]. Here, for a fixed $q \geq 2$, we study the distribution function $H_q(x,y) := \sharp\{n < x : F_q(n) < y\}$ and more precisely the case where $y = y(x,c) = \frac{\log x}{\log q} + \frac{1}{2}\log\log x + c\sqrt{\log\log x}$. We show that in this case, the expression $G(c) = \lim\limits_{x\to\infty} \frac{1}{x}H_q(x,y)$ is well defined and that $G(c) = \Phi(\sqrt{3}c)$, where $\Phi(y) := \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{y} e^{\frac{-t^2}{2}}\,dt$ is the distribution function of the standard normal law.

For real number $y \geq 0$, we let $\lfloor y \rfloor$ stand for the largest integer smaller or equal to y and we write $\{y\} := y - \lfloor y \rfloor$ for its fractional part. As usual, the letter p will always denote a prime number, while $\pi(x)$ will stand for the number of prime numbers $p \leq x$. On the other hand, $\phi(y) := \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}$ stands for the density function of the standard normal law. Moreover, we let $\omega(n)$ stand for the number of distinct prime factors of n and we let $\gamma(n) := \prod\limits_{p|n} p$ be the *kernel* of n. Finally, by $\log\log x$ we mean $\max(1, \log\log x)$.

2. The main results

It is clear that

$$F_q(n) := \sum_{p|n}\left(\left\lfloor\frac{\log p}{\log q}\right\rfloor + 1\right) + \sum_{\substack{a\geq 2\\ p^a \| n}}\left(\left\lfloor\frac{\log a}{\log q}\right\rfloor + 1\right).$$

The first sum counts the number of digits needed to write the prime factors of n while the second counts the number of digits needed to write the exponents ≥ 2. Using the identities

$$\left\lfloor\frac{\log p}{\log q}\right\rfloor = \frac{\log p}{\log q} - \left\{\frac{\log p}{\log q}\right\} \quad \text{and} \quad \sum_{p|n}\frac{\log p}{\log q} = \frac{\log\gamma(n)}{\log q},$$

it is easily seen that

$$F_q(n) := \frac{\log\gamma(n)}{\log q} + \sum_{p|n}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right) + \sum_{\substack{a\geq 2\\ p^a \| n}}\left(\left\lfloor\frac{\log a}{\log q}\right\rfloor + 1\right),$$

which can also be written as

$$(1) \qquad F_q(n) = \frac{\log n}{\log q} - h_1(n) + h_2(n) + h_3(n),$$

where

$$h_1(n) := \frac{\log(n/\gamma(n))}{\log q},$$

$$h_2(n) := \sum_{\substack{a \geq 2 \\ p^a \| n}} \left(\left\lfloor \frac{\log a}{\log q} \right\rfloor + 1 \right),$$

$$h_3(n) := \sum_{p|n} \left(1 - \left\{ \frac{\log p}{\log q} \right\} \right).$$

Let $H_q(x, y)$ be the distribution function of F_q, that is

$$H_q(x, y) := \sharp\{n < x : F_q(n) < y\}$$

and consider the function

$$G(c) = \lim_{x \to \infty} \frac{1}{x} H_q \left(x, \frac{\log x}{\log q} + \frac{1}{2} \log\log x + c\sqrt{\log\log x} \right).$$

Theorem 1. *For each real number c,*

$$G(c) = \Phi(\sqrt{3}c).$$

Remark. The fact that the function $G(c)$ is well defined is in itself an interesting result.

The following theorem reveals the interval in which the function $F_q(n)$ takes its values.

Theorem 2. *For each integer $q \geq 2$ and each integer $n \geq 2$,*

$$\left\lfloor \frac{\log\log\left(n^{1/\omega(n)}\right)}{\log q} \right\rfloor + \omega(n) \leq F_q(n) \leq \left\lfloor \frac{\log n}{\log q} \right\rfloor + 2\omega(n).$$

3. Preliminary results

The first lemma contains classical estimates on powerful numbers. Recall that a positive integer is said to be a *powerful number* if $p|n$ implies that $p^2|n$.

But first, some notation. Given a positive integer n, we shall write $n = uv$, where

$$u = u(n) := \prod_{p \| n} p$$

and

$$v = v(n) := \frac{n}{u},$$

so that u is the square free part of n and v its powerful part.

Lemma 1. *As $y \to \infty$,*

(i)
$$\sum_{\substack{n > y \\ p \mid n \Rightarrow p^2 \mid n}} \frac{1}{n} \ll \frac{1}{\sqrt{y}},$$

(ii)
$$\sharp\{n < x : v(n) > y\} \ll \frac{x}{\sqrt{y}},$$

where the implicit constant does not depend on x.

Proof of Lemma 1. For (i), see De Koninck and Kátai [2].

To establish (ii), we simply observe that it follows from (i) that

$$\sharp\{n < x : v(n) > y\} \le \sum_{\substack{v > y \\ p \mid v \Rightarrow p^2 \mid v}} \frac{x}{v} \ll \frac{x}{\sqrt{y}}.$$

Lemma 2. *There exist two positive constants c_1 and c_2 such that, as $x \to \infty$,*

$$\sum_{p \le x} \frac{1}{p} = \log\log x + c_1 + O\left(\exp\{-c_2(\log x)^{3/5}\}\right).$$

Proof of Lemma 2. It is known (see Vinogradov [8]) that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\exp\{-c_2(\log x)^{3/5}\}\right)\right),$$

where γ is Euler's constant. Taking logarithms on both sides, we easily see that

$$\sum_{p \le x} \frac{1}{p} = \log\log x + \gamma - \sum_{p \le x,\ \nu \ge 2} \frac{1}{\nu p^\nu} + \log\left(1 + O\left(\exp\{-c_2(\log x)^{3/5}\}\right)\right) =$$

$$= \log\log x + \gamma - \sum_{p,\ \nu\geq 2} \frac{1}{\nu p^\nu} + O\left(\frac{1}{x}\right) + O\left(\exp\{-c_2(\log x)^{3/5}\}\right) =$$

$$= \log\log x + c_1 + O\left(\exp\{-c_2(\log x)^{3/5}\}\right),$$

as required.

Lemma 3. (Central limit theorem) *Let $X_1, X_2, \ldots$ be independent random variables and let*

$$\begin{aligned} \mu_i &= E[X_i], \\ \sigma_i^2 &= E[(X_i - \mu_i)^2], \\ r_i^3 &= E[(X_i - \mu_i)^3]. \end{aligned}$$

If

$$\lim_{n\to\infty} \frac{\left(\sum_{i=1}^{n} r_i^3\right)^{1/3}}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} = 0,$$

then

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} < y\right) = \Phi(y).$$

Proof of Lemma 3. This is Lyapunov's condition in the Central Limit Theorem. For a proof this classical result, see Bernstein [1].

Lemma 4. *For each fixed integer $q \geq 2$ and each fixed integer $r \geq 1$, we have, as $x \to \infty$,*

$$\sum_{p<x} \frac{1}{p} \left\{\frac{\log p}{\log q}\right\} = \frac{\log\log x}{2} + O(1),$$

$$\sum_{p<x} \frac{1}{p} \left\{\frac{\log p}{\log q}\right\}^r = \frac{\log\log x}{r+1} + O\left(\sqrt{\frac{\log\log x}{r}}\right).$$

Proof of Lemma 4. We first establish the second relation. To do so, we call upon the following inequality which is valid for all positive integers k and r:

$$\sum_{p<x}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r=\sum_{j=0}^{k-1}\sum_{\substack{p<x\\ \frac{j}{k}\le\left\{\frac{\log p}{\log q}\right\}<\frac{j+1}{k}}}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r. \tag{2}$$

The sum on the right hand side can be written as

$$\sum_{\substack{p<x\\ \frac{j}{k}\le\left\{\frac{\log p}{\log q}\right\}<\frac{j+1}{k}}}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r=\sum_{\ell=0}^{\left\lfloor\frac{\log x}{\log q}\right\rfloor}\sum_{q^{\ell+\frac{j}{k}}\le p<\min\left(q^{\ell+\frac{j+1}{k}},x\right)}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r. \tag{3}$$

On the other hand, observe that

$$\sum_{q^{\ell+\frac{j}{k}}\le p<q^{\ell+\frac{j+1}{k}}}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r=\left(\frac{j}{k}+\frac{\xi}{k}\right)^r\sum_{q^{\ell+\frac{j}{k}}\le p<q^{\ell+\frac{j+1}{k}}}\frac{1}{p} \tag{4}$$

for some real ξ such that $|\xi|<1$. Using Lemma 2 (replacing the error term by $O(1/\log^2 x)$, say), we obtain

$$\begin{aligned}\sum_{q^{\ell+\frac{j}{k}}\le p<q^{\ell+\frac{j+1}{k}}}\frac{1}{p}&=\log\log\left(q^{\ell+\frac{j+1}{k}}\right)-\log\log\left(q^{\ell+\frac{j}{k}}\right)+O\left(\frac{1}{\ell^2\log^2 q}\right)=\\ &=\log\left(\ell+\frac{j+1}{k}\right)-\log\left(\ell+\frac{j}{k}\right)+O\left(\frac{1}{\ell^2\log^2 q}\right)=\\ &=\frac{1}{k\ell}+O\left(\frac{1}{k\ell^2}\right)+O\left(\frac{1}{\ell^2\log^2 q}\right).\end{aligned} \tag{5}$$

Combining relations (3), (4) and (5), we obtain that

$$\sum_{\substack{p<x\\ \frac{j}{k}\le\left\{\frac{\log p}{\log q}\right\}<\frac{j+1}{k}}}\frac{1}{p}\left\{\frac{\log p}{\log q}\right\}^r= \tag{6}$$

$$=\left(\frac{j}{k}+\frac{\xi}{k}\right)^r\sum_{\ell=1}^{\left\lfloor\frac{\log x}{\log q}\right\rfloor}\left(\frac{1}{k\ell}+O\left(\frac{1}{k\ell^2}\right)+O\left(\frac{1}{\ell^2\log^2 q}\right)\right).$$

Observe also that

$$\text{(7)} \qquad \sum_{\ell=1}^{\left\lfloor \frac{\log x}{\log q} \right\rfloor} \left(\frac{1}{k\ell} + O\left(\frac{1}{k\ell^2}\right) + O\left(\frac{1}{\ell^2 \log^2 q}\right) \right) = \frac{1}{k} \log\log x + O(1).$$

Combining relations (2), (6) and (7) with the identity

$$\left(\frac{j}{k} + \frac{\xi}{k}\right)^r = \frac{j^r}{k^r} + O\left(\frac{r(j+1)^{r-1}}{k^r}\right),$$

we obtain

$$\text{(8)} \qquad \sum_{p<x} \frac{1}{p} \left\{ \frac{\log p}{\log q} \right\}^r =$$

$$= \sum_{j=0}^{k-1} \left(\frac{j^r}{k^{r+1}} \log\log x + O\left(\frac{r(j+1)^{r-1}}{k^{r+1}} \log\log x\right) + O\left(\frac{j^r}{k^r}\right) \right) + O(1).$$

The right hand side member of (8) is equal to

$$\text{(9)} \qquad \frac{1}{r+1} \log\log x + O\left(\frac{(k+1)^r}{k^{r+1}} \log\log x\right) + O\left(\frac{k}{r}\right) + O(1).$$

Choosing $k = \lfloor \sqrt{r \log\log x} \rfloor$, the proof of the second relation of Lemma 4 then follows from relations (8) and (9).

In order to prove the first relation of Lemma 4, we first observe that, using the Prime Number Theorem in the form $\sum\limits_{p<x} \frac{\log p}{p} = \log x + O(1)$, we have

$$\sum_{p<x} \frac{1}{p} \left\{ \frac{\log p}{\log q} \right\} = \sum_{p<x} \frac{1}{p} \frac{\log p}{\log q} - \sum_{p<x} \frac{1}{p} \left\lfloor \frac{\log p}{\log q} \right\rfloor =$$
$$= \frac{\log x}{\log q} + O(1) - \sum_{p<x} \frac{1}{p} \left\lfloor \frac{\log p}{\log q} \right\rfloor .$$

Moreover,

$$\sum_{p<x} \frac{1}{p} \left\lfloor \frac{\log p}{\log q} \right\rfloor = \sum_{j=0}^{\left\lfloor \frac{\log x}{\log q} \right\rfloor} j \sum_{q^j < p \le \min(x, g^{j+1})} \frac{1}{p}.$$

Using Lemma 2 (replacing the error term by $O(1/\log^3 x)$, say) we obtain

$$\sum_{q^j < p \le q^{j+1}} \frac{1}{p} = \log\log q^{j+1} - \log\log q^j + O\left(\frac{1}{j^3 \log^3 q}\right) =$$
$$= \frac{1}{j} - \frac{1}{2j^2} + O\left(\frac{1}{j^3}\right) \qquad (j \ge 1).$$

We may therefore conclude that

$$\sum_{p<x} \frac{1}{p} \left\lfloor \frac{\log p}{\log q} \right\rfloor = \sum_{j=1}^{\left\lfloor \frac{\log x}{\log q} \right\rfloor - 1} \left(1 - \frac{1}{2j} + O\left(\frac{1}{j^2}\right)\right) + O(1) =$$
$$= \frac{\log x}{\log q} - \frac{\log\log x}{2} + O(\log\log q) + O(1),$$

which proves the first equation of Lemma 4 and thus completes the proof of the lemma.

Let x be a large fixed positive integer and set

$$R := x \prod_{p<x} p.$$

We consider the set $U = \{n < R\}$ with the probability measure

$$P(S) = \frac{\sharp S}{R}, \qquad \text{for each } \; S \subseteq U.$$

For each prime number $p < x$, we introduce the random variables

$$\xi_p(n) := \begin{cases} 1 - \left\{\frac{\log p}{\log q}\right\} & \text{if } p | n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5. *For each prime number $p < x$, the following equalities hold:*

$$\mu_p := E[\xi_p] = \frac{1}{p}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right),$$
$$\sigma_p^2 := E\left[(\xi_p - \mu_p)^2\right] = \left(\frac{1}{p} - \frac{1}{p^2}\right)\left(1 - \left\{\frac{\log p}{\log q}\right\}\right)^2,$$
$$E\left[(\xi_p - \mu_p)^3\right] = \left(\frac{1}{p} - \frac{3}{p^2} + \frac{2}{p^3}\right)\left(1 - \left\{\frac{\log p}{\log q}\right\}\right)^3.$$

Proof of Lemma 5. Since for each prime number $p < x$, we have $p|R$, the random variables ξ_p are independent. Moreover, one can easily verify the following equalities:

$$P\left(\xi_p = 1 - \left\{\frac{\log p}{\log q}\right\}\right) = \frac{1}{p},$$
$$P(\xi_p = 0) = \frac{p-1}{p}.$$

From these, it follows immediately that

$$E[\xi_p] = \frac{1}{p}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right), \tag{10}$$

$$E[\xi_p^2] = \frac{1}{p}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right)^2, \tag{11}$$

$$E[\xi_p^3] = \frac{1}{p}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right)^3 \tag{12}$$

All three equalities of Lemma 5 then easily follow from (10), (11) and (12).

Lemma 6. *For each real number y,*

$$\lim_{x\to\infty} P\left(\frac{\sum\limits_{p<x} \xi_p - \frac{1}{2}\log\log x}{\sqrt{\frac{1}{3}\log\log x}} < y\right) = \Phi(y).$$

Proof of Lemma 6. This result follows from Lemmas 4, 5 and 3 (Central Limit Theorem).

On the same probability space $\{n < R\}$, we define the random variables

$$\chi_p(n) := \begin{cases} 1 - \left\{\frac{\log p}{\log q}\right\} & \text{if } p|a, \\ 0 & \text{otherwise.} \end{cases}$$

where a is the smallest positive integer such that $a \equiv n \pmod{x}$.

Lemma 7. *As* $x \to \infty$,

$$E\left[\left|\sum_p \xi_p - \sum_p \chi_p\right|\right] < \frac{\pi(x)}{x} = \frac{1+o(1)}{\log x}.$$

Proof of Lemma 7. We need to observe that

$$P\left(\chi_p = 1 - \left\{\frac{\log p}{\log q}\right\}\right) = \frac{1}{R}\sharp\{n < R : p|a\} = \frac{1}{R}\frac{R}{x}\left\lfloor\frac{x}{p}\right\rfloor = \frac{1}{x}\left\lfloor\frac{x}{p}\right\rfloor.$$

Indeed, it then follows that

$$E[\chi_p] = \left(1 - \left\{\frac{\log p}{\log q}\right\}\right)\frac{1}{x}\left\lfloor\frac{x}{p}\right\rfloor = \frac{1}{p}\left(1 - \left\{\frac{\log p}{\log q}\right\}\right) + \frac{\xi}{x}$$

for some $|\xi| < 1$. Hence

$$| E[\chi_p] - E[\xi_p] | < \frac{1}{x}$$

and therefore

$$E\left[\left|\sum_p \xi_p - \sum_p \chi_p\right|\right] \leq \sum_{p<x} |E[\chi_p] - E[\xi_p]| < \frac{\pi(x)}{x},$$

which completes the proof of Lemma 7.

Lemma 8. *As* $x \to \infty$,

$$P\left(\left|\sum_{p<x} \xi_p - \sum_{p<x} \chi_p\right| > 1\right) < \frac{1+o(1)}{\log x}.$$

Proof of Lemma 8. This result is an immediate consequence of Lemma 7 and the Markov inequality (see for instance Galambos [6], p.150).

Lemma 9. *Given a fixed integer* $N \geq 2$, *let* $\alpha_i \geq t \geq N^{1/(N-1)}$ *for* $i = 1, \ldots, N$. *Then*

$$\sum_{i=1}^{N} \alpha_i \leq \frac{1}{c}\prod_{i=1}^{N} \alpha_i,$$

where $c = \dfrac{t^{N-1}}{N}$.

Proof of Lemma 9. Assume that $\alpha_i \geq t \geq N^{1/(N-1)}$ for $i = 1, \ldots, N$ and that

$$\sum_{i=1}^{N} \alpha_i > \frac{1}{c} \prod_{i=1}^{N} \alpha_i.$$

We then have

$$\sum_{i=1}^{N} \alpha_i > \left(\frac{N}{t^{N-1}} \frac{\prod_{i=1}^{N} \alpha_i}{\alpha_j} \right) \alpha_j \qquad (j = 1, \ldots, N).$$

Observe that, using the fact that $\alpha_i \geq t$,

$$\frac{N}{t^{N-1}} \frac{\prod_{i=1}^{N} \alpha_i}{\alpha_j} \geq N \qquad (j = 1, \ldots, N).$$

We therefore obtain that for each integer $j = 1, \ldots, N$,

$$\sum_{i=1}^{N} \alpha_i > N\alpha_j,$$

which contradicts the fact that

$$\sum_{i=1}^{N} \alpha_i \geq N \max_i \alpha_i,$$

thus completing the proof of Lemma 9.

4. The proofs of the main results

Proof of Theorem 1. Assume that $n \leq x$ satisfies the inequality

$$v(n) < \log\log n.$$

By Lemma 1 (ii), we thus omit at most $\frac{x}{\sqrt{\log\log x}}$ integers $n \leq x$. By the definition of the function $h_1(n)$, we then obtain

$$h_1(n) = O(\log\log\log n). \tag{13}$$

Moreover, by definition, we have

$$h_2(n) \le \omega(v(n)) \left(\left\lfloor \frac{\log\left(\frac{\log v(n)}{\log 2}\right)}{\log q} \right\rfloor + 1 \right).$$

It follows from this that

$$h_2(n) \ll \frac{\log v(n)}{\log\log v(n)} \log\log v(n) \ll \log\log\log n. \tag{14}$$

Hence, combining (13) and (14), we have

$$h_1(n) + h_2(n) = O(\log\log\log n). \tag{15}$$

Assume also that $\frac{x}{\log\log x} < n < x$, so that

$$\frac{\log n}{\log q} = \frac{\log x}{\log q} + O(\log\log\log x). \tag{16}$$

Combining (1), (13), (15) and (16), we obtain

$$\sharp\left\{n < x : F_q(n) < \frac{\log x}{\log q} + w\right\} = \tag{17}$$

$$= \sharp\{n < x : h_3(n) < w + O(\log\log\log x)\} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

Calling upon the identity

$$h_3(n) = \sum_{p<x} \chi_p(n), \tag{18}$$

it follows from (17) and (18) that

$$\sharp\left\{n < x : F_q(n) < \frac{\log x}{\log q} + w\right\} = \tag{19}$$

$$= \sharp\left\{n < x : \sum_{p<x} \chi_p(n) < w + O(\log\log\log x)\right\} + O\left(\frac{x}{\sqrt{\log\log x}}\right).$$

By the definition of the $\chi_p(n)$, we have that

$$\sharp\left\{n<x:\sum_{p<x}\chi_p(n)<w+O(\log\log\log x)\right\}= \tag{20}$$

$$=\frac{x}{R}\sharp\left\{n<R:\sum_{p<x}\chi_p(n)<w+O(\log\log\log x)\right\}.$$

On the other hand, by Lemma 8, we have that

$$\sharp\left\{n<R:\sum_{p<x}\chi_p(n)<w+O(\log\log\log x)\right\}= \tag{21}$$

$$=\sharp\left\{n<R:\sum_{p<x}\xi_p(n)<w+O(\log\log\log x)\right\}+O\left(\frac{R}{\log x}\right).$$

From (21) and Lemma 7, it then follows that

$$\sharp\left\{n<R:\sum_{p<x}\chi_p(n)<\frac{1}{2}\log\log x+c\sqrt{\log\log x}+O(\log\log\log x)\right\}= \tag{22}$$

$$=R(1+o(1))\Phi(\sqrt{3}c).$$

Combining (19), (20) and (22), we finally obtain

$$\sharp\left\{n<x:F_q(n)<\frac{\log x}{\log q}+\frac{1}{2}\log\log x+c\sqrt{\log\log x}\right\}=x(1+o(1))\Phi(\sqrt{3}c),$$

thus completing the proof of Theorem 1.

Proof of Theorem 2. We first proof the upper bound. We have

$$F_q(n)=\sum_{p|n}\left(\left\lfloor\frac{\log p}{\log q}\right\rfloor+1\right)+\sum_{\substack{p^a\|n\\ a\geq 2}}\left(\left\lfloor\frac{\log a}{\log q}\right\rfloor+1\right)\leq$$

$$\leq\frac{\log\left(\prod_{p^a\|v}ap\right)}{\log q}+2\omega(n).$$

Since $a^{\frac{1}{a-1}} \leq 2$ for each $a \geq 2$, we have that $ap \leq p^a$ for each prime $p \geq 2$. Hence,

$$F_q(n) \leq \frac{\log n}{\log q} + 2\omega(n),$$

thus establishing the upper bound.

We now prove the lower bound. As before, we write $n = u(n)v(n)$. Since $(u(n), v(n)) = 1$, we have

$$\begin{aligned} F_q(n) &\geq \sum_{p|u(n)} \max\left(1, \frac{\log p}{\log q}\right) + \sum_{p^a \| (n)} \max\left(2, \frac{\log\log p^a}{\log q}\right) = \\ &= \frac{1}{\log q}\left(\sum_{p|u(n)} \max(\log q, \log p) + \sum_{p^a \| v(n)} \max\left(\log q^2, \log\log p^a\right)\right) \geq \\ &\geq \frac{1}{\log q} \sum_{p^a \| n} \max(\log q, \log\log p^a) = \\ &= \frac{1}{\log q} \log\left(\prod_{p^a \| n} \max(q, \log p^a)\right). \end{aligned}$$

Using Lemma 9, we then have

$$\begin{aligned} F_q(n) &\geq \frac{1}{\log q} \log\left(\frac{q^{\omega(n)-1}}{\omega(n)} \sum_{p^a \| n} \max(q, \log p^a)\right) \geq \\ &\geq \frac{1}{\log q} \log\left(\frac{q^{\omega(n)-1}}{\omega(n)} \log n\right) = \\ &= \frac{\log\log n}{\log q} + \omega(n) - 1 - \frac{\log \omega(n)}{\log q}. \end{aligned}$$

Moreover, since $\dfrac{\log\log n^{\frac{1}{\omega(n)}}}{\log q}$ is not an integer for $n, q \geq 2$, it follows that

$$\begin{aligned} F_q(n) &\geq \left\lceil \frac{\log\log n^{\frac{1}{\omega(n)}}}{\log q} + \omega(n) - 1 \right\rceil = \\ &= \left\lceil \frac{\log\log n^{\frac{1}{\omega(n)}}}{\log q} \right\rceil + \omega(n) - 1 = \\ &= \left\lfloor \frac{\log\log n^{\frac{1}{\omega(n)}}}{\log q} \right\rfloor + \omega(n), \end{aligned}$$

thus establishing the lower bound and completing the proof of Theorem 2.

5. Final remarks

The study of the behavior of the function $H_q(x, y)$ is still very much uncharted. For instance, for any fixed value of y, Theorem 2 only reveals that $H_q(\infty, y) < \infty$. Hence obtaining a general fairly good estimate for $H_q(x, y)$ is certainly an interesting challenge. On the other hand, we believe that the result for economical numbers could be generalized to yield

$$H_q\left(x, \frac{\log x}{\log q} + c\log\log x\right) = \frac{x}{(\log x)^{R(q,c)+o(1)}} \qquad \left(x \to \infty,\ -\infty < c < \frac{1}{2}\right).$$

To prove or disprove this claim and moreover to describe the behavior of the function $R(q, c)$ in the eventuality that the claim is true would also be very interesting.

Acknowledgement. The first author was supported in part by a grant from NSERC.

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