

INFINITE PRODUCTS WITH STRONGLY B -MULTIPLICATIVE EXPONENTS

J.-P. Allouche (Orsay, France)

J. Sondow (New York, USA)

To Professor Kátai on the occasion of his 70th birthday

Abstract. Let $N_{1,B}(n)$ denote the number of ones in the B -ary expansion of an integer n . Woods introduced the infinite product

$$P := \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(-1)^{N_{1,2}(n)}}$$

and Robbins proved that $P = 1/\sqrt{2}$. Related products were studied by several authors. We show that a trick for proving that $P^2 = 1/2$ (knowing that P converges) can be extended to evaluating new products with (generalized) strongly B -multiplicative exponents. A simple example is

$$\prod_{n \geq 0} \left(\frac{Bn+1}{Bn+2} \right)^{(-1)^{N_{1,B}(n)}} = \frac{1}{\sqrt{B}}.$$

1. Introduction

In 1985 the following infinite product, for which no closed expression is known, appeared in [8, p.193 and p.209]:

$$R := \prod_{n \geq 1} \left(\frac{(4n+1)(4n+2)}{4n(4n+3)} \right)^{\varepsilon(n)},$$

where $(\varepsilon(n))_{n \geq 0}$ is the ± 1 Prouhet-Thue-Morse sequence, defined by

$$\varepsilon(n) = (-1)^{N_{1,2}(n)}$$

with $N_{1,2}(n)$ being the number of ones in the binary expansion of n . (For more on the Prouhet-Thue-Morse sequence, see for example [5].)

On the one hand, it is not difficult to see that $R = \frac{3}{2Q}$, where

$$Q := \prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{\varepsilon(n)}$$

Namely, splitting the simpler product into even and odd indices and using the relations $\varepsilon(2n) = \varepsilon(n)$ and $\varepsilon(2n+1) = -\varepsilon(n)$, we get

$$\begin{aligned} Q &= \left(\prod_{n \geq 1} \left(\frac{4n}{4n+1} \right)^{\varepsilon(n)} \right) \left(\prod_{n \geq 0} \left(\frac{4n+2}{4n+3} \right)^{-\varepsilon(n)} \right) = \\ &= \frac{3}{2} \prod_{n \geq 1} \left(\frac{4n(4n+3)}{(4n+1)(4n+2)} \right)^{\varepsilon(n)} = \frac{3}{2R}. \end{aligned}$$

(Note that, whereas the logarithm of R is an absolutely convergent series, the logarithm of Q - and similarly the logarithm of the product P below - is a conditionally convergent series, as can be seen by partial summation, using the fact that the sums $\sum_{0 \leq k \leq n} \varepsilon(k)$ only take the values $+1$, 0 and -1 , hence are bounded.)

On the other hand, the product Q reminds us of the Woods-Robbins product [18, 12]

$$P := \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{\varepsilon(n)} = \frac{1}{\sqrt{2}}$$

(generalized for example in [13, 1, 2, 3, 4, 14]).

In 1987 during a stay at the University of Chicago, the first author, convinced that the computation of the infinite product Q should not resist

the even-odd splitting techniques he was using with J. Shallit, discovered the following trick. First write QP as

$$QP = \left(\frac{1}{2}\right)^{\varepsilon(0)} \prod_{n \geq 1} \left(\frac{2n}{2n+1} \cdot \frac{2n+1}{2n+2}\right)^{\varepsilon(n)} = \frac{1}{2} \prod_{n \geq 1} \left(\frac{n}{n+1}\right)^{\varepsilon(n)}$$

Now split the indices as we did above, obtaining

$$\prod_{n \geq 1} \left(\frac{n}{n+1}\right)^{\varepsilon(n)} = \left(\prod_{n \geq 1} \left(\frac{2n}{2n+1}\right)^{\varepsilon(n)}\right) \left(\prod_{n \geq 0} \left(\frac{2n+1}{2n+2}\right)^{-\varepsilon(n)}\right) = QP^{-1}.$$

This gives $QP = \frac{1}{2}QP^{-1}$: as the hope of computing Q fades, the trick at least yields an easy way to compute $P = 1/\sqrt{2}$. By extending this trick to B -ary expansions, the second author [14] found the generalization of $P = 1/\sqrt{2}$ given in Corollary 5 of Section 4.2.

It happens that the sequence $(\varepsilon(n))_{n \geq 0}$ is strongly 2-multiplicative (see Definition 1 in the next section). The purpose of this paper is to extend the trick to products with more general exponents. For example, we prove the following.

Let $B > 1$ be an integer. For $k = 1, \dots, B-1$ define $N_{k,B}(n)$ to be the number of occurrences of the digit k in the B -ary expansion of the integer n . Also, let

$$s_B(n) := \sum_{0 < k < B} k N_{k,B}(n)$$

be the sum of the B -ary digits of n , and let $q \nmid 1$ be an integer. Then

$$\prod_{n \geq 0} \left(\frac{Bn+k}{Bn+k+1}\right)^{(-1)^{N_{k,B}(n)}} = \frac{1}{\sqrt{B}},$$

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv 0 \pmod{q}}} \left(\frac{Bn+k}{Bn+k+1}\right)^{\sin \frac{\pi k}{q} \sin \frac{\pi(2s_B(n)+k)}{q}} = \frac{1}{\sqrt{B}},$$

and

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv 0 \pmod{q}}} \left(\frac{Bn+k}{Bn+k+1}\right)^{\sin \frac{\pi k}{q} \cos \frac{\pi(2s_B(n)+k)}{q}} = 1.$$

Note that the use of the trick is not necessarily the only way to compute products of this type: real analysis is used for computing P in [12] and to compute products more general than P in [13]; the core of [1] is the use of Dirichlet series, while [2] deals with complex power series and the second part of [3] with real integrals. It may even happen that, in some cases, the use of the trick gives less general results than other methods. For example, in Remark 5 we show that Corollary 5 or [14] can also be obtained as an easy consequence of [2, Theorem 1].

2. Strongly B -multiplicative sequences

We recall the classical definition of a strongly B -multiplicative sequence. (For this and for the definitions of B -multiplicative, B -additive, and strongly B -additive, see [6, 9, 7, 11, 10].)

Definition 1. Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geq 0}$ is *strongly B -multiplicative* if $u(0) = 1$ and, for all $n \geq 0$ and all $k \in \{0, 1, \dots, B-1\}$,

$$u(Bn + k) = u(n)u(k).$$

Example 1. If z is any complex number, then the sequence u defined by $u(0) := 1$ and $u(n) := z^{s_B(n)}$ for $n \geq 1$ is strongly B -multiplicative.

Remark 1. If we do not impose the condition $u(0) = 1$ in Definition 1, then either $u(0) = 1$ holds, or the sequence $(u(n))_{n \geq 0}$ must be identically 0. To see this, note that the relation $u(Bn + k) = u(n)u(k)$ implies, with $n = k = 0$, that $u(0) = u(0)^2$. Hence $u(0) = 1$ or $u(0) = 0$. If $u(0) = 0$, then taking $n = 0$ in the relation gives $u(k) = 0$ for all $k \in \{0, 1, \dots, B-1\}$, which by (1) implies $u(n) = 0$ for all $n \geq 0$.

Proposition 1. If the sequence $(u(n))_{n \geq 0}$ is strongly B -multiplicative, and if the B -ary expansion of $n \geq 1$ is $n = \sum_j e_j(n)B^j$, then $u(n) = \prod_j u(e_j(n))$.

In particular, the only strongly B -multiplicative sequence with $u(1) = u(2) = \dots = u(B-1) = \theta$, where $\theta = 0$ or 1 , is the sequence $1, \theta, \theta, \dots$.

Proof. Use induction on the number of base B digits of n .

We now generalize the notion of a strongly B -multiplicative sequence different from $1, 0, 0, 0, \dots$.

Definition 2. Let $B \geq 2$ be an integer. A sequence of complex numbers $(u(n))_{n \geq 0}$ satisfies *Hypothesis \mathcal{H}_B* if there exist an integer $n_0 \geq B$ and complex

numbers $v(0), v(1), \dots, v(B-1)$ such that $u(n_0) \neq 0$ and, for all $n \geq 1$ and all $k = 0, 1, \dots, B-1$,

$$u(Bn + k) = u(n)v(k).$$

Proposition 2. (1) If a sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis \mathcal{H}_B , then values $v(0), v(1), \dots, v(B-1)$ are uniquely determined.

(2) A sequence $(u(n))_{n \geq 0}$ has $u(0) = 1$ and satisfies Hypothesis \mathcal{H}_B with $u(Bn + k) = u(n)v(k)$ not only for $n \geq 1$ but also for $n=0$, if and only if the sequence is strongly B -multiplicative and not equal to $1, 0, 0, 0, \dots$. In that case, $v(k) = u(k)$ for $k = 0, 1, \dots, B-1$.

Proof. If the sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis \mathcal{H}_B , then $v(k) = u(Bn_0 + k)/u(n_0)$ for $k = 0, 1, \dots, B-1$. This implies (1).

To prove the "only if" part of (2), take $n = 0$ in the relation $u(Bn + k) = u(n)v(k)$, yielding $u(k) = u(0)v(k) = v(k)$ for $k = 0, 1, \dots, B-1$. Hence $u(Bn + k) = u(n)u(k)$ for all $n \geq 0$ and $k = 0, 1, \dots, B-1$. Thus $(u(n))_{n \geq 0}$ is strongly B -multiplicative. Since $u(n_0) \neq 0$ for some $n_0 \geq B$, the sequence is not $1, 0, 0, 0, \dots$.

Conversely, suppose that $(u(n))_{n \geq 0}$ is strongly B -multiplicative and is not $1, 0, 0, 0, \dots$. Then there exists an integer $\ell_0 \geq 1$ such that $u(\ell_0) \neq 0$. Hence $n_0 := B\ell_0 \geq B$ and $u(n_0) = u(B\ell_0) = u(\ell_0)u(0) = u(\ell_0) \neq 0$. Defining $v(k) := u(k)$ for $k = 0, 1, \dots, B-1$, we see that $(u(n))_{n \geq 0}$ satisfies Hypothesis \mathcal{H}_B , and the proposition follows.

Example 2. We construct a sequence which satisfies Hypothesis \mathcal{H}_B but is not strongly B -multiplicative. Let z be a complex number, with $z \notin \{0, 1\}$, and define $u(n) := z^{N_{0,B}(n)}$, where $N_{0,B}(n)$ counts the number of zeros in the B -ary expansion of n for $n > 0$, and $N_{0,B}(0) := 0$ (which corresponds to representing zero by the empty sum, that is, the empty word). Note that for all $n \geq 1$ the relation $N_{0,B}(Bn) = N_{0,B}(n) + 1$ holds, and for all $k \in \{1, 2, \dots, B-1\}$ and all $n \geq 0$ the relation $N_{0,B}(Bn + k) = N_{0,B}(n) = N_{0,B}(n) + N_{0,B}(k)$ holds. Hence the nonzero sequence $(u(n))_{n \geq 0}$ satisfies Hypothesis \mathcal{H}_B , with $v(0) := z$ and $v(k) := 1 = u(k)$ for $k = 1, 2, \dots, B-1$. But the sequence is not strongly B -multiplicative: $u(B \times 1 + 0) = z \neq 1 = u(1)u(0)$.

Remark 2. The alternative definition $N_{0,B}(0) := 1$ (which would correspond to representing zero by the single digit 0 instead of by the empty word) would also not lead to a strongly B -multiplicative sequence u , since then $u(0) = z \neq 1$, which does not agree with Definition 1 (see also Remark 1). On the other hand, the new sequence would still satisfy Hypothesis \mathcal{H}_B , with the same values $v(k)$, as the same proof shows, since $u(0)$ does not appear in it.

3. Convergence of infinite products

Inspired by the Woods-Robbins product P , we want to study products given in the following lemma.

Lemma 1. *Let $B > 1$ be an integer. Let $(u(n))_{n \geq 0}$ be a sequence of complex numbers with $|u(n)| \leq 1$ for all $n \geq 0$. Suppose that it satisfies Hypothesis \mathcal{H}_B with $|v(k)| \leq 1$ for all $k \in \{0, 1, \dots, B-1\}$, and that*

$$\left| \sum_{0 \leq k < B} v(k) \right| < B. \text{ Then for each } k \in \{0, 1, \dots, B-1\}, \text{ the infinite product}$$

$$\prod_{n \geq \delta_k} \left(\frac{Bn + k}{Bn + k + 1} \right)^{u(n)}$$

converges, where δ_k - a special case of the Kronecker delta - is defined by

$$\delta_k := \begin{cases} 0 & \text{if } k \neq 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Proof. For $N = 1, 2, \dots$, let

$$F(N) := \sum_{0 \leq n < N} u(n).$$

Also define for $j = 1, 2, \dots, B-1$

$$G(j) := \sum_{0 \leq n < j} v(n)$$

and set $G(0) := 0$. Then, for each $b \in \{0, 1, \dots, B-1\}$, and for every $N \geq 1$

$$\begin{aligned} F(BN + b) &= \sum_{0 \leq n < BN} u(n) + \sum_{BN \leq n < BN+b} u(n) = \\ &= \sum_{0 \leq n < N} \sum_{0 \leq \ell < B} u(Bn + \ell) + \sum_{0 \leq \ell < b} u(BN + \ell) = \\ &= \sum_{0 \leq \ell < B} u(\ell) + \sum_{1 \leq n < N} \sum_{0 \leq \ell < B} u(n)v(\ell) + u(N) \sum_{0 \leq \ell < b} v(\ell). \end{aligned}$$

Hence, using $|u(N)| \leq 1$ and $|G(b)| \leq B-1 < B$,

$$\begin{aligned} |F(BN + b)| &= |F(B) + (F(N) - u(0))G(B) + u(N)G(b)| < \\ &< |F(B) - u(0)G(B)| + |F(N)||G(B)| + B. \end{aligned}$$

This gives the case $d = 1$ of the following inequality, which holds for $d \geq 1$ and $e_t \in \{0, 1, \dots, B-1\}$, and which is proved by induction on d using $|F(e_t)| \leq B$:

$$\begin{aligned} &\left| F \left(\sum_{0 \leq t \leq d} e_t B^t \right) \right| < \\ &< |F(B) - u(0)G(B)| \left(1 + \sum_{1 \leq t \leq d-1} |G(B)|^t \right) + B \left(1 + \sum_{1 \leq t \leq d} |G(B)|^t \right). \end{aligned}$$

Hence

$$\left| F \left(\sum_{0 \leq t \leq d} e_t B^t \right) \right| < \begin{cases} B(3d+1) & \text{if } |G(B)| \leq 1, \\ 3B \frac{|G(B)|^{d+1}-1}{|G(B)|-1} & \text{if } |G(B)| > 1. \end{cases}$$

This implies that for some constant $C = C(B, v)$, and for every N large enough,

$$|F(N)| < \begin{cases} C \log N & \text{if } |G(B)| \leq 1, \\ C|G(B)|^{\frac{\log N}{\log B}} = CN^{\frac{\log |G(B)|}{\log B}} & \text{if } |G(B)| > 1. \end{cases}$$

Since $|G(B)| < B$ by hypothesis, we can define $\alpha \in (0, 1)$ by

$$\alpha := \begin{cases} \frac{1}{2} & \text{if } |G(B)| \leq 1, \\ \frac{\log |G(B)|}{\log B} & \text{if } |G(B)| > 1. \end{cases}$$

Hence for every N large enough $|F(N)| < CN^\alpha$. It follows, using summation by parts, that the series $\sum_n u(n) \log \frac{Bn+k}{Bn+k+1}$ converges, hence the lemma.

Remark 3. (1) Here and in what follows, expressions of the form a^z , where a is a positive real number and z a complex number, are defined by $a^z := e^{z \log a}$, and $\log a$ is real.

(2) For more precise estimates of summatory functions of (strongly) B -multiplicative sequences, see for example [7, 10]. (In [10] strongly B -multiplicative sequences are called completely B -multiplicative.)

4. Evaluation of infinite products

This section is devoted to computing some infinite products with exponents that satisfy Hypothesis \mathcal{H}_B , including some whose exponents are strongly B -multiplicative.

4.1. General results

Theorem 1. *Let $B > 1$ be an integer. Let $(u(n))_{n \geq 0}$ be a sequence of complex numbers with $|u(n)| \leq 1$ for all $n \geq 0$. Suppose that u satisfies Hypothesis \mathcal{H}_B , with complex numbers $v(0), v(1), \dots, v(B-1)$ such that $|v(k)| \leq 1$ for $k \in \{0, 1, \dots, B-1\}$ and $\left| \sum_{0 \leq k < B} v(k) \right| < B$. Then the following relation between nonempty products holds:*

$$\prod_{\substack{0 \leq k < B \\ v(k) \neq 1}} \prod_{n \geq \delta_k} \left(\frac{Bn + k}{Bn + k + 1} \right)^{u(n)(1-v(k))} = \frac{1}{B^{u(0)}} \prod_{0 < k < B} \left(\frac{k}{k+1} \right)^{u(k)-u(0)v(k)}$$

Proof. The condition $\left| \sum_{0 \leq k < B} v(k) \right| < B$ prevents v from being identically equal to 1 on $\{0, 1, \dots, B-1\}$, so the left side of the equation is not empty. Since $B > 1$, so is the right.

We first show that

$$(*) \quad \prod_{0 \leq k < B} \prod_{n \geq \delta_k} \left(\frac{Bn + k}{Bn + k + 1} \right)^{u(n)} = \frac{1}{B^{u(0)}} \prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{u(n)}$$

(note that, by Lemma 1, all the products converge). To see this, write the left side as

$$\left(\frac{1}{2} \frac{2}{3} \cdots \frac{B-1}{B} \right)^{u(0)} \prod_{n \geq 1} \left(\frac{Bn}{Bn+1} \frac{Bn+1}{Bn+2} \cdots \frac{Bn+B-1}{Bn+B} \right)^{u(n)}$$

and use telescopic cancellation. Now, splitting the product on the right side of (*) according to the values of n modulo B gives

$$\begin{aligned}
& \prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{u(n)} = \\
&= \prod_{0 \leq k < B} \prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{u(Bn+k)} = \\
&= \prod_{0 < k < B} \left(\frac{k}{k+1} \right)^{u(k)} \prod_{0 \leq k < B} \prod_{n \geq 1} \left(\frac{Bn+k}{Bn+k+1} \right)^{u(n)v(k)} = \\
&= \prod_{0 < k < B} \left(\frac{k}{k+1} \right)^{u(k)-u(0)v(k)} \prod_{0 \leq k < B} \prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{u(n)v(k)}
\end{aligned}$$

Using (*) and the fact that convergent infinite products are nonzero, the theorem follows.

Example 3. As in Example 2, the sequence u defined by $u(n) = z^{N_{0,B}(n)}$, with $z \neq \{0, 1\}$, satisfies Hypothesis \mathcal{H}_B , and $\sum_{0 \leq k < B} v(k) = z + B - 1$. If furthermore $|z| \leq 1$, then

$$\prod_{n \geq 1} \left(\frac{Bn}{Bn+1} \right)^{(1-z)z^{N_{0,B}(n)}} = B.$$

Corollary 1. Fix an integer $B > 1$. If $(u(n))_{n \geq 0}$ is strongly B -multiplicative, satisfies $|u(n)| \leq 1$ for all $n \geq 0$, and is not equal to either of the sequences $1, 0, 0, 0, \dots$ or $1, 1, 1, \dots$, then

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ u(k) \neq 1}} \left(\frac{Bn+k}{Bn+k+1} \right)^{u(n)(1-u(k))} = \frac{1}{B}.$$

Proof. Using Theorem 1 and Proposition 2 part (2) it suffices to prove that $\left| \sum_{0 \leq k < B} u_k \right| < B$. Since $|u_n| \leq 1$ for all $n \geq 0$, we have $\left| \sum_{0 \leq k < B} u_k \right| \leq B$. From the equality case of the triangle inequality, it thus suffices to prove that the numbers u_0, u_1, \dots, u_{B-1} are not all equal to a same complex number z with $|z| = 1$. If they were, then, since $u_0 = 1$, we would have $u_0 = u_1 = \dots = u_{B-1} = 1$. Hence $(u(n))_{n \geq 0} = 1, 1, 1, \dots$ from Proposition 1, a contradiction.

Addendum. Theorem 1 and Corollary 1 can be strengthened, as follows.

(1) If B , u , and v satisfy the hypotheses of Theorem 1, then

$$\begin{aligned} & \sum_{\substack{0 \leq k < B \\ v(k) \neq 1}} (1 - v(k)) \sum_{n \geq \delta_k} u(n) \log \frac{Bn + k}{Bn + k + 1} = \\ & = -u(0) \log B + \sum_{0 < k < B} (u(k) - u(0)v(k)) \log \frac{k}{k + 1}. \end{aligned}$$

(2) If B and u satisfy the hypotheses of Corollary 1, then

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ u(k) \neq 1}} u(n)(1 - u(k)) \log \frac{Bn + k}{Bn + k + 1} = -\log B.$$

Proof. Write the proofs of Theorem 1 and Corollary 1 additively instead of multiplicatively.

Remark 4. The Addendum cannot be proved by just taking logarithms in the formulas in Theorem 1 and Corollary 1. To illustrate the problem, note that while

$$\prod_{n \geq 0} e^{\frac{(-1)^n 8i}{2n+1}} = 1$$

(because the product converges to $e^{2\pi i}$), the log equation is false:

$$\sum_{n \geq 0} \frac{(-1)^n 8i}{2n+1} = 2\pi i \neq 0 = \log 1.$$

Example 4. With the same u and z as in Example 3, Addendum (1) yields

$$\sum_{n \geq 1} z^{N_{0,B}(n)} \log \frac{Bn}{Bn + 1} = \frac{\log B}{z - 1}.$$

Hence

$$\prod_{n \geq 1} \left(\frac{Bn}{Bn + 1} \right)^{z^{N_{0,B}(n)}} = B^{\frac{1}{z-1}}.$$

(Note the similarity between this product and the one in Example 3. Neither implies the other, but of course the preceding log equation implies both.)

If we modify the sequence u as in Remark 2, we get the same two formulas, because the value $N_{0,B}(0)$ does not appear in them.

Corollary 2. *Fix integers B, q, p with $B > 1$, $q > p > 0$, and $B \equiv 1 \pmod{q}$. Then*

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv \text{mod } q}} \left(\frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi k p}{q} \sin \frac{\pi(2n+k)p}{q}} = \frac{1}{\sqrt{B}}$$

and

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv \text{mod } q}} \left(\frac{Bn + k}{Bn + k + 1} \right)^{\sin \frac{\pi k p}{q} \cos \frac{\pi(2n+k)p}{q}} = 1.$$

Proof. Let $\omega := e^{2\pi i p/q}$. Since $B \equiv 1 \pmod{q}$, we may take $u(n) := \omega^n$ in Addendum (2), yielding the formula

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv \text{mod } q}} \omega^n (1 - \omega^k) \log \frac{Bn + k}{Bn + k + 1} = -\log B.$$

Writing $\omega^n (1 - \omega^k) = -2i\omega^{n+\frac{k}{2}} \sin \frac{\pi k p}{q}$, and multiplying the real and imaginary parts of the formula by $1/2$, the result follows.

Example 5. Take $B = 5$, $p = 1$, and $q = 4$. Squaring the products, we get

Define $\sigma(n)$ to be $+1$ if n is a square modulo 4, and -1 otherwise, that is

$$\sigma(n) := \begin{cases} +1 & \text{if } n \equiv 0 \text{ or } 1 \pmod{4}, \\ -1 & \text{if } n \equiv 2 \text{ or } 3 \pmod{4}. \end{cases}$$

Then

$$\prod_{n \geq 0} \left(\frac{5n+1}{5n+2} \right)^{\sigma(n)} \left(\frac{5n+2}{5n+3} \right)^{\sigma(n)+\sigma(n+1)} \left(\frac{5n+3}{5n+4} \right)^{\sigma(n+1)} = \frac{1}{5}$$

and

$$\prod_{n \geq 0} \left(\frac{5n+1}{5n+2} \right)^{\sigma(n-1)} \left(\frac{5n+2}{5n+3} \right)^{\sigma(n-1)+\sigma(n)} \left(\frac{5n+3}{5n+4} \right)^{\sigma(n)} = 1.$$

4.2. The sum-of-digits function $s_B(n)$

Other products can also be obtained from Corollary 1. We give three corollaries, each of which generalizes the Woods-Robbins formula $P = 1/\sqrt{2}$ in the Introduction. Recall that $s_B(n)$ denotes the sum of the B -ary digits of the integer n .

Corollary 3. *Fix an integer $B > 1$ and a complex number z with $|z| \leq 1$. If $z \notin \{0, 1\}$, then*

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ z^k \neq 1}} \left(\frac{Bn+k}{Bn+k+1} \right)^{z^{s_B(n)}(1-z^k)} = \frac{1}{B}.$$

Proof. Take $u(n) := z^{s_B(n)}$ in Corollary 1 and note that $s_B(k) = k$ when $0 < k < B$.

Example 6. Take $B = 2$ and $z = 1/2$. Squaring the product, we obtain

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(1/2)^{s_2(n)}} = \frac{1}{4}.$$

Corollary 4. *Let B, p, q be integers with $B > 1$ and $q > p > 0$. Then*

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \not\equiv \text{mod } q}} \left(\frac{Bn+k}{Bn+k+1} \right)^{\sin \frac{\pi kp}{q} \sin \frac{\pi(2s_B(n)+k)p}{q}} = \frac{1}{\sqrt{B}}$$

and

$$\prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \equiv \text{mod } q}} \left(\frac{Bn+k}{Bn+k+1} \right)^{\sin \frac{\pi kp}{q} \cos \frac{\pi(2s_B(n)+k)p}{q}} = 1.$$

Proof. Use the proof of Corollary 2, but replace $B \equiv 1 \pmod q$ with $s_B(Bn+k) = s_B(n) + k$ when $0 \leq k < B$, and replace ω^n with $\omega^{s_B(n)}$.

Example 7. Take $B = 2$, $q = 4$, and $p = 1$. Squaring the products and defining $\sigma(n)$ as in Example 5, we get

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{\sigma(s_2(n))} = \frac{1}{2} \quad \text{and} \quad \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{\sigma(s_2(n)+1)} = 1.$$

In the same spirit, we recover a result from [3, p.369-370].

Example 8. Taking $B = q = 3$ and $p = 1$ in Corollary 4, we obtain two infinite products. Raising the second to the power $-2/\sqrt{3}$ and multiplying by the square of the first, we get:

Define $\theta(n)$ by

$$\theta(n) := \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod{3}, \\ -2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Then

$$\prod_{n \geq 0} (3n+1)^{\theta(s_3(n))} (3n+2)^{\theta(s_3(n)+1)} (3n+3)^{\theta(s_3(n)+2)} = \frac{1}{3}.$$

Corollary 5 ([14]). Let $B > 1$ be an integer. Then

$$\prod_{n \geq 0} \prod_{\substack{0 \leq k < B \\ k \text{ odd}}} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^{s_B(n)}} = \frac{1}{\sqrt{B}}.$$

Proof. Take $z = -1$ in Corollary 3 (or take $q = 2$ and $p = 1$ in Corollary 4).

Example 9. With $B = 2$, since $s_2(n) = N_{1,2}(n)$, we recover the Woods-Robbins formula $P = 1/\sqrt{2}$. Taking $B = 6$ gives

$$\prod_{n \geq 0} \left(\frac{(6n+1)(6n+3)(6n+5)}{(6n+2)(6n+4)(6n+6)} \right)^{(-1)^{s_6(n)}} = \frac{1}{\sqrt{6}}.$$

Remark 5. Corollary 5 can also be obtained from [2, Theorem 1], as follows. Taking x equal to -1 and j equal to 0 in that theorem gives

$$\sum_{n \geq 0} (-1)^{s_B(n)} \log \frac{n+1}{B \lfloor n/B \rfloor + B} = -\frac{1}{2} \log B,$$

where $\lfloor x \rfloor$ is the integer part of x . But the series is equal to

$$\begin{aligned} & \sum_{m \geq 0} \sum_{0 \leq k < B} (-1)^{s_B(Bm+k)} \log \frac{Bm+k+1}{Bm+B} = \\ &= \sum_{m \geq 0} (-1)^{s_B(m)} \sum_{0 \leq k < B} (-1)^k \log \frac{Bm+k+1}{Bm+B} = \\ &= \sum_{m \geq 0} (-1)^{s_B(m)} \sum_{\substack{k \text{ odd} \\ 0 < k < B}} \log \frac{Bm+k}{Bm+k+1}, \end{aligned}$$

where the last equality follows by looking separately at the cases B even and B odd.

4.3. The counting function $N_{j,B}(n)$

We can also compute some infinite products associated with counting the number of occurrences of one or several given digits in the base B expansion of an integer.

Definition 3. If B is an integer ≥ 2 and if j is in $\{0, 1, \dots, B-1\}$, let $N_{j,B}(n)$ be the number of occurrences of the digit j in the B -ary expansion of n when $n > 0$, and set $N_{j,B}(0) := 0$.

Corollary 6. Let B, q, p be integers with $B > 1$ and $q > p > 0$. Let J be a nonempty, proper subset of $\{0, 1, \dots, B-1\}$. Define $N_{J,B}(n) := \sum_{j \in J} N_{j,B}(n)$.

Then the following equalities hold:

$$\prod_{k \in J} \prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\sin \frac{\pi(2N_{J,B}(n)+1)p}{q}} = B^{-\frac{1}{2 \sin \frac{\pi p}{q}}}$$

and

$$\prod_{k \in J} \prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\cos \frac{\pi(2N_{J,B}(n)+1)p}{q}} = 1.$$

Proof. Let $\omega := e^{2\pi i p/q}$. We denote $u_{q,j,B}(n) := \omega^{N_{j,B}(n)}$ and $u_{q,J,B}(n) := \prod_{j \in J} u_{q,j,B}(n) = \omega^{N_{J,B}(n)}$. Note that, for every j in $\{1, 2, \dots, B-1\}$,

the sequence $(u_{q,j,B}(n))_{n \geq 0}$ is strongly B -multiplicative and nonzero, hence satisfies Hypothesis \mathcal{H}_B . The sequence $(u_{q,0,B}(n))_{n \geq 0}$ also satisfies Hypothesis \mathcal{H}_B , as is seen by taking $z = \omega$ in Example 2. Therefore the sequence

$(u_{q,j,B}(n))_{n \geq 0}$ satisfies Hypothesis \mathcal{H}_B , with, for $k = 0, 1, \dots, B-1$, the value $v(k) := \omega$ if $k \in J$ and $v(k) := 1$ otherwise.

Now $|u_{q,J,B}(n)| = 1$ for $n \geq 0$, and $|v(k)| = 1$ for $k = 0, 1, \dots, B-1$. Furthermore, $\left| \sum_{0 \leq k < B} v(k) \right| < B$, since v is not constant on $\{0, 1, \dots, B-1\}$. Thus we may apply Addendum (1) with $u(n) := u_{q,J,B}(n)$, obtaining

$$(1 - \omega) \sum_{k \in J} \sum_{n \geq \delta_k} \omega^{N_{J,B}(n)} \log \frac{Bn + k}{Bn + k + 1} = -\log B.$$

Writing $(1 - \omega)\omega^{N_{J,B}(n)} = -2i\omega^{N_{J,B}(n) + \frac{1}{2}} \sin \frac{\pi p}{q}$, and taking the real and imaginary parts of the summation, the result follows.

Example 10. Taking $q = 2$ and $p = 1$ in the first formula gives

$$\prod_{k \in J} \prod_{n \geq \delta_k} \left(\frac{Bn + k}{Bn + k + 1} \right)^{(-1)^{N_{J,B}(n)}} = \frac{1}{\sqrt{B}}.$$

An application is an alternate proof of Corollary 5: take J to be the set of odd numbers in $\{1, 2, \dots, B-1\}$; since $s_B(n) = \sum_{0 < k < B} kN_{k,B}(n)$, it follows that

$$\sum_{j \in J} N_{j,B}(n) = (-1)^{s_B(n)}.$$

Remark 6. Corollary 6 requires that J be a proper subset of $\{0, 1, \dots, B-1\}$. Suppose instead that $J = \{0, 1, \dots, B-1\}$. Then $N_{J,B}(n)$ is the number of B -ary digits of n if $n > 0$ (that is, $N_{J,B}(n) = \left\lfloor \frac{\log n}{\log B} \right\rfloor + 1$), and $N_{J,B}(0) = 0$. In that case, Corollary 6 does not apply, and the products may diverge. For example, when $B = q = 2$ and $p = 1$ the logarithm of the first product is equal to the series

$$-\log 2 + \sum_{n \geq 1} (-1)^{\left\lfloor \frac{\log n}{\log 2} \right\rfloor} \log \frac{n+1}{n},$$

which does not converge. However, note its resemblance with Vacca's (convergent) series for Euler's constant [16]

$$\gamma = \sum_{n \geq 1} \left\lfloor \frac{\log n}{\log 2} \right\rfloor \frac{(-1)^n}{n}.$$

Corollary 7. *Let B, q, p be integers with $B > 1$ and $q > p > 0$. Then for $k = 0, 1, \dots, B-1$ the following equalities hold:*

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\sin \frac{\pi(2N_{k,B}(n)+1)p}{q}} = B^{-\frac{1}{2 \sin \frac{\pi p}{q}}}$$

and

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{\cos \frac{\pi(2N_{k,B}(n)+1)p}{q}} = 1.$$

Proof. Take $J := \{k\}$ in Corollary 6. (The case $k = 0$ and $p = 1$ is Example 4 with $z = e^{2\pi i/q}$.)

Example 11. Taking $q = 2$ and $p = 1$ in the first formula (or taking $J = \{k\}$ in Example 10) yields

$$\prod_{n \geq \delta_k} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^{N_{k,B}(n)}} = \frac{1}{\sqrt{B}}.$$

In particular, if $B = 2$ the choice $k = 1$ gives the Woods-Robbins formula $P = 1/\sqrt{2}$, and $k = 0$ gives

$$\prod_{n \geq 1} \left(\frac{2n}{2n+1} \right) (-1)^{N_{0,2}(n)} = \frac{1}{\sqrt{2}}.$$

Remark 7. For base $B = 2$, the formulas in Example 11 are special cases of results in [4], where $N_{j,2}(n)$ is generalized to counting the number of occurrences of a given *word* in the binary expansion of n . On the other hand, the value of the product Q in the Introduction,

$$Q = \prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{(-1)^{N_{1,2}(n)}},$$

remains a mystery.

Example 12. Take $B = q = 3$ and $p = 1$. Raising the first product to the power $2/\sqrt{3}$ and squaring the second, we obtain:

Define $\eta(n)$ by

$$\eta(n) := \begin{cases} +1 & \text{if } n \equiv 0 \pmod{3}, \\ 0 & \text{if } n \equiv 1 \pmod{3}, \\ -1 & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

and define $\theta(n)$ as in Example 8. Then for $k=0,1$, and 2

$$\prod_{n \geq \delta_k} \left(\frac{3n+k}{3n+k+1} \right)^{\eta(N_{k,3}(n))} = \frac{1}{3^{2/3}} \quad \text{and} \quad \prod_{n \geq \delta_k} \left(\frac{3n+k}{3n+k+1} \right)^{\theta(N_{k,3}(n)+1)} = 1.$$

4.4. The Gamma function

It can happen that the exponent in some of our products is a periodic function of n . For example, this is obviously the case in Corollary 2. To take another example, it is not hard to see that if B odd, then $(-1)^{2_B(n)} = (-1)^n$. Hence Corollary 5 gives

$$(**) \quad \prod_{n \geq 0} \prod_{\substack{0 < k < B \\ k \text{ odd}}} \left(\frac{Bn+k}{Bn+k+1} \right)^{(-1)^n} = \frac{1}{\sqrt{B}} \quad (B \text{ odd}).$$

(This formula can also be obtained from Corollary 2 with $q = 2$ and $p = 1$.)
For instance

$$P_{1,3} := \prod_{n \geq 0} \left(\frac{3n+1}{3n+2} \right)^{(-1)^n} = \frac{1}{\sqrt{3}}.$$

The product $P_{1,3}$ can also be computed using the following corollary of the Weierstrass product for the Gamma function [17, Section 12.13].

If d is a positive integer and $a_1 + a_2 + \cdots + a_d = b_1 + b_2 + \cdots + b_d$, where the a_j and b_j are complex numbers and no b_j is zero or a negative integer, then

$$\prod_{n \geq 0} \frac{(n+a_1) \cdots (n+a_d)}{(n+b_1) \cdots (n+b_d)} = \frac{\Gamma(b_1) \cdots \Gamma(b_d)}{\Gamma(a_1) \cdots \Gamma(a_d)}.$$

Combining this with the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ gives $P_{1,3} = 1/\sqrt{3}$.

The computation can be generalized, using Gauss' multiplication theorem for the Gamma function, to give another proof of Corollary 5 for B odd. Likewise, an analog of the odd- B case of Corollary 5 can be proved for even k :

$$\prod_{n \geq 1} \prod_{\substack{0 \leq k < B \\ k \text{ even}}} \left(\frac{Bn + k}{Bn + k + 1} \right)^{(-1)^n} = \frac{\pi \sqrt{B}}{2^B} \left(\frac{B-1}{(B-1)/2} \right) \quad (B \text{ odd}).$$

Multiplying this by the formula

$$\prod_{n \geq 1} \prod_{\substack{0 \leq k < B \\ k \text{ odd}}} \left(\frac{Bn + k}{Bn + k + 1} \right)^{(-1)^n} = \frac{2^{B-1}}{\sqrt{B}} \left(\frac{B-1}{(B-1)/2} \right)^{-1} \quad (B \text{ odd}),$$

which is (**) rewritten, yields Wallis' product for π . (For an evaluation of preceding two products when $B = 2$, see [15, Example 7].)

References

- [1] **Allouche J.-P. and Cohen H.**, Dirichlet series and curious infinite products, *Bull. Lond. Math. Soc.*, **17** (1985), 531-538.
- [2] **Allouche J.-P., Cohen H., Mendès France M. and Shallit J.**, De nouveaux curieux produits infinis, *Acta Arith.*, **49** (1987), 141-153.
- [3] **Allouche J.-P., Mendès France M. and Peyrière J.**, Automatic Dirichlet series, *J. Number Theory*, **81** (2000), 359-373.
- [4] **Allouche J.-P. and Shallit J.O.**, Infinite products associated with counting blocks in binary strings, *J. Lond. Math. Soc.*, **39** (1989), 193-204.
- [5] **Allouche J.-P. and Shallit J.**, The ubiquitous Prouhet-Thue-Morse sequence, *Sequences and their Applications, Proceedings of SETA '98*, eds. C. Ding, T. Helleseth and H. Niederreiter, Springer, 1999, 1-16.
- [6] **Bellman R. and Shapiro H.N.**, On a problem in additive number theory, *Ann. Math.*, **49** (1948), 333-340.
- [7] **Delange H.**, Sur les fonctions q -additives ou q -multiplicatives, *Acta Arith.*, **21** (1972), 285-298.
- [8] **Flajolet P. and Martin G.N.**, Probabilistic counting algorithms for data base applications, *J. Comput. Sys. Sci.*, **31** (1985), 182-209.
- [9] **Gel'fond A.O.**, Sur les nombres qui ont des propriétés additives et multiplicatives données, *Acta Arith.*, **13** (1968), 259-265.

- [10] **Grabner P.**, Completely q -multiplicative functions: the Mellin transform approach, *Acta Arith.*, **65** (1993), 85-96.
- [11] **Mendès France M.**, Les suites à spectre vide et la répartition modulo 1, *J. Number Theory*, **5** (1973), 1-15.
- [12] **Robbins D.**, Solution to problem E 2692, *Amer. Math. Monthly*, **86** (1979), 394-395.
- [13] **Shallit J.O.**, On infinite products associated with sums of digits, *J. Number Theory*, **21** (1985), 128-134.
- [14] **Sondow J.**, Problem 11222, *Amer. Math. Monthly*, **113** (2006), 459.
- [15] **Sondow J. and Hadjicostas P.**, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, *J. Math. Anal. Appl.*, **332** (2007), 292-314.
- [16] **Vacca G.**, A new series for the Eulerian constant $\gamma = .577\dots$, *Quart. J. Pure Appl. Math.*, **41** (1910), 363-364.
- [17] **Whittaker E.T. and Watson G.N.**, *A course of modern analysis*, Cambridge University Press, Cambridge, 1978.
- [18] **Woods D.R.**, Problem E 2692, *Amer. Math. Monthly*, **85** (1978), 48.

J.-P. Allouche

CNRS, LRI, Bâtiment 490
F-91405 Orsay Cedex, France
allouche@lri.fr

J. Sondow

209 West 97th Street
New York, NY10025, USA
jsondow@alumni.princeton.edu

