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ON THE STABILITY OF THE R-RATIONAL CHOICE FUNCTION

B. Bodó (Gabčíkovo, Slovakia)

M. Kovács (Budapest, Hungary)

Dedicated to Professor Imre Kátai
on the occasion of his 70th birthday

Abstract. In this paper our discussion attaches the R-max- and the extended R-max-rational decision mechanisms which save the choices on the sets of the given optional set system. We execute under what conditions will reveal an R-max-rational or an extended R-max-rational decision mechanism nonempty choices for the sets not included into the optional set system. We give conditions for the existence, decisiveness and stability of these structures.

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1. Introduction

The early phase of the theory of choice is due to Samuelson [10], [12] and Houthakker [5]. They applied it for the analysis of the consumer demand on competitive markets introducing the revealed preference. This line of research which is based on the revealed preferences has been continued by lots of researchers, e.g. Bossert-Sprumont-Suzumura [3], Hansson [4], Richter [8], [9], Sen [13], Suzumura [14], etc. In our days this theory plays role in decision theory and the different fields of behavior sciences, too.

Our investigation is motivated by the decision theory, namely by tender-evaluation. In the tender-problem there is a task for which a solver is demanded. From the possible solvers is built the set of alternatives. The different solvers can use different technologies to solve the subproblems, the prices and the termination times for the subproblems may also be different. So, on the base of the mentioned characteristics as decision standpoints it is possible to classify the alternatives. The subsets composed by this classification will be the subsets of the optional set system, from which the decision maker chooses. (Remark, that the economist often refer to the optional set system as *budget set*, but we do not use this terminology because we want to avoid the illusion that the basis of the choice can only be the money.) The question is whether is it possible to build such preference on the base of choices, which decides about the choice of the most appropriate alternative.

There are a lot of papers dealing with this problem, which characterize the revealed by the choices preferences. The advantage of the revealed preferences is the following: The optional set system can be incomplete, only a subset of all possible subsets of alternatives, nevertheless by the revealed preference derives the set of the best elements of the subsets not included into the optional set system.

However, the incompleteness of the optional set system propounds new questions:

- Is it suitable for the choice from the whole set of alternatives, i.e. when the revealed preference will be decisive for the whole set of alternatives;
- Is there such revealed preference which derives nonempty choice for all subsets of the alternatives, i.e. when the revealed preference will be stable;
- If the revealed preference is not decisive or not stable, is there any weakening of this preference making it to be decisive, and what is more to be stable without changing the choice on the set of optional set system.

These problems will be in the focus of our investigations. Mainly the last problem is essential, because the positive answer makes possible to take into the best alternatives new alternatives using new decision standpoints.

2. Preliminaries

In this section we survey the basic definitions and well known theorems related to the decision structures and mechanisms, mainly those which are connected with the choice functions. We touch upon only those earlier results, which will be used in our investigations.

Let Ω be a final *set of alternatives* with the cardinality $|\Omega|$ and let denote 2^{Ω} the powerset of Ω. Let $\mathcal{B} \subseteq 2^{\Omega} \setminus \emptyset$ be given. We will refer to this set system as *optional set system.*

In the practice the elements of the optional set system are such subsets of the alternatives, which can be treated by the same decision standpoints. For example, in a tender-problem we can collect those alternatives into an element of the optional set system, which use the same technology in the solution of a subtask of the project.

Definition 2.1. *The set-to-set function* $C : \mathcal{B} \to 2^{\Omega}$ $C(X) \subseteq X$ *will be called* choice function *on* $\mathcal{B}$. *If* $C : \mathcal{B} \to \mathcal{B}$, *then we say, that* C *injective.*

For illustration of the choice function let again refer to a tender problem. The choice function on the subset of alternatives using the same technology in the solution of a subtask of the project tells us, which alternatives are more appropriate for the user. The terms of the tender in this question can be, for example, the choice of the cheaper variants or the alternatives realizing the shorter production time.

Definition 2.2. *The decision structure given by the triplet* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *will be called* real decision mechanism *if it satisfies the following conditions:*

1. $\emptyset \notin \mathcal{B}$;
2. *The optional set system* $\mathcal{B} \subseteq 2^{\Omega}$ *covers the set* Ω *i.e.* $\Omega = \bigcup_{X \in \mathcal{B}} X$;
3. $C(X) \neq \emptyset \quad \forall X \in \mathcal{B}$;
4. *The optional set system may contain a set of* $\mathcal{B} \subseteq 2^{\Omega}$ *at most once.*

Moreover, if the condition

5. $\mathcal{B} = 2^{\Omega} \setminus \emptyset$

is also fulfilled, then we say about perfect decision mechanism.

Definition 2.3. *We say that the decision structure* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *is* P-max-rational, normal *or* free of P-max-contradictions *if there exists a binary relation* P *on* Ω *such that for all* $X \in \mathcal{B}$ *the choice* $C(X)$ *is the set of the maximal elements of* X, *i.e.*

$$C(X) = C_P^{MAX}(X) \quad \forall X \in \mathcal{B},$$

where the set of maximal elements is defined by the following formula:

$$C_P^{MAX}(X) = \{x \in X : xPy \ \ \forall y \in X\}. \tag{2.1}$$

We say that the relation P which corresponds to the definition is a *rationalization* of the decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$.

Here and in the following P will denote an arbitrary binary relation on Ω and $\overline{P}$ will denote its complement. If we use a special relation, the notation will be indicative of this speciality.

In this paper will be used some relations with special property, namely

- P is *reflexive* if aPa for all $a \in \Omega$;
- P is *complete* if for all pairs $(a, b) \in \Omega \times \Omega$ either aPb or bPa or both hold.

A great part of publications deals with only the perfect decision mechanism. Since in this paper we analyze the effect of the modification of the optional set system for the rationality of the decision structure, therefore we always assume that $\mathcal{B} \neq 2^{\Omega} \setminus \emptyset$, and what is more, we usually assume that $\Omega \notin \mathcal{B}$. This latter condition is meaningful from practical point of view. Indeed, the task of the decision making is to choose the best alternative(s) from the set of all possible alternatives using the given decision. If the mechanism directly can define this choice for the whole set of alternatives, then the task of decision making looses its meaning.

Every real decision mechanism reveals two binary relation on Ω, namely (see in [8] and [11]):

Definition 2.4. *R is the C* revealed Richter-relation *on $\mathcal{B}$ if*

$$xRy \Leftrightarrow \exists X \in \mathcal{B} : x \in C(X), \quad y \in X. \tag{2.2}$$

Definition 2.5. *S is the C* revealed Samuelson-relation *on $\mathcal{B}$ if*

$$xSy \Leftrightarrow \exists X \in \mathcal{B} : x \in C(X), \quad y \in X \setminus C(X). \tag{2.3}$$

The revealed relations depend very much on the optional set system $\mathcal{B}$ which defines the real decision mechanism, and on the choice function $C(X)$, $X \in \mathcal{B}$ given on the optional set system.

Otherwise, we have to mention that a real decision mechanism given by $\mathcal{D} = (\Omega, \mathcal{B}, C)$ is not always rational by neither the Richter- nor the Samuelson-relation. However, we have the following easily verifiable lemmas (see, e.g. in [3] and [7]):

Lemma 2.1. *If P is a P-*max*-rationalization of the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$, then $R \subseteq P$, where R is the revealed Richter-relation by $\mathcal{D} = (\Omega, \mathcal{B}, C)$. Consequently, if the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$ is R-*max*-rational, then R is the weakest* max*-rationalization of $\mathcal{D}$.*

Lemma 2.2. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism. Then for all $X \in \mathcal{B}$ the following inclusions are valid*

$$C_S^{ND}(X) \subseteq C(X) \subseteq C_R^{MAX}(X), \tag{2.4}$$

where

$$C_S^{ND}(X) = \{x \in X : y\overline{S}x \ \ \forall y \in X\}, \tag{2.5}$$

Let us remark that for any relation P

$$C_P^{ND}(X) = C_{P^d}^{MAX}(X) \ \ \forall X \in \mathcal{B},$$

where $P^d = \overline{P^{-1}}$ is the dual of the relation P. So (2.5) can be defined as the set of maximal elements corresponding to the dual relation S^d of S.

According to the Definition 2.3 the terminology R-max-contradictory will be used not only for the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$ if

$$\exists X \in \mathcal{B} \text{ such that } C(X) \subset C_R^{MAX}(X),$$

but also for those subsets from $\mathcal{B}$, where the strict inclusion accomplishes.

The revealing power of the Richter- and Samuelson-relations resides in that they can assign choice to those subsets of alternatives which do not appear in the optional set system. This fact will be used in the following discussions of the rationality and rationalizability of the decision mechanism, but our investigation will be restricted to the R-max-rationality.

Lots of papers deal with the problem of the rationality of a decision mechanism. Without the claim to the entirety we cite some significant papers, namely [1], [3], [4], [7], [8], [9], [13], [14], etc.

3. Characterization of the extended R-max-rationalizations

The starting point of our discussion will be an R-max-rational decision mechanism.

Definition 3.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism. The relation P_R will be called* extended R-max rationalization revealed by *$\mathcal{D}$, if $R \subset P_R$ and $C_{P_R}^{MAX}(X) = C_R^{MAX}(X) \ \forall X \in \mathcal{B}$. The set of all extended R-max rationalizations revealed by $\mathcal{D}$ will be denoted by $\mathcal{P}_R$.*

It is not too difficult to find such R-max-rational real decision mechanism, which has no extended R-max rationalization revealed by $\mathcal{D}$. Otherwise, it is possible that a real decision mechanism has several different R-max extensions revealed by $\mathcal{D}$.

These situations are illustrated with the Examples 3.1. and 3.2.

Example 3.1. Let us consider the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$, where: $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ for all $X \in \mathcal{B}$ and the revealed Richter-relation are given by the Table 1.

Table 1:

$X \in \mathcal{B}$	$C(X)$
a b	b
a c	a
a d	d
b c	c
b d	b
c d	c d

Decision structure of Example 3.1

R	a	b	c	d
a	1	0	1	0
b	1	1	0	1
c	0	1	1	1
d	1	0	1	1

Richter relation for Example 3.1

It is easy to see, that this real decision mechanism is R-max-rational and one can also verify that any modification of the Richter-relation can not be max-rationalization.

Example 3.2. Let us consider the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$, where: $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ for all $X \in \mathcal{B}$ and the revealed Richter-relation are given by the Table 2.

Table 2:

$X \in \mathcal{B}$	$C(X)$
a b c	c
a b	b
b c	c
b c d	d
a d	a

Decision structure of Example 3.2

R	a	b	c	d
a	1	0	0	1
b	1	1	0	0
c	1	1	1	0
d	0	1	1	1

Richter relation for Example 3.2

One can control that in the Example 3.2 the Richter-relation is a max-rationalization, but it has three other max-rationalizations saving the choice on $\mathcal{B}$. These rationalizations are given in the Table 3.

Table 3:

P_1	a	b	c	d
a	1	0	1	1
b	1	1	0	0
c	1	1	1	0
d	0	1	1	1

P_1-max-rationalization

P_2	a	b	c	d
a	1	0	0	1
b	1	1	0	1
c	1	1	1	0
d	0	1	1	1

P_2-max-rationalization for Example 3.2

P_3	a	b	c	d
a	1	0	1	1
b	1	1	0	1
c	1	1	1	0
d	0	1	1	1

P_3-max-rationalization

From these examples it can be seen that there are some pairs $(x, y) \in \Omega \times \Omega$ of alternatives such that $x\overline{R}y$ stands between them, but can not be changed in P after all. To find the condition of this fact let us introduce the relations on $\Omega \times \Omega$ given in the following two definitions.

Definition 3.2. *We say that the relation P_{sc} on $\Omega \times \Omega$ is the* strict complementary part *of the relation P if*

$$xP_{sc}y \Leftrightarrow \quad x\overline{P}y \text{ and } \exists Y \in \mathcal{B} : x, y \in Y, x \notin C_P^{MAX}(Y), \text{ but } xPz\ \forall z \in Y \setminus \{y\}. \tag{3.1}$$

The strict complementary part of a relation P defines those pairs of alternatives between which the complement relation can not be changed without changing the P-max-choice on the sets of $\mathcal{B}$.

Definition 3.3. *We will call* atomization *of a relation P the set $\mathcal{Q}$ containing the not empty relations Q^{ab}, which are defined as follows:*

$$xQ^{ab}y \Leftrightarrow x = a, \quad y = b \text{ and } aPb. \tag{3.2}$$

For the Example 3.2 the strict complementary part of R and the atomizations of $\overline{R} \cap \overline{R_{sc}}$ are shown in the Table 4.

Proposition 3.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism. P_R is an R-max-extended rationalization revealed by $\mathcal{D} = (\Omega, \mathcal{B}, C)$ if and only if there exists a finite chain of R-max-relations $\{P^{(i)},\ i = 0, \ldots, k\}$, with $k \geq 1$ such that*

1. $P^{(0)} = R;\quad P^{(i-1)} \subset P^{(i)},\ i = 1, \ldots, k;\quad P^{(k)} = P_R;$

Table 4:

R_{sc}	a	b	c	d
a	0	1	0	0
b	0	0	1	0
c	0	0	0	1
d	1	0	0	0

R_{sc}
for Example 3.2

Q^{ac}	a	b	c	d
a	0	0	1	0
b	0	0	0	0
c	0	0	0	0
d	0	0	0	0

Q^{bd}	a	b	c	d
a	0	0	0	0
b	0	0	0	1
c	0	0	0	0
d	0	0	0	0

Atomizations of $\overline{R} \cap \overline{R_{sc}}$

2. *for all $i = 1, \ldots, k$ there exists a pair of alternatives $(a_i, b_i) \in \Omega \times \Omega$ such that*
$$P^{(i)} \cap \overline{P^{(i-1)}} = Q_{i-1}^{a_i b_i},$$
where Q_{i-1}^{ab} is the atomization of $\overline{P^{(i-1)}} \cap \overline{P_{sc}^{(i-1)}}$ belonging to the pair (a, b) and $P_{sc}^{(i-1)}$ is the strict complementary part of $P^{(i-1)}$.

The length k of the chain will be called the distance between R and P_R *and will be denoted by $d(R, P_R)$.*

Proof. *Sufficiency.* Let P be defined as it is given in the proposition. According to the definition of the strict complementary part for all $i = 1, \ldots, k$ the atomization $Q_{i-1}^{a_i b_i}$ changes the relation in $P^{(i-1)}$ only such pair, which does not effect for the choice from the sets of the optional set system $\mathcal{B}$. Therefore, $C_{P^{(i)}}^{MAX}(X) = C_{P^{(i-1)}}^{MAX}(X)$ for all $X \in \mathcal{B}$ and for all $i = 1, \ldots, k$. Consequently,

$$\begin{aligned} C(X) &= C_R^{MAX}(X) = C_{P^{(0)}}^{MAX}(X) = \ldots = C_{P^{(i-1)}}^{MAX}(X) = C_{P^{(i)}}^{MAX}(X) = \\ &= \ldots = C_{P^{(k)}}^{MAX}(X) = C_{P_R}^{MAX}(X) \quad \forall X \in \mathcal{B}. \end{aligned}$$

So, for all $i = 1, \ldots, k$ the relation $P^{(i)}$ is an extended R-max-rationalization.

Necessity. Let P_R be an extended R-max-rationalization. Then

$$\emptyset \neq P_R \cap \overline{R} = \bigcup_{i=1}^{k} Q^{a_i b_i} \subseteq \overline{R_{sc}} \cap \overline{R},$$

where $Q^{a_i b_i}$ is an atomization of $\overline{R_{sc}} \cap \overline{R}$. Otherwise,

$$\begin{aligned} P_R &= R \cup \left(\bigcup_{i=1}^{k} Q^{a_i b_i} \right) = \\ &= ((R \cup Q^{a_1 b_1}) \cup Q^{a_2 b_2}) \ldots \cup Q^{a_k b_k}. \end{aligned}$$

Introducing the relations

$$P^{(0)} = R;\ \ P^{(i-1)} = P^{(i)} \cup Q^{a_i b_i},\ i = 1\ldots,k;\ \ P^{(k)} = P_R$$

we get a sequential generation of P_R. Since P_R is an extended R-max-rationalization, the changes in R can not change the choice from the set of optional set system $\mathcal{B}$, therefore no one $P^{(i)}$ can change them, so $Q^{a_i b_i}$ must be the atomization of $\overline{P^{(i-1)}} \cap \overline{P_{sc}^{(i-1)}}$ for all $i = 1, \ldots, k$, i.e. $Q^{a_i b_i} = Q^{a_i b_i}_{i-1}$ for all $i = 1, \ldots, k$. □

To create the set $\mathcal{P}_R$ of all possible extended R-max-rationalization we can follow the Algorithm 1.

Algorithm 1 Create all extending R-max-rationalizations

$\mathcal{P}_R \Leftarrow \emptyset$ {initialize the set of all R-max-rationalization}
$\mathcal{P} \Leftarrow \{R\}$ {start with the Richter relation}
while $\mathcal{P} \neq \emptyset$ **do**
  select $P \in \mathcal{P}$
  define P_{sc}
  $\mathcal{Q} \Leftarrow$ the set of atomizations Q^{ij} of $\overline{P} \cap \overline{P_{sc}}$
  while $\neq \emptyset$ **do**
    select $Q^{ij} \in \mathcal{Q}$
    $P_R \Leftarrow P_R \cup (\overline{P} \cap Q^{ij})$
    $\mathcal{P}_R \Leftarrow \mathcal{P}_R \cup \{P_R\}$
    $\mathcal{P} \Leftarrow \mathcal{P} \cup \{P_R\}$
  end while{atomizations}
  $\mathcal{P} \Leftarrow \mathcal{P} \setminus \{P_R\}$
end while{rationalizations}
return $\mathcal{P}_R$

The next question is, when the extended R-max-rationalization revealed by $\mathcal{D} = (\Omega, \mathcal{B}, C)$ does not exist. We give a sufficient condition to answer this question.

Proposition 3.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism, where $\mathcal{B}$ satisfies the following conditions:*

1. *$\mathcal{B}$ contains all subsets of two elements of Ω;*
2. *For all $a \in \Omega$ there exists $X \in \mathcal{B}$ such that $a \in C(X)$.*

Then $\mathcal{P}_R = \emptyset$.

Proof. Firstly, let us observe that R is a reflexive relation. Indeed, let $a \in \Omega$ be an arbitrarily chosen alternative. From the second assumption of the proposition follows that there exists $X \in \mathcal{B}$ such that $a \in C(X)$, then according to the definition of the Richter-relation aRa.

Let us now assume on the contrary, that there exist $P_R \supset R$ such that

$$C_{P_R}^{MAX}(X) = C_R^{MAX}(X) \;\; \forall X \in \mathcal{B}.$$

$P_R \supset R$ means that there exist $a, b \in \Omega$ such that aP_Rb, but $a\overline{R}b$ and $a\overline{R_{sc}}b$. From the reflexivity follows that $a \neq b$. But, $\{a, b\} \setminus \{b\} = \{a\}$ and aRa, and this leads to the contradicting $aR_{sc}b$. □

Corollary 3.1.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-*max*-rational real decision mechanism, where $\mathcal{B}$ contains all subsets of one or two elements of Ω. Then $\mathcal{P}_R = \emptyset$.*

Proof. Trivial. □

4. Decisive and stable R-max-rationalizations

In this section we will execute how effect the R-max-rationalization and the extended R-max-rationalization revealed by the real decision mechanism $\mathcal{D} = = (\Omega, \mathcal{B}, C)$ for the revealed choice from the subsets not included in the optional set system $\mathcal{B}$. A specific interest has the revealed choice from the whole set Ω of alternatives since the final aim of the decision maker is to tell which alternatives are the best or the most acceptable from all alternatives. For any alternative $x^* \in \Omega$ let us introduce the set systems

$$\begin{aligned} \mathcal{B}(x^*) &= \{X \in \mathcal{B} : x^* \in X\}, \\ \mathcal{B}^*(x^*) &= \{X \in \mathcal{B} : x^* \in C(X)\} \end{aligned}$$

and the sets

$$Z(x^*) = \bigcup \{Y : Y \in \mathcal{B}^*(x^*)\}.$$

In this and the following sections we will use the following assumptions:

(A-1) For all $a \in \Omega$ $\{a\} \in \mathcal{B}$ with the choice $C(\{a\}) = \{a\}$.

(A-2) $|\Omega| \geq 2$.

(A-1) guaranties that $Z(x^*) \neq \emptyset$ for all $x^* \in \Omega$ and the Richter relation is reflexive. However, for the simplicity, in the examples we omit the one element subsets from the description of the real decision mechanisms.

(A-2) is technical assumption which excludes the trivial, in the practice uninteresting cases.

Proposition 4.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)** *and let $X \in 2^{\Omega} \setminus \emptyset$. Then $C_R^{MAX}(X) \neq \emptyset$ if and only if there exists $x^* \in X$ such that $X \subseteq Z(x^*)$.*

Proof. *Necessity.* Let $x^* \in C_R^{MAX}(X)$. Then $x^* R y \quad \forall y \in X$. It means that for all $y \in X$ there exists $Y \in \mathcal{B}(y)$ such that $x^* \in C(Y)$. Consequently, for all $y \in X$ we have that $y \in Y \in \mathcal{B}^*(x^*)$, therefore

$$X \subseteq \bigcup \{Y : Y \in \mathcal{B}^*(x^*)\} = Z(x^*).$$

Sufficiency. Let us now assume that there exist $x^* \in \Omega$ such that the inclusion $Z(x^*) \supseteq X$ holds. Then

$$x^* R y \quad \forall y \in Y \text{ and } \forall Y \in \mathcal{B}^*(x^*).$$

From it we obtain that

$$x^* R y \quad \forall y \in \bigcup_{Y \in \mathcal{B}^*(x^*)} Y = Z(x^*).$$

Consequently, $x^* R y \quad \forall y \in X$, i.e. $x^* \in C_R^{MAX}(\Omega)$. □

Definition 4.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)**, *where $\Omega \notin \mathcal{B}$. It is* R-max-decisive *if $C_R^{MAX}(\Omega) \neq \emptyset$.*

Proposition 4.2. *The real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$ satisfying the assumptions* **(A-1)** *and* **(A-2)** *is* R-max-*decisive if and only if there exists $x^* \in \Omega$ such that $Z(x^*) = \Omega$.*

Proof. The proposition is a corollary of the Proposition 4.1 applying it for the set $X = \Omega$. □

The real decision mechanisms defined in the Example 3.1 and Example 3.2 are not R-max-decisive.

We have to remark that the decisiveness does not imply that the revealed R-max-choice $C_R^{MAX}(X') \neq \emptyset$ if $X' \notin \mathcal{B}$. To see it let us consider Example 4.1.

Example 4.1. Let us consider the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$, where $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ for all $X \in \mathcal{B}$ and the revealed Richter-relation are given by the Table 5.

In this example the real decision mechanism is decisive, since $C_R^{MAX}(\Omega) = \{c\}$, but $C_R^{MAX}(\{a, d\}) = \emptyset$.

Table 5:

$X \in \mathcal{B}$	$C(X)$
a b c	b c
b c d	c d

Decision structure of Example 4.1

R	a	b	c	d
a	1	0	0	0
b	1	1	1	0
c	1	1	1	1
d	0	1	1	1

Richter relation for Example 4.1

Definition 4.2. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism. It is R-max-stable if $C_R^{MAX}(X) \neq \emptyset$ for all $X \in 2^\Omega \setminus \emptyset$.*

Proposition 4.3. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an real decision mechanism, which satisfies the assumptions* **(A-1)** *and* **(A-2)**. *It is R-max-stable if and only if for all $X \in 2^\Omega \setminus \emptyset$ there exists $x_X^* \in X$ such that $X \subseteq Z(x_X^*)$.*

Proof. The proposition is a consequence of the Proposition 4.1 using it for all $X \in 2^\Omega \setminus \emptyset$. □

Proposition 4.4. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-stable real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)**. *Then R is complete.*

Proof. Let us assume on contrary that R is not complete. Then there exists at least one pair $(x, y) \in \Omega \times \Omega$ such that $x\overline{R}y$ and $y\overline{R}x$. Then neither x nor y can not be chosen from the subset $\{x, y\} \in 2^\Omega \setminus \emptyset$, i.e. $C_R^{MAX}(\{x, y\}) = \emptyset$ contradicting to the stability. □

As it has been seen in Example 4.1 the decisiveness does not imply, in general, the stability. But it is obvious, the absence of the decisiveness implies the instability. To see an R-max-decisive R-max-stable real decision mechanism let us consider the Example 4.2.

Example 4.2. Let us consider the real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$, where $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ for all $X \in \mathcal{B}$ and the revealed Richter-relation are given by the Table 6.

Table 6:

$X \in \mathcal{B}$	$C(X)$
$a\ b\ c$	$b\ c$
$a\ b\ \ d$	b
$b\ c$	$b\ c$
$a\ \ c$	c
$a\ \ \ d$	d

Decision structure of Example 4.2

R	a	b	c	d
a	1	0	0	0
b	1	1	1	1
c	1	1	1	0
d	1	0	1	1

Richter relation for Example 4.2

5. Weakly R-max-decisive and weakly R-max-stable real decision mechanisms

In the most practical models neither the decisiveness nor the stability can not be guaranteed. Therefore we formulate the analogous concepts of decisiveness and stability for the extended R-max-rationalizations.

Definition 5.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism, where $\Omega \notin \mathcal{B}$. It is* weakly R-max-decisive *if it is not R-max-decisive but there exists an extended R-max-rationalization $P_R \supset R$ such that $C_{P_R}^{MAX}(\Omega) \neq \emptyset$ and the distance $d(R, P_R)$ is minimal.*

Definition 5.2. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be an R-max-rational real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)**. *It is* weakly R-max-stable *if it is not R-max-stable, but there exists an extended R-max-rationalization $P \supset R$ such that $C_P^{MAX}(X) \neq \emptyset$ for all $X \in 2^\Omega \setminus \emptyset$ and the distance $d(R, P_R)$ is minimal.*

Proposition 5.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)**. *Let $X \in (2^\Omega \setminus \emptyset) \setminus \mathcal{B}$. There exists extended R-max-rationalization $P_R \supset R$ with minimal distance $d(R, P_R)$ such that $C_{P_R}^{MAX}(X) \neq \emptyset$ if and only if there exists $x^* \in X$ for which one of the following conditions fulfills:*

1. *$X \subseteq Z(x^*) = \Omega$ and $\exists (x, y) \in \Omega \times \Omega$ such that $x \neq x^*$, $x\overline{R}y$ and $x\overline{R_{sc}}y$;*
2. *$X \subseteq Z(x^*)$, $\Omega \setminus Z(x^*) \neq \emptyset$ and there exists $y \in \Omega \setminus Z(x^*)$ such that $x^*\overline{R}y$ and $x^*\overline{R_{sc}}y$;*

3. $X \setminus Z(x^*) \neq \emptyset$ *and for all* $Y \in \mathcal{B}$ *if* $x^* \in Y$, *then either* $Y \subseteq Z(x^*)$ *or* $(Y \setminus Z(x^*)) \setminus (X \setminus Z(x^*)) \neq \emptyset$.

Proof. According to the Proposition 4.1. if $X \subset Z(x^*)$ then $C_R^{MAX}(X) \neq \emptyset$, therefore the minimal distance $d(R, P_R) = 1$. It means that

$$P^{(1)} = P^{(0)} \cup Q_0^{x,y} = R \cup Q_0^{x,y},$$

where $Q_0^{x,y}$ can be any atomization of the relation $\overline{R} \cap \overline{R_{sc}}$ belonging to the pair $(x,y) \in \Omega \times \Omega$ satisfying the relations $x\overline{R}y$ and $x\overline{R_{sc}}y$.

If $\Omega = Z(x^*)$ then it is necessary that $(x,y) \in \Omega \times (\Omega \setminus \{x^*\})$, and if $\Omega \subset Z(x^*)$, then the pairs $(x,y) \in (\{x^*\} \times (\Omega \setminus Z(x^*)$ are possible for changes.

Let now $X \setminus Z(x^*) = \{y_1, \dots, y_k\}$. The minimal number of changes in R to guarantee $x^* \in C_P^{MAX}(X)$ with some $P \supset R$ may only be realized by the chain

$$P^{(0)} = R,\ P^{(i-1)} = P^{(i)} \cup Q^{x^*,y_i},\ P_R = P^{(k)},$$

where the relation $Q^{a,b}$ is defined by (3.2), i.e.

$$P_R = R \bigcup \Big(\bigcup_{y \in X \setminus Z(x^*)} Q^{x^*,y} \Big). \tag{5.1}$$

This relation will be an extended R-max rationalization if and only if there does not exist $Y \in \mathcal{B}$ for which $x^* \notin C(Y) = C_R^{MAX}(Y)$, but $x^* \in C_{P_R}^{MAX}(Y)$.

Since

$$Y = (Y \cap Z(x^*)) \cup ((Y \setminus Z(x^*)) \cap (X \setminus Z(x^*))) \cup ((Y \setminus Z(x^*)) \setminus (X \setminus Z(x^*))),$$

we have

$$x^* R y \qquad \forall y \in Y \cap Z(x^*), \tag{5.2}$$

$$x^* \overline{R} y \ \text{ but } \ x^* P_R y \qquad \forall y \in (Y \setminus Z(x^*)) \cap (X \setminus Z(x^*)), \tag{5.3}$$

$$x^* \overline{R} y \ \text{ but } \ x^* \overline{P_R} y \qquad \forall y \in (Y \setminus Z(x^*)) \setminus (X \setminus Z(x^*)). \tag{5.4}$$

From this follows that the choice from Y does not change in and only in the following two cases:

If $Y \setminus Z(x^*) = \emptyset$, i.e. only (5.2) holds. In this case $Y \subseteq Z(x^*)$ and according to the Proposition 4.1. $x^* \in C(X)$.

If there exists $y \in (Y \setminus Z(x^*)) \setminus (X \setminus Z(x^*))$, i.e. at least one $y \in Y$ satisfies (5.4) then $x^* \overline{P_R} y$, so $x^* \notin C_{P_R}^{MAX}(Y)$. □

Proposition 5.2. *The real decision mechanism* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *satisfying the assumptions* **(A-1)** *and* **(A-2)** *is weakly* R-max-*decisive if and only if there exists* $x^* \in \Omega$ *such that for all* $Y \in \mathcal{B}$ *if* $x^* \in Y$ *then* $Y \subseteq Z(x^*)$. *In this case the extending* R-max-*rationalization is given by (5.1) and (3.2).*

Proof. This proposition is an outcome of the Proposition 5.1 applying it for $X = \Omega$. The other than $Y \subseteq Z(x^*)$ conditions given in that proposition never hold. □

Proposition 5.3. *The real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$ satisfying the assumptions* **(A-1)** *and* **(A-2)** *is weakly R-max-stable if and only if for all $X \in (2^\Omega \setminus \emptyset) \setminus \mathcal{B}$ there exists $x^*_X \in X$ such that for all $Y \in \mathcal{B}$ if $x^*_X \in Y$ then either $Y \subseteq Z(x^*_X)$ or $(Y \setminus Z(x^*_X)) \cap (X \setminus Z(x^*)) \neq \emptyset$.*

Proof. This is a direct outcome of the Proposition 5.1. applying it for all $X = \Omega$. □

Proposition 5.4. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a weakly R-max-stable real decision mechanism satisfying the assumptions* **(A-1)** *and* **(A-2)***. Then the extended R-max-rationalization P_R is complete.*

Proof. The proof is analogous to the proof of the Proposition 4.4. □

To see weakly decisive and weakly stable real decision mechanisms let us consider the following examples:

Example 5.1. Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism, where $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ for all $X \in \mathcal{B}$, the revealed Richter-relation and the extended R-max-rationalization are given in the Table 7.

Table 7:

$X \in \mathcal{B}$	$C(X)$
$a\ b\ c$	a
$b\ c\ d$	b
$a\ c$	a
$b\ d$	$b\ d$
$b\ c$	$b\ c$

Decision structure of Example 5.1

R	a	b	c	d
a	1	1	1	0
b	0	1	1	1
c	0	1	1	0
d	0	1	0	1

Richter relation for Example 5.1

P_R	a	b	c	d
a	1	1	1	**1**
b	0	1	1	1
c	0	1	1	0
d	0	1	0	1

Extended R-max-rationalization for Example 5.1

The real decision mechanism given in the Example 5.1 is weakly decisive with $C^{MAX}_{P_R}(\Omega) = \{a\}$, but it is not weakly stable since $C^{MAX}_{P_R}(\{c, d\}) = \emptyset$ and neither cP_Rd nor dP_Rc can not be involved since $cR_{sc}d$ and $dR_{sc}c$.

Table 8:

$X \in \mathcal{B}$	$C(X)$
a b	a
b c	b c
c d	d

Decision structure of Example 5.2

R	a	b	c	d
a	1	1	0	0
b	0	1	1	0
c	0	1	1	0
d	0	0	1	1

Richter relation for Example 5.2

P_R	a	b	c	d
a	1	1	**1**	**1**
b	0	1	1	**1**
c	0	1	1	0
d	0	0	1	1

Extended R-max-rationalization for Example 5.2

Example 5.2. Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism, where $\Omega = \{a, b, c, d\}$, $\mathcal{B}$ and $C(X)$ $\forall X \in \mathcal{B}$, the revealed Richter-relation and the extended R-max-rationalization are given in the Table 8.

The real decision mechanism given in the Example 5.2 is weakly decisive and weakly stable.

6. Concluding remarks

The obtained results show that to reach the stability by weakening the revealed preference can lead to the inclusion new decision standpoints into the decision mechanism. However, it is possible, that weakening of the revealed preference relation is not unique, and in such a way the choice-set from the whole set of alternatives can also be not univocal. In this case we can speak about the possibility of the corruption.

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B. Bodó
Department of Operations Research
Eötvös Loránd University
Pázmány Péter sétány 1/c.
H-1117 Budapest, Hungary

Mailing address:
Bodó Beáta
Temető u. 631.
Gabčíkovo
93005 Slovakia

M. Kovács
Department of Operations Research
Eötvös Loránd University
Pázmány Péter sétány 1/c.
H-1117 Budapest, Hungary

Mailing address:
Kovács Margit
Bánkút u. 53/a.
H-1154 Budapest
Hungary