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ON THE PRIME NUMBER THEOREM

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. In this paper we give an elementary proof for the mean asymptotic behaviour of the Möbius function.

1. Introduction

Let $\pi(x)$ denote the number of primes not exceeding the real number x. In 1793 C.F. Gauß [6] and in 1798 A.M. Legendre [15] proposed independently that for large x the ratio

$$\frac{\pi(x)}{x/\log x}$$

was nearly 1 and they conjectured that this ratio would approach 1 as x approaches ∞. Both Gauß and Legendre attempted to prove this statement but did not succeed. The problem of deciding the truth or falsehood of this conjecture attracted the attention of eminent mathematicians for nearly 100 years.

In 1851 the Russian mathematician P.L. Chebychev [1] made an important step forward by proving that if the ratio did tend to a limit, then this limit must be one. Further, he succeeded in showing that the actual order of $\pi(x)$ is $x/\log x$, that is

$$\pi(x) \asymp \frac{x}{\log x}.$$

However, he was unable to prove that the ratio does tend to a limit.

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In 1859 B. Riemann [18] attacked the problem with analytic methods, using a formula discovered by L. Euler in 1737 which relates the prime numbers to the function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for real $s > 1$. Riemann considered complex values of s and outlined an ingenious method for connecting the distribution of primes to properties of the function $\zeta(s)$. The mathematics needed to justify all the details of his method had not been fully developed and Riemann was unable to completely settle the problem before his death in 1866.

Thirty years later the necessary analytic tools were at hand and in 1896 J. Hadamard [7] and C.J. de la Vallée Poussin [17] independently and almost simultaneously succeeded in proving that

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as} \quad x \to \infty.$$

This remarkable result is called the *prime number theorem*, and its proof was one of the crowning achievements of analytic number theory.

The prime number theorem was subsequently reproved and improved by others. However, a proof of this theorem, not fundamentally dependent upon the ideas of the theory of functions, seemed, not only to G.H. Hardy (cf. [8], pp. 549-550), extraordinarily unlikely.

Therefore, in 1949 A. Selberg [19] and P. Erdős [3] caused a sensation when they discovered an elementary proof of the prime number theorem. Their proof, though very intricate, makes no use of $\zeta(s)$ nor of complex function theory and in principal is accessible to anyone familiar with elementary analysis.

In 1911 E. Landau [14] showed that the prime number theorem is equivalent to the validity of the assertion that the mean value

$$\mathbf{M}(\mu) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \mu(n)$$

of the Möbius function μ exists and is equal to zero.

The function μ is multiplicative, i.e. $\mu(mn) = \mu(m)\mu(n)$ whenever $(m, n)(:= \gcd(m, n)) = 1$, and defined by

$$\mu(1) = 1$$

and

$$\mu(p^m) = \begin{cases} -1, & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases}$$

Obviously

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$

Here we give an (elementary) proof of

Theorem 1.1. *For $x \to \infty$ we have*

$$\frac{1}{x} \sum_{n \le x} \mu(n) = O\left(\frac{1}{\log x}\right).$$

The proof of Theorem 1.1 is done by using estimates which are interesting in themselves.

Theorem 1.2. *Put $M(x) = \sum_{n \le x} \mu(n)$. Then, for $x \ge 3$,*

$$\frac{|M(x)|}{x} \le \frac{2}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + O\left(\frac{1}{\log x}\right) \tag{1.1}$$

and

$$\frac{|M(x)|}{x} \le \frac{1}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + O\left(\frac{\log \log x}{\log x}\right). \tag{1.2}$$

Remark 1. The proof for (1.1) works also in the case of completely multiplicative functions of modulus ≤ 1 (see [12]), and leads to

Corollary 1. *Let $f : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function and $|f(n)| \le 1$ for all $n \in \mathbb{N}$. Put $M(x) = \sum_{n \le x} f(n)$. Then the inequality*

$$|M(x)|x \le \frac{2}{\log x} \int_1^x \frac{|M(u)|}{u^2} du + O\left(\frac{1}{\log x}\right)$$

holds for all $x \ge 3$.

The method of proof can be extended to (not necessarily completely) multiplicative functions and will be described somewhere else.

The estimate (1.2) has already been given by Postnikov and Romanov in [16].

For an arithmetical function $f:\mathbb{N}\to\mathbb{C}$ we define the generating function F of f by

$$F(s) := \sum_{n=1}^{\infty} f(n)n^{-s},$$

where $s=\sigma+it$.

Then, integration by parts shows for $\sigma>1$

$$s^{-1}F(s) = \int_0^{\infty} e^{-\omega}\left(\sum_{n\le e^{\omega}} f(n)\right) e^{-\omega(\sigma-1)}e^{-i\omega t}\,d\omega$$

and, by Parseval's formula,

$$\int_{-\infty}^{\infty}\left|\frac{F(s)}{s}\right|^2 dt = 2\pi\int_0^{\infty}\left|e^{-\omega}\sum_{n\le e^{\omega}} f(n)\right|^2 e^{-2\omega(\sigma-1)}\,d\omega\,. \tag{1.3}$$

Theorem 1.3. *Let $M(x)=\sum_{n\le x}\mu(n)$. Then, as $x\to\infty$,*

$$\frac{1}{\log x}\int_1^x \frac{|M(u)|}{u^2}du \ll \left(\frac{1}{\log x}\int_{-\infty}^{\infty}\frac{1}{|\zeta(s)|^2}\frac{dt}{|s|^2}\right)^{\frac12} \tag{1.4}$$

and

$$\frac{1}{\log x}\int_1^x \frac{|M(u)|}{u^2}du \ll \frac{1}{\log x}\left(\int_{-\infty}^{\infty}\left|\frac{\zeta'(s)}{\zeta^2(s)}\right|^2\frac{dt}{|s|^2}\right)^{\frac12}, \tag{1.5}$$

where $s=1+\dfrac{1}{\log x}+it$.

2. Convolution

Our treatment of this topic follows that of Shapiro's book [20].

The classes of functions that are distinguished are denoted by $\mathcal{S}$ and $\mathcal{A}$, and are defined as follows

$$\mathcal{S} := \{f : \mathbb{R} \to \mathbb{C}, \quad f(x) = 0 \quad \text{for} \quad x < 1\},$$
$$\mathcal{A} := \{f \in \mathcal{S} : \, f(x) = 0 \quad \text{for} \quad x \notin \mathbb{N}\}.$$

Then, for $f, g \in \mathcal{S}$, the *convolution* $f * g$ in $\mathcal{S}$ is defined by

$$(f * g)(x) = \sum_{1 \le n \le x} f\left(\frac{x}{n}\right) g(n). \tag{2.1}$$

The "action" of this definition on functions of $\mathcal{A}$ is given by the following: if $f \in \mathcal{A}$, $g \in \mathcal{S}$ then $f * g \in \mathcal{A}$ and for $n \in \mathbb{N}$,

$$(f * g)(n) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d) \,. \tag{2.2}$$

In general the binary operation $*$ is not commutative in $\mathcal{S}$, but if $f, g \in \mathcal{A}$ then $f * g = g * f$.

Consider the function ε defined by

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\varepsilon \in \mathcal{A}$, and

$$f * \varepsilon = f \qquad \text{for } f \in \mathcal{S}$$

and

$$(\varepsilon * f)(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } f \in \mathcal{S}. \tag{2.3}$$

Thus ε serves as a *right identity under convolution* for all of $\mathcal{S}$, but is a *left identity* only in $\mathcal{A}$.

The relation (2.3) suggests that for each $f \in \mathcal{S}$ we define an image $f_0 \in \mathcal{A}$ by

$$f_0 = \varepsilon * f \qquad \text{for } f \in \mathcal{S}.$$

The Möbius function μ is defined by

$$\mathbf{1}_0 * \mu = \varepsilon,$$

where $\mathbf{1}_0 = \varepsilon * \mathbf{1}$ and $\mathbf{1} \in \mathcal{S}$ with

$$\mathbf{1}(x) = \begin{cases} 1 & x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The wellknown *Möbius inversion formula* says that if $f, g \in \mathcal{S}$ then $f = g * \mathbf{1_0}$ if and only if $g = f * \mu$.

Examples. (i) Let $g = \mathbf{1}$. Then $f(x) = [x]$ and $\sum_{n \leq x} \left[\frac{x}{n}\right] \mu(n) = 1$ which implies $x \sum_{n \leq x} \frac{\mu(n)}{n} = O(x)$, i.e.

$$\sum_{n \leq x} \frac{\mu(n)}{n} = O(1). \tag{2.4}$$

(ii) Let $g(x) = x$ for $x \geq 1$. Then

$$f(x) = \sum_{n \leq x} \frac{x}{n} = x \log x + c_1 x + O(1)$$

and

$$\begin{aligned} x = g(x) &= \sum_{n \leq x} \mu(n) \left\{ \frac{x}{n} \log \frac{x}{n} + c_1 \frac{x}{n} \right\} + O(x) = \\ &= x \sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} + c_1 x \sum_{n \leq x} \frac{\mu(n)}{n} + O(x) \end{aligned}$$

which implies

$$\sum_{n \leq x} \frac{\mu(n)}{n} \log \frac{x}{n} = O(1). \tag{2.5}$$

The constant c_1 equals Euler‘s constant γ.

(iii) Let $g(x) = x \log x$. By a straightforward calculation (partial summation) we deduce

$$
\begin{aligned}
f(x) &= \sum_{n\le x} \frac{x}{n}\log\frac{x}{n} = \\
&= x\log x \sum_{n\le x}\frac{1}{n} - x\sum_{n\le x}\frac{\log n}{n} = \\
&= \frac{1}{2}x\log^2 x + c_1 x\log x - c_2 x + O(\log x)
\end{aligned}
$$

since with some constant c_2

$$
\sum_{n\le x}\frac{\log n}{n} = \frac{1}{2}\log^2 x + c_2 + O\left(\frac{\log x}{x}\right).
$$

This implies, by (2.4) and (2.5)

$$
x\log x = g(x) = \frac{1}{2}x\sum_{n\le x}\frac{\mu(n)}{n}\log^2\frac{x}{n} + O(x)
$$

and

$$
\sum_{n\le x}\frac{\mu(n)}{n}\log^2\frac{x}{n} = 2\log x + O(1). \tag{2.6}
$$

Let $L\in\mathcal{S}$ denote the logarithm function. Then obviously L acts as a derivation on $\mathcal{S}$, that is

$$
L\cdot(f*g) = (L\cdot f)*g + f*(L\cdot g) \quad \text{for all } f,g\in\mathcal{S}. \tag{2.7}
$$

Further, we introduce the von Mangoldt function $\Lambda\in\mathcal{A}$ by

$$
\varepsilon * L = L_0 = \Lambda * \mathbf{1}_0, \tag{2.8}
$$

i.e.

$$
\Lambda = L_0 * \mu\,. \tag{2.9}
$$

The relation (2.8) and (2.9) immediately show

$$
\begin{aligned}
L_0^2 &= L_0\cdot(\mathbf{1}_0*\Lambda) = \\
&= L_0*\Lambda + \mathbf{1}_0 * L_0\Lambda = \\
&= \mathbf{1}_0*(\Lambda*\Lambda + L_0\Lambda)
\end{aligned}
$$

and

$$\mu * L_0^2 = \Lambda * \Lambda + L_0 \Lambda. \tag{2.10}$$

On the other hand, by (2.4) and (2.5)

$$\mathbf{1} * (\mu * L_0^2)(x) = \sum_{n \le x} \sum_{d|n} \mu(d) \log^2 \frac{n}{d} = \sum_{dd' \le x} \mu(d) \log^2 d' =$$

$$= \sum_{d \le x} \mu(d) \sum_{d' \le \frac{x}{d}} \log^2 d' =$$

$$= x \sum_{d \le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} - 2x \sum_{d \le x} \frac{\mu(d)}{d} \log \frac{x}{d} + 2x \sum_{d \le x} \frac{\mu(d)}{d} + O(x) =$$

$$= x \sum_{d \le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} + O(x),$$

since

$$\sum_{n \le y} \log^2 n = \int_1^y \log^2 t \, dt + O(\log^2 y) =$$

$$= y \log^2 y - 2y \log y + 2y + O(\log^2 y).$$

Considering (2.10) and (2.6) produces

$$\mathbf{1}*(L_0\Lambda+\Lambda*\Lambda)(x) = \sum_{n \le x} \Lambda(n) \log n + \sum_{dd' \le x} \Lambda(d)\Lambda(d') = 2x \log x + O(x) \tag{2.11}$$

which is known as *Selberg's Symmetry Formula.*

Remark 2. Putting $\psi = \mathbf{1} * \Lambda$ and summing by parts we have

$$\mathbf{1} * (L_0\Lambda)(x) = \sum_{n \le x} \Lambda(n) \log n =$$

$$= \psi(x) \log x + O(x).$$

Using this in (2.11) it becomes, since $\mathbf{1} * (\Lambda * \Lambda) = \psi * \Lambda$,

$$\psi(x) \log x + \sum_{d \le x} \Lambda(d) \psi\left(\frac{x}{d}\right) = 2x \log x + O(x). \tag{2.12}$$

This form of the Selberg Formula was the basis of the first elementary proofs of the prime number theorem due to Selberg and Erdős.

3. Proof of Lemmas

The first Lemma is based on a summation formula the proof of which uses standard techniques already described by E.M. Wright in 1951 (see, for example, [9]).

Lemma 3.1. *Let $R \in \mathcal{S}$ and $g \in \mathcal{A}$ such that $\sum_{n \le x} g(n) = cx(\log x)^m + {}$ $+O\left(x(\log x)^{m-1}\right)$ for some $m \ge 0$. Assume that there is a steadily increasing function $H \in S$, $H(x) = O(x)$ such that for $1 \le t' < t$*

$$\left| |R(t)| - |R(t')| \right| \le H(t) - H(t') \ .$$

Then

$$(|R| * g)(x) = \sum_{n \le x} \left| R\left(\frac{x}{n}\right) \right| g(n) =$$

$$= c \int_1^x \left| R\left(\frac{x}{t}\right) \right| (\log t)^m \, dt + O(x(\log x)^m).$$

Proof. We put

$$h(n) = g(n) - c \int_{n-1}^{n} (\log t)^m \ dt \ .$$

Then

$$\sum_{n \le x} h(n) = \sum_{n \le x} g(n) - c \int_1^{[x]} (\log t)^m \, dt = O\left(x(\log x)^{m-1}\right) \ .$$

By partial summation we have

$$\sum_{n \le x} \left| R\left(\frac{x}{n}\right) \right| g(n) - c \sum_{2 \le n \le x} \left| R\left(\frac{x}{n}\right) \right| \int_{n-1}^{n} (\log t)^m \, dt = \tag{3.1}$$

$$= \sum_{n \le x-1} \sum_{m \le n} h(m) \left\{ \left| R\left(\frac{x}{n}\right) \right| - \left| R\left(\frac{x}{n+1}\right) \right| \right\} + \sum_{n \le x} h(n) \left| R\left(\frac{x}{[x]}\right) \right| =$$

$$= O\left((\log x)^{m-1}\sum_{n\le x-1} n\left(H\left(\frac{x}{n}\right)-H\left(\frac{x}{n+1}\right)\right)\right)+O\left(x(\log x)^{m-1}\right)=$$

$$= O\left((\log x)^{m-1}\sum_{n\le x} H\left(\frac{x}{n}\right)\right)+O\left(x(\log x)^{m-1}\right)=$$

$$= O\left(x(\log x)^{m}\right)\ .$$

In the same way we obtain

$$\left|\left|R\left(\frac{x}{n}\right)\right|\int\limits_{n-1}^{n}(\log t)^m\,dt-\int\limits_{n-1}^{n}\left|R\left(\frac{x}{t}\right)\right|(\log t)^m\,dt\right|\le$$

$$\le\int\limits_{n-1}^{n}\left|\left|R\left(\frac{x}{n}\right)\right|-\left|R\left(\frac{x}{t}\right)\right|\right|(\log t)^m\,dt\le$$

$$\le\int\limits_{n-1}^{n}\left\{H\left(\frac{x}{t}\right)-H\left(\frac{x}{n}\right)\right\}(\log t)^m\,dt\le$$

$$\le(\log n)^{m-1}(n-1)\left\{H\left(\frac{x}{n-1}\right)-H\left(\frac{x}{n}\right)\right\}$$

and

$$\sum_{2\le n\le x}\left|R\left(\frac{x}{n}\right)\right|\int\limits_{n-1}^{n}(\log t)^m\,dt-\int\limits_{1}^{x}\left|R\left(\frac{x}{t}\right)\right|(\log t)^m\,dt=$$

(3.2)

$$= O\left((\log x)^{m-1}\sum_{n\le x-1} n\left\{H\left(\frac{x}{n}\right)-H\left(\frac{x}{n+1}\right)\right\}\right)+O\left(x(\log x)^{m}\right)=$$

$$= O\left(x(\log x)^{m}\right).$$

Equations (3.1) and (3.2) give the assertion of Lemma 3.1.

Next we collect some elementary facts about the Riemann zeta function.

Lemma 3.2. *Let $s=\sigma+it$ and $\sigma>1$. Then there exists a positive number $t_0, 0<t_0\le 2$, such that the following holds.*
(a) If $0\le t\le t_0$ then

$$|\zeta(s)|\asymp\frac{1}{|s-1|}\ \ \textit{and}\ \ |\zeta'(s)|\asymp\frac{1}{|s-1|^2}\ .$$

(b) For $t_0 \le t$ and $\sigma \ge 1$ the estimates

$$\zeta(s) = O(\log t) \qquad (t \ge t_0), \tag{3.3}$$

$$\zeta'(s) = O(\log^2 t) \qquad (t \ge t_0), \tag{3.4}$$

$$\frac{1}{\zeta(s)} = O(\log^7 t) \qquad (t \ge t_0) \tag{3.5}$$

and

$$1/\zeta(s) = O(1) \qquad (0 \le t \le t_0) \tag{3.6}$$

are valid.

Proof. The assertions of Lemma 3.2 are well-known, but for the sake of completeness we give the (elementary) proof.

Integration by parts shows that for every $\sigma > 1$ and positive integer N

$$\zeta(s) - \sum_{n=1}^{N} n^{-s} = \frac{N^{1-s}}{s-1} + s \int_N^\infty \frac{([u]-u)}{u^{s+1}} du. \tag{3.7}$$

Putting $N = 1$ gives (a). For arbitrary N

$$|\zeta(s)| \le \sum_{n=1}^{N} n^{-1} + \frac{1}{|s-1|} + |s| \int_N^\infty \frac{du}{u^{\sigma+1}} \le$$

$$\le \log N + \frac{1}{|s-1|} + \frac{|s|}{\sigma} N^{-\sigma} + O(1)$$

and the desired result (3.3) is obtained by choosing N suitably. In the same way it is easy to see that

$$\zeta'(s) = O(\log^2 t)$$

in the above region.

For the estimate of $\dfrac{1}{|\zeta(s)|}$ we use the well-known relation

$$\zeta^3(\sigma)|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \ge 1$$

for $\sigma > 1$. Then, by (3.3)

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \le |\zeta(\sigma)|^{3/4}|\zeta(\sigma+2it)|^{1/4} =$$

$$= O\left(\frac{(\log t)^{1/4}}{(\sigma-1)^{3/4}}\right).$$

If $\sigma \geq 1 + c_1 \log^{-9} t$ this implies

$$\left| \frac{1}{\zeta(\sigma + it)} \right| = O(\log^7 t) \qquad (\sigma \geq 1 + c_1 \log^{-9} t). \tag{3.8}$$

Now let $1 < \sigma < 1 + c_1 \log^{-9} t$. Then by (3.4)

$$\zeta(1 + it) - \zeta(\sigma + it) = -\int_1^\sigma \zeta'(u + it)du = O((\sigma - 1)\log^2 t). \tag{3.9}$$

Hence

$$|\zeta(1 + it)| > c_2 \frac{(\sigma - 1)^{3/4}}{\log^{1/4} t} - c_3(\sigma - 1)\log^2 t.$$

The two terms on the right are of the same order if $\sigma = 1 + \log^{-9} t$. Hence, taking $\sigma = 1 + c_4 \log^{-9} t$, where c_4 is sufficiently small,

$$|\zeta(1 + it)| > c_5 \log^{-7} t. \tag{3.10}$$

Next (3.9) and (3.10) together give

$$|\zeta(\sigma + it)| > c_5 \log^{-7} t - c_6(\sigma - 1)\log^2 t$$

and the right-hand side is positive if $\sigma < 1 + c_1 \log^{-9} t$ and c_1 is sufficiently small. This and (3.8) prove Lemma 3.2.

4. Proof of Theorem 1.2

We put $M = \mathbf{1} * \mu$, i.e. $M(x) = \sum_{n \leq x} \mu(n)$. Since

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d$$

we arrive at

$$\Lambda = L_0 * \mu = -L_0\mu * \mathbf{1}_0$$

and

$$L_0\mu = -\Lambda * \mu.$$

Thus

$$LM = L(\mathbf{1} * \mu) = L * \mu + \mathbf{1} * L_0\mu = \\ = -\mathbf{1} * (\Lambda * \mu) + L * \mu = -M * \Lambda + L * \mu \tag{4.1}$$

and

$$L^2 M = -LM * \Lambda - M * L_0\Lambda + L(L * \mu).$$

Putting $R = L * \mu$ and replacing LM by (4.1) yield

$$L^2 M = (M * \Lambda) * \Lambda - R * \Lambda - M * L_0\Lambda + LR = \\ = M * (\Lambda * \Lambda - L_0\Lambda) - R * \Lambda + LR.$$

We observe $R(x) = O(x)$ and $(R * \Lambda)(x) = O\left(x \sum_{n \le x} \frac{\Lambda(n)}{n}\right) = O(x \log x)$ and obtain

$$\log^2 x |M(x)| \le \tag{4.2}$$

$$\le \sum_{n \le x} \left| M\left(\frac{x}{n}\right)\right| \left(\Lambda(n)\log n + \sum_{dd'=n} \Lambda(d)\Lambda(d')\right) + O(x \log x).$$

Lemma 3.1 can be applied, because of Selberg's Symmetry Formula (2.11) and since $||M(t)| - |M(t')|| \le t - t'$ for $1 \le t' < t$. Then we arrive at

$$\log^2 x |M(x)| \le 2 \int_1^x \left| M\left(\frac{x}{t}\right)\right| \log t \, dt + O(x \log x) \le \\ \le 2 \log x \int_1^x \left| M\left(\frac{x}{t}\right)\right| dt + O(x \log x) = \\ = 2 (\log x) \int_1^x \frac{|M(u)|}{u^2} du + O(x \log x)$$

which gives, for $x \ge 3$, inequality (1.1) of Theorem 1.2.

Let us write Selberg's Formula (2.11) in the form

$$L\psi + \mathbf{1} * \Lambda * \Lambda = R_1$$

with $R_1(x) = 2x\log x + O(x)$. Then

$$\psi + \frac{1}{L}(\mathbf{1} * \Lambda * \Lambda) = \frac{R_1}{L}. \tag{4.3}$$

We have

$$\frac{1}{L}(\mathbf{1} * \Lambda * \Lambda) = \mathbf{1} * \frac{\Lambda * \Lambda}{L_0} + R_2,$$

where

$$\begin{aligned} R_2(x) &= \sum_{n\le x}(\Lambda*\Lambda)(n)\left(\frac{1}{\log x} - \frac{1}{\log n}\right) = \\ &= -\sum_{n\le x}(\Lambda*\Lambda)(n)\int_n^x \frac{du}{u\log^2 u} = \\ &= -\int_4^x \frac{\sum_{n\le u}(\Lambda*\Lambda)(n)}{u\log^2 u}du = \\ &= O\left(\int_4^x \frac{du}{\log u}\right) = O\left(\frac{x}{\log x}\right) \end{aligned} \tag{4.4}$$

since

$$\sum_{dd'\le u}\Lambda(d)\Lambda(d') = O\left(u\sum_{d'\le u}\frac{\Lambda(d')}{d'}\right) = O(u\log u).$$

Collecting the estimates (4.3) and (4.4) yields

$$\psi = -1 * \frac{\Lambda*\Lambda}{L_0} + \frac{R_1}{L} + R_3 \quad \text{with} \quad R_3(x) = O\left(\frac{x}{\log x}\right)$$

and

$$\begin{aligned} LM &= -\mathbf{1} * \frac{\Lambda*\Lambda}{L_0} * \mu + R + R_3 * \mu + \frac{R_1}{L} * \mu = \\ &= M * \frac{\Lambda*\Lambda}{L_0} + R_4, \end{aligned} \tag{4.5}$$

where

$$R_4(x) = O(x) + O\left(x\sum_{n\le x}\frac{1}{n\log\left(2\frac{x}{n}\right)}\right) = O\left(x\log\log x\right).$$

Addition of (4.1) and (4.3) gives

$$2\log x\,|M(x)| \;\le \sum_{2\le n\le x}\left|M\left(\frac{x}{n}\right)\right|\left(\Lambda(n)+\frac{(\Lambda*\Lambda)(n)}{\log n}\right)$$

and, since by (4.3) and (4.4)

$$\sum_{n\le x}\Lambda(n)+\sum_{n\le x}\frac{(\Lambda*\Lambda)(n)}{\log n}=2x+\left(\frac{x}{\log x}\right).$$

Lemma 3.1 proves assertion (1.2) of Theorem 1.2.

Proof of Theorem 1.3

Observing

$$\int_1^x \frac{|M(u)|}{u^2}du \le \left(\int_1^x \frac{|M(u)|^2}{u^3}du\right)^{\frac12}\left(\int_1^x \frac{du}{u}\right)^{\frac12}$$

shows

(5.1) $$\frac{|M(x)|}{x} \ll \left(\frac{1}{\log x}\int_1^x \frac{|M(u)|^2}{u^3}du\right)^{\frac12}.$$

Since $1\le u^{2/\log x}\le e^2$ for $1\le u\le x$ we get

$$\int_1^x \frac{|M(u)|^2}{u^3}du \ll \int_1^x \frac{|M(u)|^2}{u^{3+2\alpha}}du \le \int_1^\infty \frac{|M(u)|^2}{u^{3+2\alpha}}du,$$

where $\alpha=\dfrac{1}{\log x}$. Substituting $u=e^\omega$ and using Parseval's formula (1.3) gives

(5.2)

$$\frac{1}{(\log x)^{\frac12}}\int_1^x \frac{|M(u)|}{u^2}du \ll \left(\int_0^\infty \frac{|M(e^\omega)|^2}{e^{2\omega(1+\alpha)}}dw\right)^{\frac12} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\zeta(s)s|^2}\,ds\right)^{\frac12},$$

where $s = 1 + \dfrac{1}{\log x} + it$.

Putting $K(u) = \sum_{n \le u} \mu(n) \log n$ partial summation shows that for $u \ge 2$

$$M(u) = \frac{K(u)}{\log u} + \int_2^u \frac{K(t)}{t(\log t)^2} dt, \tag{5.3}$$

so that

$$\begin{aligned} \int_2^x \frac{|M(u)|}{u^2} du &\le \int_2^x \frac{|K(u)|}{u^2 \log u} du + \int_2^x \frac{|K(t)|}{t(\log t)^2} \int_t^x \frac{du}{u^2} dt \le \\ &\le \left(1 + \frac{1}{\log 2}\right) \int_2^x \frac{|K(u)|}{u^2 \log u} du. \end{aligned} \tag{5.4}$$

Observing

$$\int_2^x \frac{|K(u)|}{u^2 \log u} du \le \left(\int_2^x \frac{|K(u)|^2}{u^3} du \right)^{\frac{1}{2}} \left(\int_2^x \frac{du}{u \log^2 u} \right)^{\frac{1}{2}}$$

shows

$$\frac{|M(x)|}{x} \ll \frac{1}{\log x} \left(\int_1^x \frac{|K(u)|^2}{u^3} du \right)^{\frac{1}{2}} \tag{5.5}$$

and in the same way as above we arrive at

$$\int_1^x \frac{|M(u)|}{u^2} du \ll \left(\int_0^\infty \frac{|K(e^\omega)|^2}{e^{2\omega(1+\alpha)}} d\omega \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{F(s)}{s} \right|^2 ds \right)^{1/2}, \tag{5.6}$$

where $s = 1 + \dfrac{1}{\log x} + it$ and $F(s) = \sum_{n=1}^\infty \dfrac{\mu(n) \log n}{n^s} = -\dfrac{\zeta'(s)}{\zeta^2(s)}$.

6. Proof of Theorem 1.1

We shall describe some variants for the proof of the prime number theorem in the form $\sum_{n\le x} \mu(n) = o(x)$.

(I) We know that

$$\frac{1}{|\zeta(s)|}\cdot\frac{1}{\zeta(\sigma)} \longrightarrow 0 \ \text{ as } \sigma \to 1^+$$

uniformly for all t belonging to a given bounded interval. Then a straightforward calculation shows (see [12] and [13]) that

$$\int_{-\infty}^{\infty} \frac{1}{|\zeta(s)|^2}\frac{dt}{|s|^2} = o\left(\frac{1}{\sigma-1}\right) = o\left(\log x\right)$$

and by (1.4) and (1.1)

$$x^{-1}\sum_{n\le x}\mu(n) = o(1).$$

(II) Using part (a) and (3.5) and (3.6) of Lemma 3.2 gives

$$\int_{-\infty}^{\infty} \left|\frac{1}{\zeta(s)}\right|^2 \frac{dt}{|s|^2} = O(1) \ \text{ as } \sigma \to 1^+$$

and, by (1.1) and (1.4),

$$x^{-1}\sum_{n\le x}\mu(n) = O\left(\frac{1}{(\log x)^{\frac{1}{2}}}\right).$$

(III) Again by Lemma 3.2 we conclude

$$\int_{-\infty}^{\infty} |F(s)|^2\frac{dt}{|s|^2} = O(1) \ \text{ as } \sigma \to 1^+,$$

where $F(s) = \sum_{n=1}^{\infty}\mu(n)\log n\, n^{-s} = \left(\frac{1}{\zeta(s)}\right)'$, and this proves by (1.1) and (1.5) the assertion of Theorem 1.1.

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